Time-Varying SDPs

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Joint work with Amir Ali Ahmadi
Problem:

Find a path $x : [0, 1] \rightarrow \mathbb{R}$

with minimum length

$$\int_0^1 \sqrt{1 + x'(t)^2} \, dt$$

Such that

$$p(t) \leq x(t) \leq q(t)$$

$$-\gamma \leq x''(t) \leq \gamma$$

\forall t \in [0, 1]$$
**Framework**

**SDP:** Find $x \in \mathbb{R}^n$ that minimizes $\langle c, x \rangle$

subject to $A_0 + \sum A_i x_i \succeq 0 \quad (A_i \in \mathbb{R}^{m \times m})$

**TV-SDP:** Find $x : [0, 1] \rightarrow \mathbb{R}^n$ that minimizes $\int_0^1 \langle c(t), x(t) \rangle \, dt$

subject to

$A_0(t) + \sum A_i(t) x_i(t) \preceq \sum \int_0^1 D_i(t, s)x_i(s)ds \succeq 0$

$\forall t \in [0, 1]$

**Data:** $c(t), A_i(t), D_i(t, s)$ polynomials.
How to get good solutions?

How to get lower bounds on the optimal value?

- Length: 1.64
- Found via a small SDP
- No other curve has length < 1.63

This is a TV-SDP!
Optimal value of TV-SDP

Plan

Primal approach: Search for polynomial solutions of degree d, increase d

Dual approach: Finite dimensional relaxation

Focus on polynomials:
- Smooth
- Tractable

When are polynomial solutions optimal?

When are these lower bounds tight?
Our Constraint:

\[ A_0(t) + \sum A_i(t)x_i(t) + \sum \int_0^1 D_i(t, s)x_i(s)ds \geq 0 \]

\[ X(t) \]

\[ X(t) \geq 0 \quad \forall t \in [0, 1] \quad \iff \quad y^TX(t)y \geq 0 \quad \forall y \quad \forall t \in [0, 1] \]

\[ y^TX(t)y = \sigma_0(t, y) + t \sigma_1(t, y) + (1 - t)\sigma_2(t, y) \]

where the \( \sigma_i(t, y) \) are SOS

Takeaway: (Small) SDP can find the best polynomial solution of a given degree to a TV-SDP
Primal Approach (2/2)

What can go wrong?

A "discontinuous" TV-SDP

\[(t - \frac{1}{2})x(t) \geq 0, \quad (t - \frac{1}{2})(x(t) - 1) \geq 0\]

Strict Feasibility:

\[X(t) \succeq \varepsilon I \quad \forall t \in [0, 1]\]

Theorem (Ahmadi, BEK)

Strict feasibility \(\implies\) Optimality of polynomials
Main Idea:

Restrict $Y := Y(t)$ to be a polynomial of degree $\leq d$.

$$S_d^+ := \{X \mid \langle X, Y \rangle \geq 0 \ \forall Y \in S^+ \cap \mathbb{R}^{m \times m}[t]\}$$

Takeaway: Lower bound at level $d$ can be found via SDP
Theorem (Ahmadi, BEK)

Boundedness of TV-SDP $\Rightarrow$ Lower bounds tight
Let’s see some applications!
TV-Maxflow

\[
\max \sum_{(i,j) \in E} \int_0^1 \sum f_{ij} f_1 f_{1j}(t) dt \\
0 \leq f_{ij} f_1 \leq b_{ij}(t) \\
\forall (i,j) \in E \\
\sum_{j: (i,j) \in E} f_{ij}(t) - \sum_{j: (j,i) \in E} f_{ji}(t) \geq 0 \\
\forall i \neq i_1 \\
1, 9
\]

\[
|f_{14}'(t)| \leq \frac{1}{2} \\
|f_{59}'(t)| \leq \frac{1}{2} \\
\int_0^t \sum_{j=1}^n f_{1j}(t) \leq t^2
\]

Obtained from dual problem

<table>
<thead>
<tr>
<th>degree</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>10</th>
<th>\infty</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.61</td>
<td>0.81</td>
<td></td>
<td>0.90</td>
<td>&lt; 0.91</td>
</tr>
</tbody>
</table>
One-Shot Approximation of the Pareto Curve

\[ \min \langle c_1, x \rangle \text{ and } \langle c_2, x \rangle \quad \text{s.t.} \quad x \in C \]

Pareto Curve:
\[ \left\{ \left( \langle c_1, x(t) \rangle, \langle c_2, x(t) \rangle \right) \mid t \in [0, 1] \right\} \]
\[ x(t) := \arg \min \langle c_1, x(t) \rangle \quad \langle c_2, x(t) \rangle \leq t \quad x(t) \in C \]

Markowitz Portfolio Theory
\[ \max \langle r, x \rangle \quad \min \langle x, \Sigma x \rangle \]
\[ \sum x_i = 1 \quad x \geq 0 \]

Idea:
\[ \min \int_0^1 \langle c_1, x(t) \rangle dt \]
\[ \langle c_2, x(t) \rangle \leq t \quad x(t) \in C \]

Takeaway: This is a TV-SDP!
TV-SDP for Imitation Learning

Joint work with
- Jake Varley
- Vikas Sindhwani
Demonstration

\[ x : [0, 1] \rightarrow \mathbb{R}^n \]
\[ \dot{x} = f(x) \]
\[
\dot{x} = f(x)
\]
\[
\|\partial x\| \to 0 \quad \text{as} \quad t \to \infty
\]
\[
\partial \dot{x} = \frac{\partial f}{\partial x} \partial x
\]

**Takeaway:** Contraction is a TV-SDP constraint
Primal approach: Search for poly

Dual approach: Finite dim. relaxation

TV-SDP!

(boundedness) ≤ (strict feasibility)

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Thanks!