A shear-lag model for a broken fiber embedded in a composite with a ductile matrix

Chad M. Landis, Robert M. McMeeking*

Department of Mechanical and Environmental Engineering, College of Engineering, University of California, Santa Barbara, CA 93106 5070, USA

Received 3 November 1997; received in revised form 17 March 1998; accepted 20 April 1998

Abstract

A shear-lag model has been developed for the prediction of stress recovery in a broken fiber embedded in a ductile-matrix composite. The model builds on the original shear-lag model of (Cox HL. Br J Appl Phys 1952:3:72–9) by introducing plasticity constitutive behavior into the matrix. The matrix is assumed to be an elastic/perfectly-plastic material that deforms according to \( J^2 \) flow theory. The use of a flow rule to govern the matrix deformation in this model differs from previous attempts to represent plasticity in the matrix. A non-linear partial differential equation is obtained from the model. Numerical solutions to the equation are obtained and compared to simpler shear-lag models which assume sliding at the fiber/matrix interface controlled by a uniform shear stress. Axisymmetric finite-element calculations were done to assess the validity of the shear-lag model. It proves to be in good agreement with the finite-element analysis. Predictions of the shear-lag calculations suggest that the global load-sharing (GLS) strength model of (Curtin WA. J Am Ceram Soc 1991;74:2837–45) is valid for a composite with a yielding matrix that is elastically rigid. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: D. Fiber composite; Shear-lag; Stress analysis; Plasticity

1. Introduction

Fibrous composite materials utilize a weak matrix reinforced by strong, elastically stiff fibers. Prediction of the mechanical properties of the composite requires a detailed understanding of the stress and strain states in the constituent materials. Owing to the complex geometry of composite materials, it is a formidable endeavor to solve the governing equations of solid mechanics in closed form. Further complications arise when the statistical nature of the fibers is considered. During loading, weak fibers will fail first, and an understanding of the resulting stresses in the reinforcements is required to predict the strength of the composite.

The term shear lag has been used to describe models that represent fibers as one-dimensional, axial load-carrying springs. Other simplifications can be made, but the universal characteristic of shear-lag models is that a three-dimensional fiber is assumed to act like a one-dimensional entity. Shear-lag models of varying degrees of complexity have been used to determine the stresses in a broken fiber. The first model we discuss is the simple shear-sliding model ([1]). This version assumes that the stress in the broken fiber recovers to the far field applied stress through the action of a constant sliding stress. The slip length, \( L_s \), axial stress, \( \sigma_f \), and shear stress, \( \tau \), along the fiber are:

\[
L_s = \frac{D E_f \varepsilon}{4 \tau_0},
\]

(1)

\[
\sigma_f = \begin{cases} 
4 \tau_0 \hat{\varepsilon}, & 0 \leq z \leq L_s \\
E_f \hat{\varepsilon}, & L_s \leq z \leq \infty,
\end{cases}
\]

(2)

\[
\tau = \begin{cases} 
-\tau_0, & 0 \leq z \leq L_s \\
0, & L_s \leq z \leq \infty,
\end{cases}
\]

(3)

where \( D \) is the fiber diameter, \( E_f \) is the Young’s modulus of the fiber, \( \varepsilon \) is the strain applied to the composite in the fiber direction, \( \tau_0 \) is the level of the shear-sliding resistance, and \( z \) is the axial coordinate along the fiber with the origin located at the break as shown in Fig. 1. This model of the fiber stress distributions was used by Curtin [2] to predict the strength of a uniaxially reinforced composite under the assumptions of global load-sharing (GLS).

However, the first shear-lag model in the literature was developed by Cox [3], to determine the stresses in a fiber embedded in an elastic matrix. The governing
The equation for this model is a second order linear ordinary differential equation. The results for the axial and shear stresses along an infinite fiber are:

$$\sigma_f = E_f \varepsilon \left[ 1 - \exp \left( -2 \frac{G_m D}{E_f W} z \right) \right], \quad (4)$$

$$\tau = -\frac{1}{2} E_f \varepsilon \left[ \frac{G_m D}{E_f W} \exp \left( -2 \frac{G_m D}{E_f W} z \right) \right], \quad (5)$$

where $G_m$ is the shear modulus of the matrix, and $w$ is a measure of the fiber spacing, also as shown in Fig. 1. The Cox model assumes that the interface between the fiber and matrix is well bonded. Hedgepeth and van Dyke [4] modeled plasticity in a multiple-fiber system by limiting the allowable shear stress on a fiber. Recently, Beyerlein and Phoenix [5] studied plasticity and debonding in a similar fashion in two-dimensional composite systems. This type of model for plasticity does not require a flow rule and we will refer to it as shear sliding. Shear sliding can be added to Cox's model by limiting the shear stress at the interface to a prescribed value of $\tau_0$. We have derived the solution to this model of our own accord but note that it may have previously appeared elsewhere. The results for the slip length, and stresses for this model are:

$$L_s = \frac{D}{4} \left( \frac{E_f}{\tau_0} - 2 \frac{G_m W}{E_m D} \right), \quad (6)$$

$$\sigma_f = \begin{cases} \frac{4\tau_0}{E_f} & \frac{G_m W}{E_m D}, \quad 0 \leq z \leq L_s \\ E_f - 2\tau_0 \sqrt{\frac{E_m W}{G_m D}} \exp \left( 2 \sqrt{G_m D} (\frac{L_s}{w} - \frac{1}{2}) \right), \quad L_s \leq z \leq \infty, \end{cases} \quad (7)$$

$$\tau = \begin{cases} -\tau_0, & 0 \leq z \leq L_s \\ -\tau_0 \exp \left( 2 \sqrt{G_m D} (\frac{L_s}{w} - \frac{1}{2}) \right), \quad L_s \leq z \leq \infty. \end{cases} \quad (8)$$

Note that if

$$\varepsilon \leq \frac{\tau_0}{2} \frac{E_f W}{E_m D}$$

then $L_s = 0$ and the axial stress and shear stress are given by Eqs. (4) and (5). While the model does limit the shear stress to a prescribed yield strength, no flow rule is used to determine the evolution of the deformation. In Cox's model equilibrium in the matrix requires that the shear stress in the matrix decreases with radial distance from the fiber. This implies that plasticity occurs in a very thin layer around the broken fiber, and that the rest of the matrix is elastic.

Alumina fibers in an aluminum matrix is an example of a composite that uses strong, brittle fibers to reinforce a weak, ductile matrix. During cooling from high temperatures after processing, the thermal expansion mismatch of the alumina fibers and the aluminum matrix causes the fibers to be in a state of residual compression, and the matrix to be in tension. When the composite is loaded no fibers will break at least until the residual compression is relieved. At the applied strain when the first fiber breaks it is very likely that the matrix has yielded in tension. In this system it is unrealistic to assume that the matrix is elastic, and that yielding only occurs locally near fiber breaks. In order to account for load redistribution around a broken fiber in the presence of gross yielding, a flow rule must be introduced to govern the deformation of the matrix. This work attempts to introduce such a constitutive behavior in the context of the shear-lag framework. The model is then compared to the other shear-lag models, as well as to axisymmetric finite-element calculations.

2. Shear-lag models

In this section the basic shear-lag governing equations are developed. The differences between shear-lag models arise as a result of differences in the assumed matrix constitutive behavior (elastic, plastic, creeping), and the interface behavior (weakly bonded versus well bonded). The assembly of a shear-lag model results in a governing differential equation for the fiber displacement, $u$, as a function of the distance from the break, $z$, and possibly a time variable, $t$. Simplifications must be made such that all of the field variables, $\sigma_f, \varepsilon_f, \varepsilon$, and $u_f$ can be related to the fiber displacement, $u$, and other variables that are parameters of the model.

2.1. Model geometry

The derivation in this paper is for a cylindrical fiber system, but the equations are easily adapted to a two-dimensional slab geometry, representative of laminated composites. Consider the cell shown in Fig. 1. The fiber break is located at $z = 0$. For a hexagonal array of fibers the cell is idealized to be circular and cylindrical. The outer surface of the cell just touches the edges of the six nearest neighbor fibers. The distance between fiber surfaces can be related to the fiber volume fraction, $f$, by
\[ w = D \left( \frac{1}{\sqrt{\pi}} \sqrt{\frac{\pi}{2\sqrt{3}}} - 1 \right). \]  

(9)

2.2. Equations common to all shear-lag models

Equilibrium of forces acting on a differential element of fiber requires that
\[ \frac{\partial \sigma_f}{\partial z} = -\frac{4 \tau}{D}, \]  

(10)

where \( \sigma_f \) is the average axial stress in the fiber, and \( \tau \) is the shear stress acting on the interface between the fiber and the matrix. Hooke’s Law for the fiber gives
\[ \sigma_f = E_f \frac{\partial u}{\partial z}. \]  

(11)

where \( u \) is the axial displacement of the fiber, and \( \partial u / \partial z \) is the axial strain in the fiber. The standard shear-lag statement that relates the shear stress at the fiber/matrix interface, \( \tau \), to the fiber displacement, \( u \), is obtained by substituting Eq. (11) into Eq. (10). It is given by
\[ \tau = -\frac{1}{4} D E_f \frac{\partial^2 u}{\partial z^2}. \]  

(12)

The fiber is considered to be a one-dimensional entity such that the only quantities of interest are the axial stress, strain, and displacement. The radial, hoop, and shear stresses within the fiber are not considered in the model. Eqs. (11) and (12) characterize the stress states of the fiber and fiber/matrix interface in terms of the fiber displacement. To complete the model, the field quantities in the matrix must now be related to the fiber displacement.

The first approximation common to many shear-lag models [3,6] is that the axial displacement at the outer boundary of the cell is equivalent to the displacement that would exist in an undamaged composite at the same boundary of the cell. The second assumption involves approximating the shear strain at the fiber/matrix interface. The simplest approximation, used by Du and McMeeking [6], is that
\[ \gamma = \frac{u_c - u}{w}. \]  

(13)

where \( u_c \) is the displacement of the outer boundary, and \( \varepsilon \) is the nominal composite strain.

The second assumption involves approximating the shear strain at the fiber/matrix interface. The simplest approximation, used by Du and McMeeking [6], is that
\[ \gamma = \frac{u_c - u}{w}. \]  

(14)

where \( \gamma \) is the shear strain at the interface. Cox [3] accounted for the axisymmetric geometry in an approximate fashion by considering equilibrium of the matrix in the axial direction. The result is that in Eq. (14), \( w \) is replaced by
\[ \frac{D}{2} \ln \left( \frac{2w + D}{D} \right). \]  

This is a small correction for high volume fractions. For simplicity, this correction has been left out of the model that will be presented. Eqs. (13) and (14) are the assumptions subject to the most scrutiny in single fiber shear-lag models. For a multiple-fiber shear-lag model, \( u_c \) would be replaced by the displacement of an adjacent fiber. The addition of multiple fibers requires the solution to a set of coupled differential equations governing the displacements of the fibers. In order to derive the single governing equation for a single fiber shear-lag model, assumptions about the displacement at the boundary of the cell must be made.

Eqs. (9)–(14) are the starting point for any shear-lag model that attempts to describe the stress distribution in a broken fiber. The remaining pieces of information needed to formulate a governing differential equation are the matrix constitutive law, and the fiber/matrix interface behavior. When the applied strain is high enough to cause fibers to break, the matrix has usually failed in tension by cracking or yielding. It is assumed therefore, that the matrix is not able to directly carry any of the load shed by the broken fiber, but can only transmit it by shear stress to other fibers. In single fiber shear-lag models there are no other fibers so load is essentially transferred to the boundary of the cell. The primary purpose of the matrix constitutive law is to relate the shear stress in the matrix to the shear strain.

2.3. Shear-lag constitutive law for a well bonded fiber in an elastic/perfectly-plastic matrix

The constitutive law used is for an elastic/perfectly-plastic material with deformation governed by \( J_2 \) flow theory ([7]). The Young’s modulus of the material is \( E_m \), and the yield stress is \( \sigma_y \). For simplicity it is assumed that the material is always on the yield surface. For most systems of interest this is a good assumption for one of two reasons. First, residual thermal stresses can cause the matrix to yield prior to any loading. Second, if it did not yield upon cooling, the matrix has usually yielded in tension when enough strain has been applied to break a fiber. A simple state of stress is assumed to exist in the matrix near the fiber/matrix interface, namely an axial stress acting in the \( z \) direction and a shear stress as depicted in Fig. 1. Again, radial and hoop components of stress are not considered in the model. The yield condition is therefore,
\[ \sigma_y = \sqrt{\sigma_m^2 + 3\tau^2} \]  

(15)

where \( \sigma_m \) is the axial stress in the matrix and \( \tau \) is the shear stress acting on the fiber/matrix interface. The strain rates in the matrix are given by \( J_2 \) flow theory and elasticity,
\[ \dot{\varepsilon}_m = \frac{\sigma_m}{E_m} + \frac{2}{3} \sigma_m \dot{\lambda} \]  

(16)
\[ \dot{\gamma} = \frac{t}{G_m} + 2\dot{\varepsilon} \] (17)

where \( \varepsilon_m \) is the axial strain in the matrix at the interface, \( \gamma \) is the shear strain in the matrix at the interface, \( \lambda \) is a proportionality factor enforcing normality of plastic strains to the yield surface, and the superposed dot represents differentiation with respect to time. Note that this time does not necessarily correspond to natural time, but is a loading parameter.

Eqs. (16) and (17) can be combined to eliminate \( \lambda \), yielding an equation governing the matrix response

\[ \frac{3\sigma_m}{E_m} - \frac{\sigma_m}{G_m} = 3\dot{\varepsilon}_m - \sigma_m \dot{\gamma}. \] (18)

From Eq. (15), it can be shown that \( \sigma_m = -3\dot{\varepsilon}/\dot{\sigma}_m \).

With the use of \( \varepsilon_m = \partial u/\partial z \) at the interface and Eqs. (12)–(14), (18) becomes the governing partial differential equation for \( u(z,t) \)

\[ \left( 9 \frac{D}{E_m} \right) \dot{u} + \frac{1}{4} \frac{D}{G_m} \sigma_m \frac{\partial^2 \dot{u}}{\partial z^2} - 3\dot{\sigma}_m \frac{\partial \dot{u}}{\partial z} - \frac{\sigma_m}{w} (u - \dot{\varepsilon}z) = 0 \] (19)

It is important to note that \( \dot{\varepsilon} \) is the nominal composite strain rate. At first glance Eq. (19) may appear to be a linear partial differential equation for \( u(z,t) \). However, all of the coefficients of the derivatives contain \( \sigma \) or \( \sigma_m \). Eq. (15) relates \( \sigma_m \) to \( \tau \), and \( \tau \) is related to \( u \) by Eq. (12). Thus the coefficients of the derivatives are actually non-linear functions of \( \partial^2 u/\partial z^2 \), and consequently Eq. (19) is highly non-linear.

At this point it is useful to summarize and compare the constitutive laws used to obtain Eqs. (1)–(3), Eqs. (6)–(8) and (19). Fig. 2 is a schematic of the three constitutive behaviors. The figure shows a shear stress being applied to a rigid (not fiber) material which is bonded to the matrix material. The rigid material is used instead of fiber material to isolate the constitutive behaviors of the matrix and interface. For the simple shear-sliding model of Kelly and Tyson [1] the matrix is rigid, i.e. there is no shear deformation in the matrix, and the interface is able to support any shear stress up to \( \tau_0 \). When the shear stress reaches \( \tau_0 \) deformation occurs in the form of sliding at the interface. For the constitutive behavior used to derive Eqs. (6)–(8) the matrix is able to deform elastically and then sliding ensues at the interface when the shear stress reaches \( \tau_0 \). The behavior for the elastic/perfectly-plastic matrix depends on the initial state of stress in the matrix. We assume that the matrix has yielded in tension before any shear stress is applied (this assumption is used throughout this paper). Fig. 2 shows how the shear stress and axial stress change as shear strain is applied to the system while the axial strain remains constant.

To solve Eq. (19) boundary conditions are needed at \( z = 0 \) and at \( z \to \infty \). At the fiber break the axial stress in the fiber is zero, and far from the break the fiber stress must be equal to the applied composite stress. Mathematically these boundary condition are:

\[ E_f \frac{\partial u}{\partial z} (z = 0) = 0, \] (20)

\[ \frac{\partial u}{\partial z} (z \to \infty) = \varepsilon. \] (21)

A time history for the applied strain must also be specified. The key variables that must be specified in the time history are the strain at which the fiber breaks and the strain to which the composite is extended after the fiber breaks. Eq. (19) is solved by a finite difference scheme that is described in the Appendix.

Figs. 3 and 4 illustrate the stress profiles from the solution of Eq. (19) immediately after fracture for fibers that have broken at various different composite strain levels with \( E_f/E_m = 5 \), and \( f = 0.5 \). \( G_m \) has been assumed to be \( 3/8 E_m (\nu = 1/3) \) for all of these calculations. The shear stress versus distance from the break plots of Fig. 3 consist of three regions. The slip zone is where the shear stress is almost uniform at the value \( \tau = \tau_s = \sigma_s/\sqrt{3} \). The non-linear recovery region is where there is a non-linear transition from the maximum plateau shear stress to nearly zero shear stress. Finally, the
third region is where the shear stress is essentially zero. The shear stress approaches $\tau_y$ only asymptotically near the break, and approaches zero asymptotically far from the break. Therefore, the entire curve actually lies in the non-linear region. For practical considerations the non-linear recovery region is taken to be finite with boundaries at the position where $\tau = 0.99\tau_y$ and $\tau = 0.01\tau_y$. Notice that the non-linear recovery zone does not change size or shape significantly as the applied strain when the break occurs is increased. At higher strains, the non-linear recovery region is located further from the break and the length of the slipped region increases. Similarly, the axial stress profiles of Fig. 4 can be broken into three regions, a nearly linear region starting from the break, a non-linear recovery region, and a nearly uniform region. As with the shear stress, the axial stress approaches a uniform gradient near the break and a uniform value far from the break only asymptotically.

The non-linear zone in the axial stress distribution also changes location with different breaking strains while its size and shape remain the same. The positions of the non-linear recovery zones for the shear and axial stresses are coincident. For a given moduli ratio and volume fraction, the location of the non-linear recovery region depends on the composite strain. Higher applied strain implies that the region is located at a position further from the break. Notice in Fig. 3 that if the applied strain at which the break occurs is very low, then the initial uniform shear stress region will not exist and the non-linear recovery region will not be fully developed.

A closed form solution of Eq. (19) is not available so a numerical investigation was used to determine some of the characteristics of the recovery region. The size and shape of the non-linear recovery region does not depend on the applied strain (as long as the strain is sufficiently high), but it does depend on the parameters $E_i/E_m$ and $w/D$ (again note that we have chosen $G_m = 3/8 \ E_m$). Three parameters were chosen to describe the non-linear region: its length, $L_n$, the applied strain at which it is fully developed, $\varepsilon_n$, and the average axial stress in the region when the fiber breaks at an applied composite strain of $\varepsilon_n$. $\bar{\pi}_n$. Fig. 5 contains a schematic that illustrates each of these parameters. The slip length, $L_s$, can be obtained from the second parameter, $\varepsilon_n$, by the following formula,

$$L_s = \frac{D}{4} \left( \frac{E_i\varepsilon_n}{\tau_y} - \frac{E_m\varepsilon_n}{\tau_y} \right). \tag{22}$$

Notice the similarity to Eq. (6). Fig. 5 shows a schematic of these parameters. For a given value of $w/D$, all of these parameters can be fitted to a power law of the form

$$P = A \left( \frac{E_i}{E_m} \right)^n$$

The table below shows the fits to a power law for four values of the fiber volume fraction.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$w/D$</th>
<th>$f$</th>
<th>$A$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_n$</td>
<td>0.25 (0.58)</td>
<td>2.948</td>
<td>0.532</td>
<td></td>
</tr>
<tr>
<td>$0.5$ (0.40)</td>
<td>3.963</td>
<td>0.544</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1$ (0.23)</td>
<td>5.317</td>
<td>0.552</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$ (0.10)</td>
<td>7.149</td>
<td>0.554</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{E_i\varepsilon_n}{\sigma_y}$</td>
<td>0.25 (0.58)</td>
<td>1.342</td>
<td>0.600</td>
<td></td>
</tr>
<tr>
<td>$0.5$ (0.40)</td>
<td>1.582</td>
<td>0.648</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1$ (0.23)</td>
<td>1.865</td>
<td>0.680</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$ (0.10)</td>
<td>2.149</td>
<td>0.707</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{\bar{\pi}_n \ L_n}{\sigma_y}$</td>
<td>0.25 (0.58)</td>
<td>3.380</td>
<td>1.125</td>
<td></td>
</tr>
<tr>
<td>$0.5$ (0.40)</td>
<td>5.409</td>
<td>1.183</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1$ (0.23)</td>
<td>8.619</td>
<td>1.222</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2$ (0.10)</td>
<td>13.450</td>
<td>1.250</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3. Solution of Eq. (19) for the shear stress distribution, $\tau(z)$, in a broken fiber as a function of the distance from the break, $z$, for different values of applied far field strain.

Fig. 4. Solution of Eq. (19) for the axial stress distribution, $\sigma(z)$, in a broken fiber as a function of the distance from the break, $z$, for different values of applied far field strain.

Fig. 5. A schematic of the three parameters characterizing the non-linear recovery region, and their fits to a power law for four values of the fiber volume fraction.
where \( P \) is one of the three parameters, \( A \) is a constant, and \( n \) is a power law exponent. The fits for \( L_n, \sigma_n, \) and \( \sigma_w \) have maximum errors between the fitted curves and the actual data points of 2\%, 5\%, and 7\%, respectively.

Fig. 5 contains a table listing values of \( A \) and \( n \) for values of \( w/D \) between 0.25 and 2. Note that as \( E_f/E_m \) or \( w/D \) go to zero, \( L_n, \sigma_n, \) and \( \sigma_w \) also go to zero, corresponding to the simple shear-sliding approximation (Eqs. (2) and (3)). This becomes evident after careful consideration of Eq. (19). As \( E_f/E_m \) goes to zero, or \( w/D \) goes to zero the only solution available for Eq. (19) with the appropriate boundary conditions is a step function for the shear stress. As \( w/D \) goes to zero, the distance between fibers is zero. The denominator of Eq. (14) goes to zero, and therefore the shear strain in the matrix can take on only two values. Either \( u = u_c \) and the shear strain is zero and the state of stress at the interface is uniaxial extension, or \( u \neq u_c \) and the shear strain is infinite and the state of stress is pure shear. This is the physical origin for the step function shape of the shear stress distribution when \( w/D \) goes to zero. As \( E_f/E_m \) goes to zero the physical reasoning for the step function shape of the shear stress is not as simple. However, mathematically it is clear that a step function shape for the shear stress, and consequently a bilinear distribution for the axial stress, is the solution to Eq. (19) if the matrix is elastically rigid.

The three parameters listed above were not chosen arbitrarily. All three of the above parameters are needed to determine the slip length in a broken fiber and the average stress in the broken fiber. The slip length and average stress in the fiber are needed if a GLS type model is used to predict the strength of a composite with load transfer governed by Eq. (19). The parameters \( L_n \) and \( \sigma_w \) have another significance as well. Eq. (22) requires \( \sigma_w \) in order to calculate the slip (nearly constant shear stress) region for a given loading. Landis and McMeeking [8] have shown that the relative sizes of the slip length and non-linear recovery lengths govern stress concentrations in fibers neighboring a fiber break. Relatively long slip lengths cause low stress concentrations and short slip lengths (with respect to \( L_n \)) yield higher stress concentrations.

As noted above, the strain at which the break occurs and the strain to which the fiber is subsequently loaded must be specified. Owing to the strain path dependence of \( J_2 \) flow theory, different loading conditions result in different stresses in the fiber. This has a small effect on the distribution of stress in the fiber, but it is interesting since the simpler model of Eqs. (7) and (8) cannot predict it. Consider a fiber that is initially broken, i.e. the strain to cause the break is zero. The composite is then extended and stresses develop in the fiber. Figs. 6 and 7 illustrate the differences in the shear and axial stress profiles for a fiber that is initially broken and loaded to the applied strain (denoted ‘initial break’) and a fiber that breaks at the applied strain (denoted ‘loaded break’). Also on Figs. 6 and 7 are the results from the simple sliding model of Kelly and Tyson, Eqs. (1)–(3), and the model with sliding and elastic recovery, Eqs. (6)–(8). Notice that Eqs. (6)–(8) do a very good job of predicting the stress profiles for a fiber that breaks at the applied strain, the ‘loaded break’. This is in fact true for most combinations of material parameters. The stress distributions can be broken up into the same regions that were described for a fiber that breaks at finite strain. The axial and shear stress profiles will develop and then the non-linear region will move along the fiber. For a given set of parameters, \( E_f/E_m, w/D, \) and \( E_f/\sigma_f \), the length of the non-linear recovery zone for a fiber that breaks at strain \( \varepsilon \) will be shorter than the same zone for a fiber that is initially broken in a composite then extended to strain \( \varepsilon \).
Fibers that break finite strain and then subsequently loaded will have the stress profile of a loaded break initially, then the stress profile will approach the distribution of the initial break as increasing amounts of strain are applied. This behavior may have interesting effects on stress concentrations in neighboring fibers. Generally, as shown by Landis and McMeeking [8], as strain is increased when sliding is present the stress concentration in a neighboring fiber decreases. In [8] a constitutive law like the one used in Eqs. (6)–(8) was used and therefore the non-linear recovery length remains constant while the slip length grows. In Eq. (19) both $L_w$ and $L_s$ can change simultaneously. Unfortunately, the model presented here does not consider other fibers and the effect on their stresses cannot be found. The dependence of the shear and axial stress distributions on the loading sequence arises from the load path dependence of a plastically deforming material.

Returning to fibers that break at finite strain, three parameters are needed to solve Eq. (19): $E_y/E_m$, $w/D$, and $E_y/\sigma_y$ (notice that $v_m$ is taken to be 0.5 and $G_m = 3/8E_m$, otherwise $E_y/G_m$ would also have to be specified). Figs. 8 and 9 are plots that illustrate the influence of these parameters. The axial coordinate normalization $4E_y/E_m\ z/D$ maps a single simple shear sliding slip length, $L_s$, to one unit. The axial stress normalization $\sigma_y/E_y$ in Fig. 9 causes a value of unity to represent the far field stress in the fiber. The shear and axial stress profiles of the simple shear-sliding model are shown as a comparison in Figs. 8 and 9, i.e. Eqs. (3) and (2), respectively. Also, Eqs. (7) and (8) give very good approximations to the shapes of the axial and shear stress distributions respectively. As $E_y/E_m \to 0$, $w/D \to 0$, or $E_y/\sigma_y \to \infty$ the shear and axial stress profiles from the solution of Eq. (19) approach the simple shear-sliding curves. Note that the limits $E_y/E_m \to 0$ and $w/D \to 0$ will create a true step function for the shear stress distribution, while the limit $E_y/\sigma_y \to \infty$ causes the approach to a step function due to the choice of the normalization.

3. Finite-element model

Finite-element calculations were done to determine the validity of the shear-lag model with the elastic/perfectly-plastic matrix constitutive behavior. The model does not consider the exact three-dimensional geometry of the fiber composite system, but instead idealizes the geometry as axisymmetric ([9–12]). Fig. 10 is an illustration of the axisymmetric model along with the actual values used for volume fraction, moduli ratio, and applied strain to calculate solutions for the stresses. The model has a broken fiber at the center, an annulus of matrix of width $w$ surrounding the broken fiber, an annulus of fiber material representing the nearest neighbor fibers, an annulus of matrix, and finally an annulus of material that has the averaged composite properties. The width of the annulus of fiber material, denoted $m$ in Fig. 7, is chosen such that its volume is equal to the volume of the six nearest neighbor fibers. The width of the second annulus of matrix material is taken to be one half of $w$, and the thickness of the composite annulus is $2w$. The length of the model, $L_s$ is chosen such that the shear stress at the fiber/matrix interface is approximately zero at $z = 2/3 \ L_s$ for the listed applied strain. The fiber material is elastic and the matrix material is elastic/perfectly-plastic. The yield stress of the matrix, $\sigma_y$, is taken to be $E_m/1000$. The Poisson’s ratio of all materials is taken to be one third. The loading conditions are for a fiber that has broken at a finite strain. The model is extended uniformly to the strain at which the fiber fails. This also causes the matrix to yield. Thereafter, the tractions on the end of the broken fiber are removed incrementally while the displacements of the remaining boundary of the model are held fixed. No further strain is applied to the model.
after this process is completed. The model was computed with four sets of moduli ratios and three volume fractions as shown in Fig. 10. The commercial finite-element package ABAQUS was used and the results for the shear stress at the fiber/matrix interface are compared to the shear-lag model.

The shear-lag results are shown to be in excellent agreement with the more detailed finite element results. Fig. 11(a) and (b) plot the shear stress from the solution of Eq. (19) against the finite-element results. Also shown are the results from Eq. (8), which is the shear-lag model combining Cox’s original elastic version with a shear-sliding interface zone near the break. The zero point on the $z$ axis is taken to be where the non-linear recovery region begins, i.e. where $\tau = 0.99\tau_c$. This is done for two reasons. First it is easier to compare the different non-linear recovery zones. Second, any shear stress distribution for a higher strain than $\varepsilon_n$ can be constructed by adding a uniform shear stress zone from $z = 0$ to $z = L_s$ (where $L_s$ is given by Eq. (22)), and moving the beginning of the non-linear recovery region to $z = L_s$. The value of $\varepsilon_n$ is also given on Fig. 11(a) and (b). The composite properties represented were chosen such that a wide range of non-linear recovery lengths would be presented. Fig. 8(a) and (b) show that there is good agreement over a wide range of moduli ratios and fiber volume fractions between Eq. (19) and the finite-element calculations.

Eq. (8) is also a good approximation to the shear stress distribution for most of the material systems considered. Eq. (8) has a sharp kink in the plots of shear stress versus distance from the break distributions where the shear stress begins to drop off from the plateau stress. Eq. (8) cannot predict differences in the stress distribution due to different loading conditions, as the solution of Eq. (19) is able to do as discussed in the previous section. In general Eqs. (7) and (8) give a good approximation to the stresses in a broken fiber in a plastic matrix, but Eqs. (7) and (8) do not show some of the finer details of the solution to Eq. (19).

### 4. Discussion

Modeling the strength of composite materials requires an understanding of how stresses are distributed around fiber breaks. A complete solution to the governing equations (equilibrium, compatibility, and constitutive behavior) for this complicated three-dimensional system would require a great deal of numerical computation. However, the shear-lag approximation has been shown to compare well with an axisymmetric finite-element
model, and is much less onerous in computation. The shear-lag approximation allows all field quantities to be related to the axial displacement of the broken fiber. Depending on the complexity of the matrix constitutive behavior, the solution of the shear-lag model is either a closed form expression or requires minimal numerical calculations.

The model developed was one for continuous, brittle, elastic fibers well bonded to a ductile matrix with elastic/perfectly-plastic constitutive behavior. The solutions to the model were compared to a simple version where the shear stress is assumed to be constant over a finite slip region around the break. Three parameters control the shape of the stress profiles in the fiber, \( E_f/E_m \), \( w/D \), and \( E_f/\sigma_y \) (again we assume \( G_m = 3/8 E_m \)). The first two parameters dictate the size and shape of the non-linear region of the shear stress and axial stress distributions, and the third parameter specifies where the non-linear region is located along the fiber. The simple sliding model for the axial stress profile in the fiber serves as an upper bound to the axial stress predicted by the shear-lag model of Eq. (19). For volume fractions where the fibers are not in contact, as the parameter \( E_f/E_m \) goes to zero, i.e. the matrix material is elastically rigid, the stress profiles approach those of the simple sliding model.

This is an interesting result when the assumptions of the GLS composite strength model are considered (Curtin [2]). Originally, GLS was used for ceramic matrix composites but it has also been used to predict the strength of metal matrix composites [10]. The GLS model assumes that the load shed by broken fibers is...
shared equally by all intact fibers on the plane of the break. No load is carried by the matrix. It is also assumed that shear sliding occurs at a uniform sliding stress over a finite distance around the break. It has been shown by the shear-lag model that the latter assumption is true when the matrix is elastically rigid in shear and yielding occurs at the interface. It is interesting to note that the former assumption also holds when the matrix is rigid in shear. Landis and McMeeking [8] developed a model to predict stress concentrations in fibers adjacent to a broken fiber in the presence of interface sliding. They showed that when $L_a \gg L_m$, i.e. $L_a/L_m \rightarrow \infty$, the stress concentration in a fiber neighboring a break vanishes. This work, as well as [8], shows that $L_a \rightarrow 0$ when $G_m \rightarrow \infty$. Therefore the GLS composite strength model is a description of a composite with long brittle elastic fibers embedded in a matrix that will not carry axial loads, but is rigid in shear, and ‘yields’ at a prescribed level of shear stress. This ‘yielding’ can take the form of localized shear yielding for a ductile matrix, or interface sliding for a brittle matrix.

The main deficiency of this model is that it does not consider the stresses in fibers neighboring the break. While not done in this work, it is possible to generalize the governing equations developed here to include interactions with other fibers. In order to truly understand composite failure, knowledge of the stress concentrations in neighboring fibers is needed. While the model presented here does not increase our knowledge of stress concentrations in composites, it does develop the basic mechanics that are necessary to understanding failure in ductile matrix composites and also illustrates some phenomena that cannot be captured by simpler models.

Often shear-lag models are misconstrued as exact solutions. Shear-lag models make many simplifying assumptions to the governing equations of solid mechanics. The elastic/perfectly-plastic matrix response lends itself to the shear-lag description. It has been shown here that the model is in good agreement with more detailed numerical calculations and a simpler shear-lag model for a wide range of composite properties.

Acknowledgements

The work was sponsored by the Advanced Research Projects Agency through the University Research Initiative at the University of California, Santa Barbara (ONR Contract N-0014-92-J-1808)

Appendix A. Finite difference scheme

In this appendix the finite difference scheme that was used to solve Eq. (19) of the main text is described. The stresses, strains, lengths, displacements, and moduli are normalized as follows:

$$\hat{\varepsilon} = \frac{\tau}{\sigma_f}, \quad \hat{\sigma}_m = \frac{\sigma_m}{\sigma_f}, \quad \hat{\sigma}_f = \frac{\sigma_f}{\sigma_f}, \quad \dot{\varepsilon} = \frac{E_f e}{\sigma_f}, \quad \frac{\hat{\sigma}_m}{\sigma_f} = \frac{\hat{\sigma}_f}{\sigma_f}, \quad \hat{\dot{w}} = \frac{w}{D}, \quad \hat{z} = \frac{z}{D},$$

$$\hat{\dot{u}} = \frac{E_m u}{\sigma_f}, \quad \hat{\dot{E}} = \frac{E_m e}{\sigma_f}, \quad \text{and} \quad \hat{\dot{G}} = \frac{G_m}{\sigma_f}.$$

Eq. (19) becomes

$$\left(\frac{9}{4} \hat{\tau}^2 + \frac{2}{4} \hat{\sigma}_m \frac{\hat{\sigma}_m}{\sigma_f} \right) \frac{\hat{\dot{u}}}{\sigma_f} = 3 \hat{\dot{\varepsilon}} \hat{\sigma}_m \frac{\hat{\dot{w}}}{w} - \hat{\sigma}_m (\hat{\dot{u}} - \hat{\varepsilon}) = 0. \quad (A.1)$$

The fiber is discretized into $N - 1$ equal segments of length $\Delta \hat{z} = \hat{L}/N$, where $\hat{L} = L/D$, and $L$ is the total length of the fiber. For the calculations done in this paper, $L$ is taken to be sufficiently long such that the solutions represent an infinite fiber. The time history of the loading is then broken into two segments. The first segment is when the fiber just breaks and the applied strain rate is zero. During this time segment the traction at the broken end of the fiber are released gradually to zero. During the second time segment the applied strain rate is not zero, but the strain at the broken end of the fiber is always zero. Each of these time segments is discretized into a finite number of time steps $\Delta t$. It is assumed that the fiber displacements are known everywhere along the fiber at time $t$. A centered finite difference scheme yields:

$$\frac{\hat{\tau}^2}{\sigma_f} \frac{\hat{\dot{u}}}{\sigma_f} \left[ \frac{\hat{u}_{i+1}}{} - 2 \hat{u}_i + \hat{u}_{i-1} \right] \frac{\Delta \hat{z}}{} = \frac{\hat{\dot{\varepsilon}} \hat{\sigma}_m}{2 \Delta \hat{z}}, \quad (A.2)$$

$$\frac{\hat{\dot{w}}}{w} \hat{\dot{u}} = \frac{\hat{\dot{w}}}{w} \hat{u}_{i+1} - \frac{\hat{\dot{w}}}{w} \hat{u}_{i-1}, \quad (A.3)$$

$$\hat{\dot{u}} = \frac{\hat{\dot{u}}}{\Delta t}, \quad (A.4)$$

$$\hat{\dot{\varepsilon}} = -\frac{1}{4} \left( \frac{\hat{u}_{i+1}}{} - 2 \hat{u}_i + \hat{u}_{i-1} \right) \frac{\Delta \hat{z}}{}, \quad (A.5)$$

$$\hat{\sigma}_m = \sqrt{1 - 3 \hat{\tau}_i^2}. \quad (A.6)$$

Let

$$\alpha_i = -\frac{1}{4 \Delta \hat{z}} \left( \frac{9}{4} \hat{\tau}_i^2 + \frac{2}{4} \hat{\sigma}_m \right) \sigma_f + \frac{3}{2} \hat{\dot{\varepsilon}} \hat{\sigma}_m, \quad (A.7)$$

$$\beta_i = \frac{1}{2 \Delta \hat{z}} \left( \frac{9}{4} \hat{\tau}_i^2 + \frac{2}{4} \hat{\sigma}_m \right) \sigma_f + \frac{3}{2} \hat{\dot{\varepsilon}} \hat{\sigma}_m, \quad (A.8)$$

$$\gamma_i = -\frac{1}{4 \Delta \hat{z}} \left( \frac{9}{4} \hat{\tau}_i^2 + \frac{2}{4} \hat{\sigma}_m \right) \sigma_f + \frac{3}{2} \hat{\dot{\varepsilon}} \hat{\sigma}_m, \quad (A.9)$$

$$\delta_i = \frac{\hat{\sigma}_m}{w} \hat{\dot{\varepsilon}} \Delta \hat{t}_i \Delta \hat{z}, \quad (A.10)$$

where the subscripts denote positions along the fiber in $z$ space, and superscripts denote material times in $t$ space.
The solution of Eq. (19) results in the solution of a tridiagonal matrix for each time step. Notice that the coefficients \( \alpha, \beta, \gamma, \) and \( \delta \) change after each increment of time. The difference equation for an interior point on the fiber becomes

\[
\alpha_i \Delta_i u_{i+1}^{t+1} + \beta_i \Delta_i u_{i}^{t+1} + \gamma_i \Delta_i u_{i-1}^{t+1} = \alpha_i \Delta_i u_{i+1}^{t} + \beta_i \Delta_i u_{i}^{t} + \gamma_i \Delta_i u_{i-1}^{t} + \delta_i.
\]  
(A.11)

Eq. (A.11) is valid for \( 1 < i < N \).

The equation at \( i = N \) is

\[
\Delta N u_{N}^{t+1} = \Delta N u_{N}^{t} + \dot{\varepsilon} \Delta N t.
\]  
(A.12)

For the node at the fiber break, \( i = 1 \)

\[
\dot{\varepsilon}_1 = -\frac{1}{4} \left( -\frac{\Delta (\dot{\varepsilon}_1^0 + 8 \dot{\varepsilon}_1^0 - \dot{\varepsilon}_1^0)}{2 \Delta^2} \frac{3}{\Delta^2} \right).
\]  
(A.13)

where \( \varepsilon_1^0 \) is the normalized strain at the break for time step \( t \). For the first time segment when the fiber just breaks, \( \dot{\varepsilon}_1^0 \) will be a positive value. For subsequent loading \( \dot{\varepsilon}_1^0 \) will be zero. By defining

\[
\beta_1 = \frac{1}{4 \Delta^2} \left( \frac{\Delta^2}{E} + \frac{\Delta^2}{G} \right) - \frac{3}{2} \Delta \frac{\sigma_1}{\Delta^2} \frac{\Delta^2}{G},
\]  
(A.14)

\[
\gamma_1 = \frac{1}{4 \Delta^2} \left( \frac{\Delta^2}{E} + \frac{\Delta^2}{G} \right) + \frac{3}{2} \Delta \frac{\sigma_1}{\Delta^2} \frac{\Delta^2}{G},
\]  
(A.15)

\[
\delta_1 = \frac{\Delta^2}{2 \Delta^2} \frac{\Delta^2}{G} - \frac{1}{4} \left( \frac{\Delta^2}{E} + \frac{\Delta^2}{G} \right) \frac{\Delta^2}{G} \dot{\varepsilon}_1^0 \Delta t,
\]  
(A.16)

where \( \dot{\varepsilon}_1^0 \) is the rate at which strain is increasing at the break, \( \dot{\varepsilon}_1^0 \) is negative during the break process, and zero during subsequent loading. The difference equation for the node at \( i = 1 \) is

\[
\beta_i \Delta_i u_{i+1}^{t+1} + \gamma_i \Delta_i u_{i}^{t+1} + \delta_i = \beta_i \Delta_i u_{i+1}^{t} + \gamma_i \Delta_i u_{i}^{t} + \delta_i.
\]  
(A.17)

The shear stresses in Eqs. (A.5) and (A.13) must not be allowed to exceed the shear yield strength. Therefore if the value of \( \tau \) is very close to \( \tau_y \) then it is given the value \( \tau_y \) so that the next time increment will not violate the yield condition.

The initial displacement profile is \( \Delta u_1^0 = \dot{\varepsilon}_1^0 \), where \( \dot{\varepsilon}_1^0 \) is the normalized strain when the fiber breaks. Eqs. (A.11), (A.12) and (A.17) constitute a tridiagonal set of \( N \) linear algebraic equations to be solved for the displacements at time \( t + 1 \). The updated displacements are used to calculate updated shear stresses, and the process continues until the applied strain reaches its final value.

References