Verification with data-driven differential inclusions

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Overview

- How to generate guarantees for safety properties for nonlinear ODE models with nonparametric uncertainty.

Assumptions

- Consider the trajectories of a nonlinear system,

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n, \quad x(t_0) = x_0.$$

- f is a $\mathcal{C}^m$ vector field: partial derivatives $\partial^\alpha f(x)$ are uniformly bounded for $|\alpha| \leq m$.

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- We observe snapshots of the state and state velocity

$$(x_k, y_k) = \left(x, \frac{dx}{dt}\right)\Bigg|_{t=t_k}, \quad k = 0, \cdots, N.$$

  where $t_0 < t_1 < \cdots < t_N = 0$.
- Snapshots give rise to interpolation constraints on the unknown vector field:

$$f(x_k) = y_k.$$

Assumptions

- Starting at time $t = 0$, we want to verify that the trajectory will satisfy the following safety property: $x(t)$ remains in the complement of an unsafe set $\mathcal{X}_u \subset \mathbb{R}^n$ for all times $t \in [0, T]$.

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- Best-case (viability): We win if there exists at least one data-consistent vector field f so that the associated trajectory $x(t)$ remains safe.
- Worst-case (invariance): We win if for all data-consistent vector fields f the associated trajectory $x(t)$ remains safe.
- Probabilistic: Suppose the space of all functions f is endowed with a probability measure *dProb*. We win if at least α percent of the associated trajectories $x(t)$ remain safe.

Viability certificates

- A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is a "viability certificate" if the following conditions hold:

$$\begin{cases} |\partial^\alpha f(x)| \leq 1, \quad x \in \mathbb{R}^n, |\alpha| \leq m \\ f(x(t_k)) = \dfrac{dx}{dt}(t_k), \quad k = 0, \cdots, N \\ f(x) \cdot \vec{n}(x) \geq 0, \quad x \in \partial \mathcal{X}_u, \end{cases}$$

where $\vec{n}$ is the outward-directed unit normal on the boundary of $\mathcal{X}_u$.

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- This certificate encodes the viability of the safety property on an infinite time-horizon ($T = \infty$).
- Bernat and Charlie's algorithm can be used to detect the existence of such a certificate.
- We would like to produce less restrictive certificates for safety on a finite time-horizon $T < \infty$.

Invariance

- To determine whether a safety property is satisfied for all possible realizations of the system, we need to consider the set of all possible vector fields f consistent with the data.

A connection between ODEs and Differential Inclusions

- Suppose a function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\text{(Bounded Derivatives): } |\partial^\alpha f(x)| \leq 1 \quad \text{all } x \in \mathbb{R}^n,\ |\alpha| \leq m$$

and

$$\text{(Interpolation Constraints): } f(x_k) = y_k \quad \text{all } k = 1, \cdots, N.$$

For any $x \in \mathbb{R}^n$, consider

$$K(x) = \{f(x) : f \text{ has (Bounded Derivatives),} \\ \text{and satisfies (Interpolation Constraints)}\},$$

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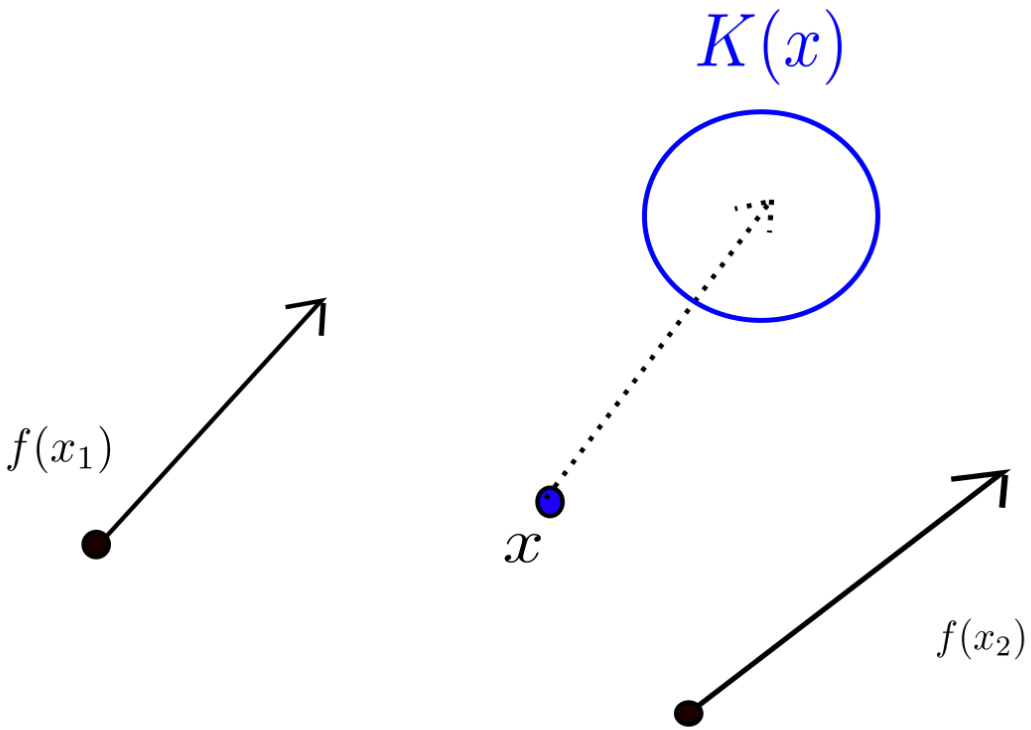

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- In addition to (3), we know that the curve $x(t)$ will be $(m+1)$-times differentiable, and

$$(4)\ \left|\frac{d^k x}{dt^k}\right| \leq \text{Const}, \quad \text{all } k = 1, \cdots, m.$$

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- Conditions (3) and (4) yield a differential inclusion satisfied by the trajectories of the original dynamical system.
- Up to constant-factor enlargements, this is the tightest differential inclusion which is consistent with the data.

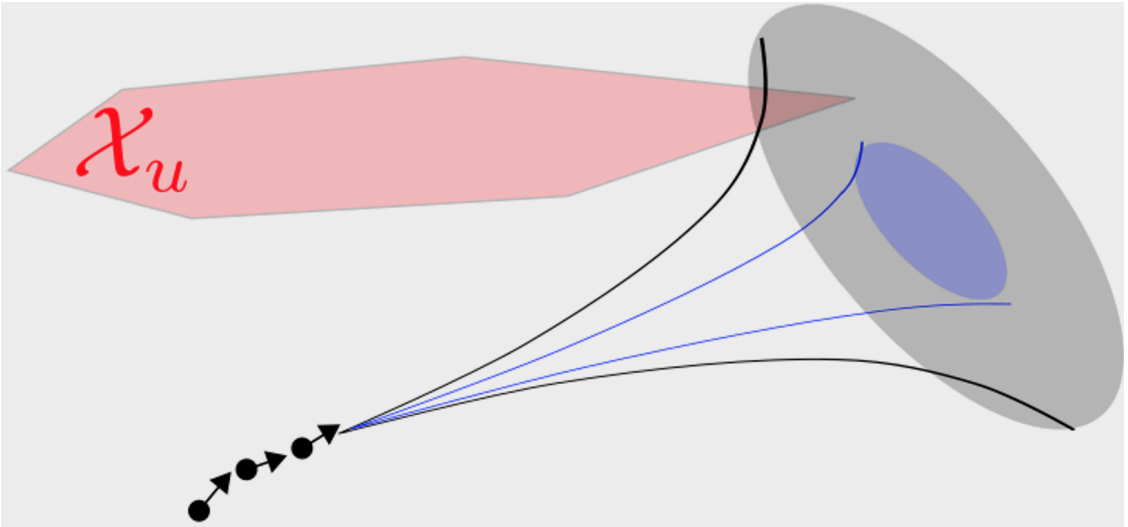
- Given a differential inclusion

$$(3)\ \frac{dx}{dt}(t) \in \widetilde{K}(x(t)), \quad \text{all } t \geq 0,$$

$$(4)\ \left|\frac{d^k x}{dt^k}\right| \leq \text{Const}, \quad \text{all } k = 1, \cdots, m,$$

$$(5)\ x(0) = x_0 \text{ (fixed)}.$$

How can we detect whether all the trajectories $x(t)$ of (3)–(5) will remain in the complement of $\mathcal{X}_u$ over a finite time-horizon?



Summary and future directions

- We defined viability certificates for safety verification over infinite-time horizons.
- We explained how to learn a Differential Inclusion from data.
- Barrier Certificates for Differential Inclusions?
- Online Interpolation?
  How to efficiently shrink the sets $\widetilde{K}(x)$ with arrival of a single new data point?