On the gap between nonnegative and sum of squares polynomials

[Connections to Graph Theory and Convex Geometry]

Joint Talk
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Nonnegativity vs. Sum of Squares

Decide if the following polynomial is nonnegative:

\[ p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_2^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \]

NP-hard for degree \( \geq 4 \)

If only we could see...

\[ p(x) = (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_2^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2 \]

**Question 1:** Is it “easy” to test for an SOS decomposition? **YES!**

**Question 2:** Is every nonnegative polynomial SOS? **NO!**
SOS Representation via Semidefinite Programming

A polynomial $p(x)$ of degree $2d$ is SOS if and only if $\exists Q \succeq 0$ such that

$$p(x) = z(x)^T Q z(x)$$

where $z = [1, x_1, \ldots, x_n, x_1x_2, \ldots, x_n^d]^T$ is the vector of monomials of degree up to $d$.

If $Q$ is a feasible matrix of the SDP, then to construct the SOS representation factorize $Q = VV^T$, and write $V = [v_1, v_2, \ldots, v_r]$ so that

$$p = z^T Q z = z^T VV^T z = \|V^T z\|^2 = \sum_{i=1}^{r} (v_i^T z)^2$$
SOS Representation via Semidefinite Programming

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$$Q = \sum_{i=1}^{3} v_i v_i^T$$

$$v_1 = [1, -3, 0, 1, 0, 2]^T$$

$$v_2 = [0, 0, 0, 1, -1, 0]^T$$

$$v_3 = [0, 0, 4, 0, 0, -1]^T$$

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### Hilbert’s Theorem (1888)

Homogeneous polynomials (forms)

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Nonnegativity ⇒ SOS

Motzkin (1967)

\[ M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \]

Robinson (1973)

\[ R(x, y, z, w) = x^2(x - w)^2 + y^2(y - w)^2 + z^2(z - w)^2 + 2xyz(x + y + z - 2w) \]
Better Understanding of the Gap?

- It seems difficult to construct nonnegative polynomials that are not SOS in relatively small dimensions and degrees.

- Especially true if additional structure is required. For example, the following is OPEN: “Construct a convex, nonnegative polynomial that is not SOS.”

- The gap between nonnegative and SOS polynomials: where do they appear?

- Connection to Graph Theory

more on this in Bachir’s part
Perfect Graphs

\( \omega(G): \text{clique number} \)
\( \chi(G): \text{chromatic number} \)

\[ \omega(G) \leq \chi(G) \]

A graph \( G \) is called **perfect** if \( \omega(H) = \chi(H) \) for every induced subgraph (isg) \( H \) of \( G \).

**Strong Perfect Graph Theorem** [Chudnovsky, Robertson, Seymour, Thomas, 2002]
A graph is perfect if and only if it contains no odd holes and no odd antiholes.
A New Notion: SOS-perfect Graphs

For a graph $G = (V, E)$ with $|V| = n$ and with clique number $\omega(G)$, define

$$p_G(x) = -2\omega(G) \sum_{ij \in E} x_i^2 x_j^2 + (\omega(G) - 1) \left( \sum_{i=1}^{n} x_i^2 \right)^2$$

We say that a graph $G$ is **sos-perfect** if $p_H(x)$ is SOS for every isg $H$ of $G$.

**Theorem:** A graph is perfect if and only if it is sos-perfect.
Proof Sketch

The Schrijver number of graph $G$

$$\vartheta'(G') = \max_{X \in S^{n \times n}} \text{tr}(JX)$$

s.t. $X_{ij} = 0$ if $ij \in E$

$$\text{tr}(X) = 1$$

$X \succeq 0$

$X \succeq 0$

$$p_G(x) = -2\omega(G) \sum_{ij \in E} x_i^2 x_j^2 + (\omega(G) - 1) \left( \sum_{i=1}^{n} x_i^2 \right)^2$$

$$p_{G,k}(x) = -2k \sum_{ij \in E} x_i^2 x_j^2 + (k - 1) \left( \sum_{i=1}^{n} x_i^2 \right)^2$$

(1) $\omega(G) \leq \vartheta'(\bar{G}) \leq \chi(G)$  \quad \text{Lovász, Schrijver}

(2) $G$ is perfect $\iff \omega(H) = \vartheta'(\bar{H})$ for every isg $H$ of $G$  \quad \sim \text{Lovász}

(3) $p_{G,k}(x) \geq 0 \iff k \geq \omega(G)$  \quad \sim \text{De Klerk, Pasechnik}

(4) $p_{G,k}(x)$ is sos $\iff k \geq \vartheta'(\bar{G})$  \quad \sim \text{De Klerk, Pasechnik}
Implication

**Theorem:** A graph is perfect if and only if it is sos-perfect.

Every odd hole and odd antihole will give a nonnegative polynomial that is not SOS.

An infinite family of degree-4 polynomials that are nonnegative but not SOS:

\[
p_{C_5} = -4(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_5^2 + x_1^2x_5^2) + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)^2
\]

\[
p_{C_7} = -4(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_5^2 + x_5^2x_6^2 + x_6^2x_7^2 + x_7^2x_1^2) + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2)^2
\]

\[
p_G = -6(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_5^2 + x_1^2x_5^2 + x_2^2x_6^2 + x_3^2x_6^2 + x_3^2x_6^2 + x_4^2x_6^2 + x_5^2x_6^2) + 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2
\]
What is next?

**Weak Perfect Graph Theorem** [Lovász]: $G$ is perfect $\iff \overline{G}$ is perfect

- Reprove WPGT using sos-perfectness, i.e. show that

$$p_H(x) \text{ is sos for every isg } H \text{ of } G \iff p_{H'}(x) \text{ is sos for every isg } H' \text{ of } \overline{G}$$

**Strong Perfect Graph Theorem** [Chudnovsky, Robertson, Seymour, Thomas]

A graph is perfect if and only if it contains no odd holes and no odd antiholes.

- Reprove SPGT using sos-perfectness, i.e. show that

$$p_H(x) \text{ is minimally non-sos-perfect } \iff H \text{ is an odd hole or an odd antihole}$$
Thank you...