Verifiable Data Driven Control

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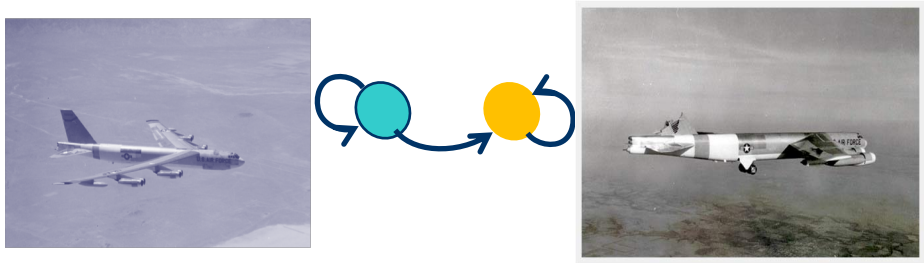
Verifiable, Control-Oriented Learning On The Fly MURI Kickoff Meeting

Big Picture: Control Oriented Learning on the Fly

- **Motivation: survive unexpected changes in dynamics.**
- **Formalization: control of unknown dynamics based on data collected from a single execution and side information**
- **Challenges**
  - **Lack of training data (single execution)**
  - **Learning while acting**
  - **Taking provably correct actions.**



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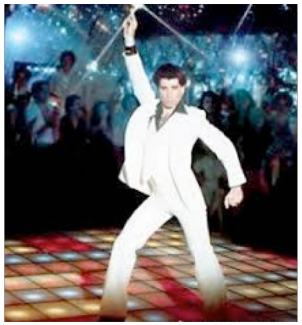
- **Challenges**
  - **Lack of training data (single execution)**
  - **Learning while acting**
  - **Taking provably correct actions.**
- **Humans can do it !**

Big Picture: Control Oriented Learning on the Fly

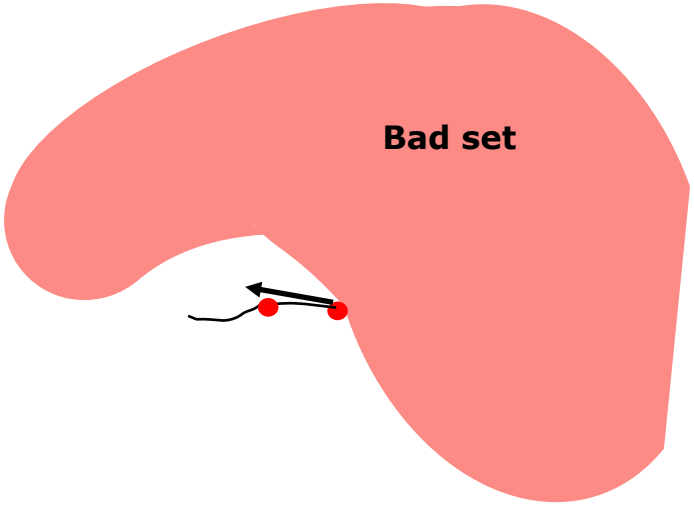
- **Three Stages:**
- **Stayin' Alive**



Big Picture: Control Oriented Learning on the Fly

- **Three Phases**

- **Stayin' Alive**
  - Maintain viability by avoiding a known bad set
  - Myopic control (Topcu & Ornik)
  - Alternative: Rantzer's dual control?
  - Use this phase to learn about the plant



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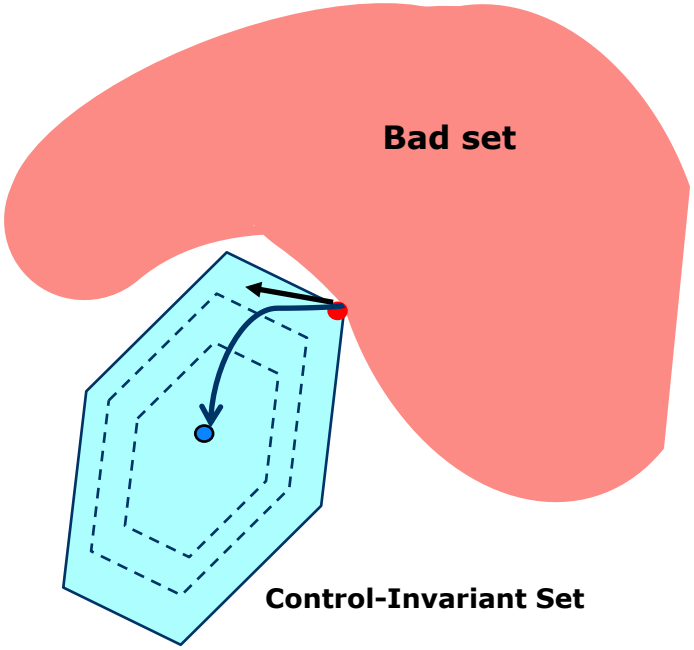
- **Verifiable Data Driven Control**
  - Find a good robust control invariant set and associated control action (Sznaier & Dai)



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- **Verifiable Data Driven Control**
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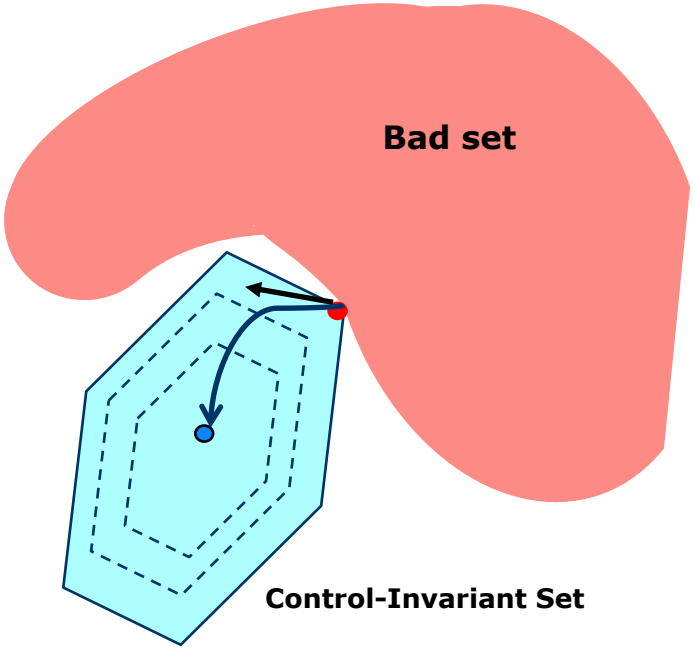
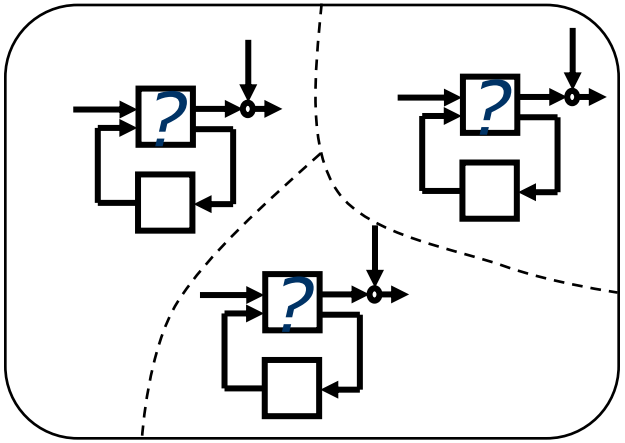
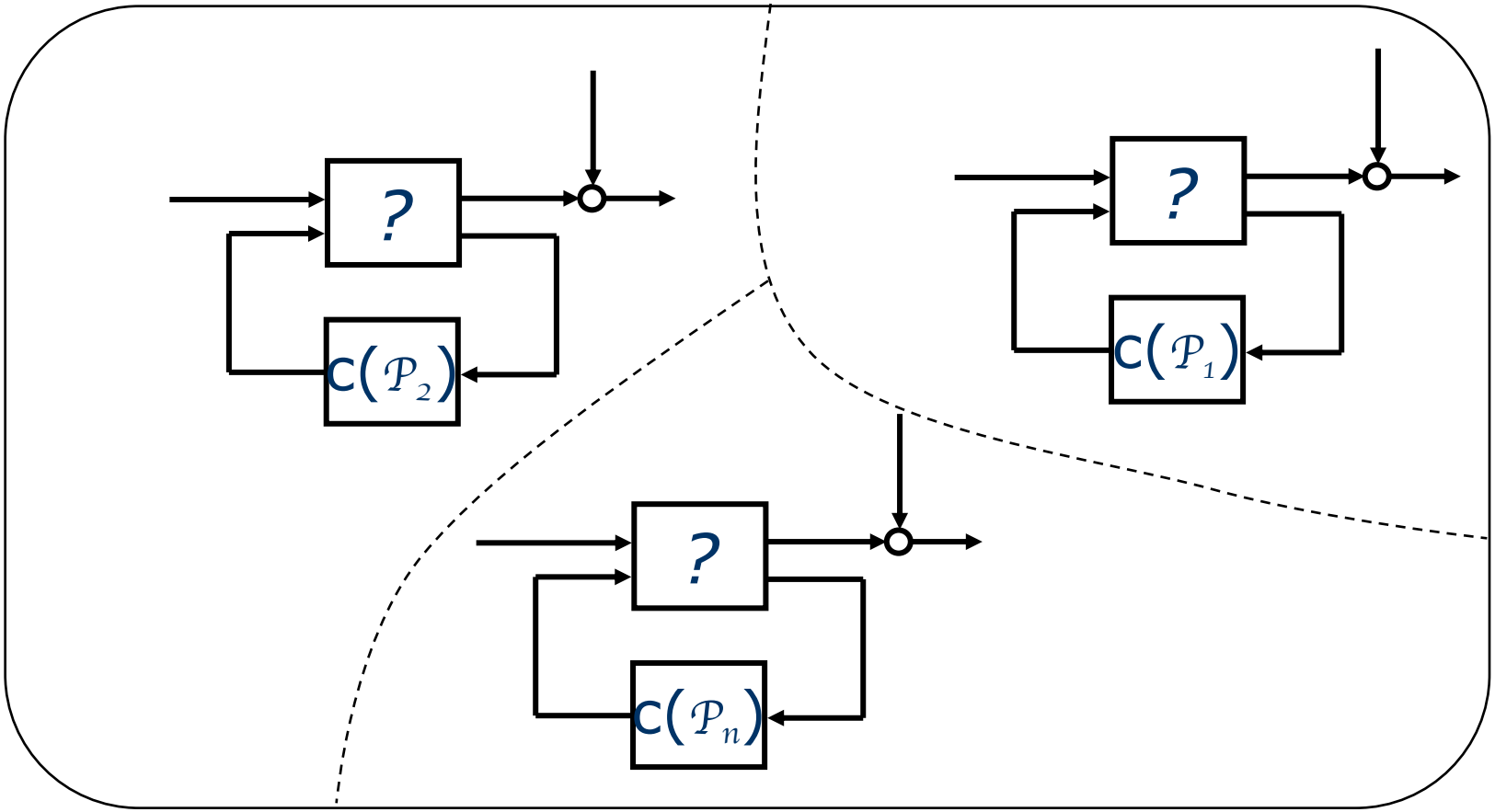
- **Learning for Performance**
  - Keep learning while DD keeps you safe
  - Use these models to optimize performance



Data Driven Control



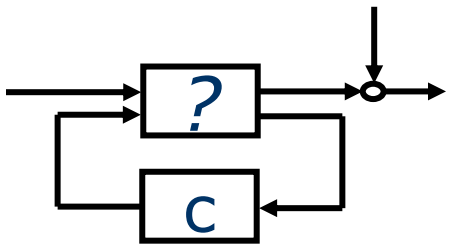
Problem setup

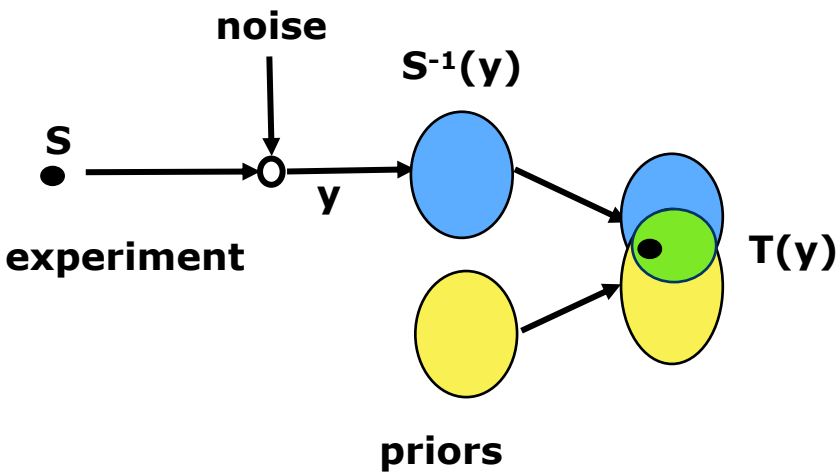


Design a switched stabilizing controller C($\mathcal{P}_i$) based solely on experimental data

From data to control (LTI)

- **Find the consistency set T(y):**
  - **All plants that could have generated the data**





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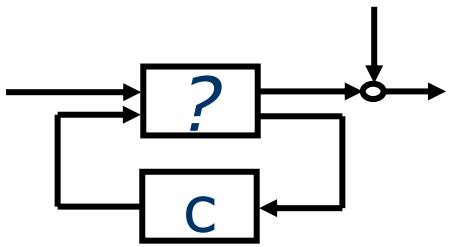

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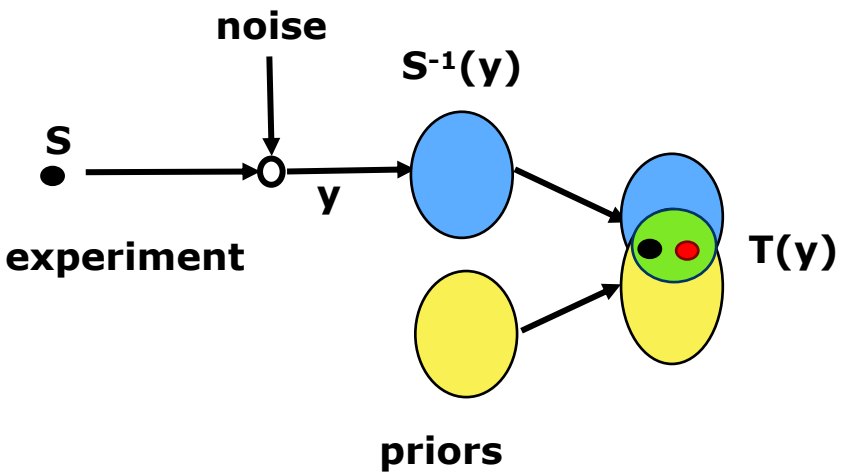

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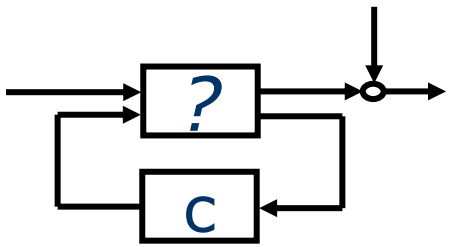

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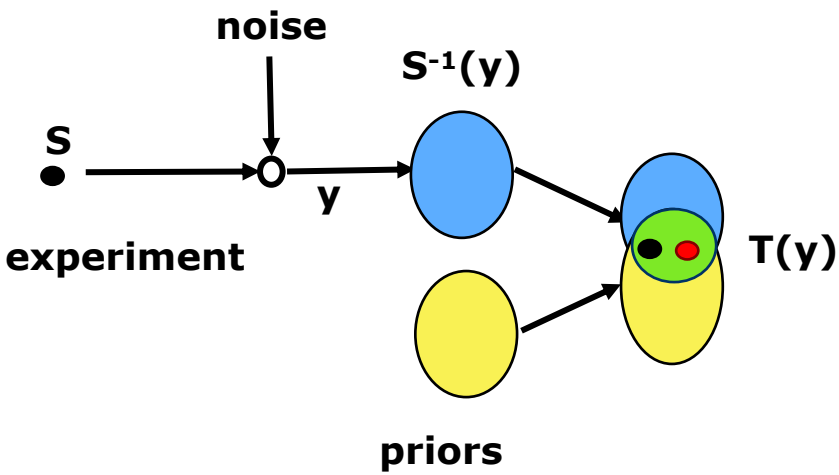

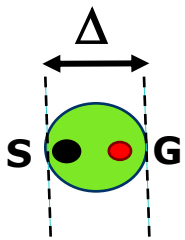

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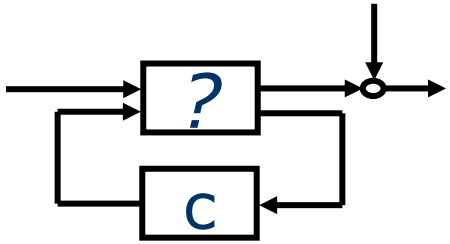

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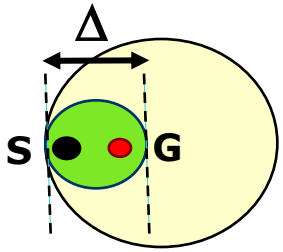
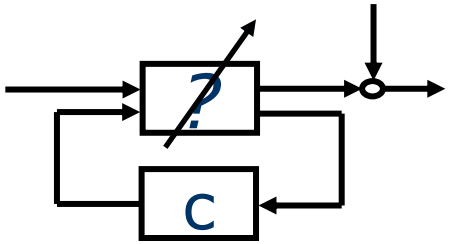
- **Bound the worst case error and design a robust controller**
- **Works, but**
  - **Conservative (!)**
  - **Computational issues**
  - **Hard to extend beyond LTI**



Switched Data Driven Control:



- **Given:**
  - **Prior: the plant admits a model of the form**

$$\mathbf{x}_{k+1}^i = \mathbf{A}_i\mathbf{x}_k^i + \mathbf{B}_i\mathbf{u}_k + \mathbf{w}_k,\ \|\mathbf{w}_k\|_\infty \leq \epsilon,\ i \in \{1,\ldots,s\}$$

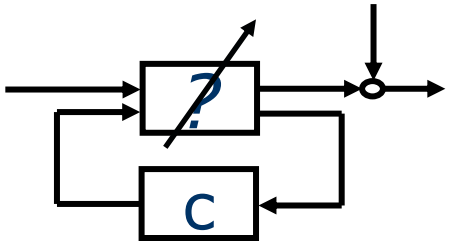
  - **Experimental data (labeled):**

$$\{\mathbf{u}_k, \mathbf{x}_k^i, \mathbf{x}_{k+1}^i\}_{k=0}^T$$

- **Find a switched controller so that** $\mathrm{A}_{\sigma_t} + \mathrm{B}_{\sigma_t}\mathbf{F}_{\sigma_t}$ **is G.A.S**

for all switching sequences and for all plants that could have generated the experimental data

Switched Data Driven Control:



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  - **Experimental data (labeled):**

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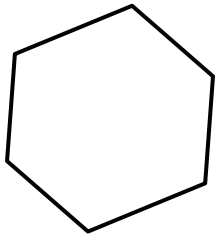
- **Consistency set is a collection of polytopes:**



$$\mathcal{P}_i \doteq \{\mathbf{A}_i, \mathbf{B}_i : \|\mathbf{x}_{t_k^i+1} - \mathbf{A}_i\mathbf{x}_{t_k^i} - \mathbf{B}_i\mathbf{u}_{t_k^i}\|_\infty \leq \epsilon \text{ for all } t_k^i \in \mathcal{K}_i\}$$

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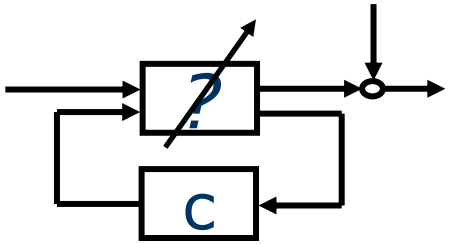
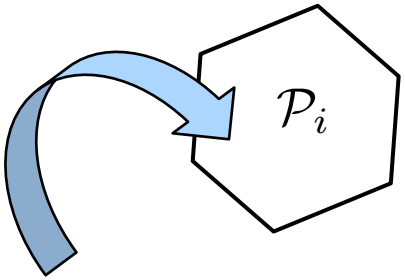


- **Find a switched controller $\mathbf{F}_i$ that stabilizes the set $\{\mathcal{P}_i\}$ *for any arbitrary switching***



All (A_i,B_i) compatible with the available information live here

Detour: control of switched systems:

$$\mathbf{x}_{k+1} = \mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k$$

can be stabilized by a switched linear controller

$$\Updownarrow$$

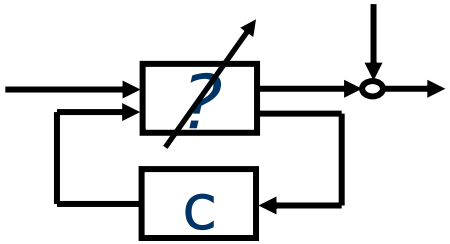
There exist

a) **A full row rank matrix T**
b) **Matrices U_i, P_i such that**

$$\begin{aligned} \mathbf{A}_i \mathbf{T} + \mathbf{B}_i \mathbf{U}_i &= \mathbf{T}\mathbf{P}_i, \\ \|\mathbf{P}\|_{\ell_1 \to \ell_1} &< 1 \end{aligned}$$

Blanchini et. al., 2009

Switched Data Driven Control as a Robust Optimization.



- **Find matrices T, U_i, P_i such that**

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for all matrices in the polytopes $\{\mathcal{P}_i\}$

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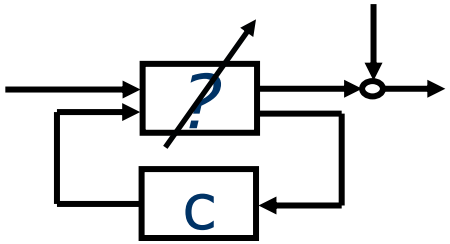

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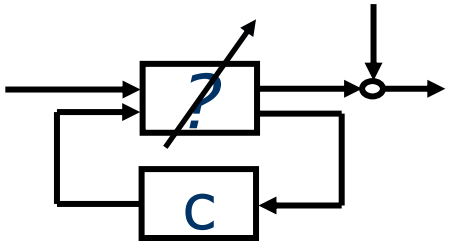
  for all matrices in the polytopes $\{\mathcal{P}_i\}$

- **A very challenging non-convex min-max problem**

- **Main result: solvable via Farkas' Lemma+ SoS**

Switched Data Driven Control as a Robust Optimization.

- **Find matrices V, U_i, P_i such that**



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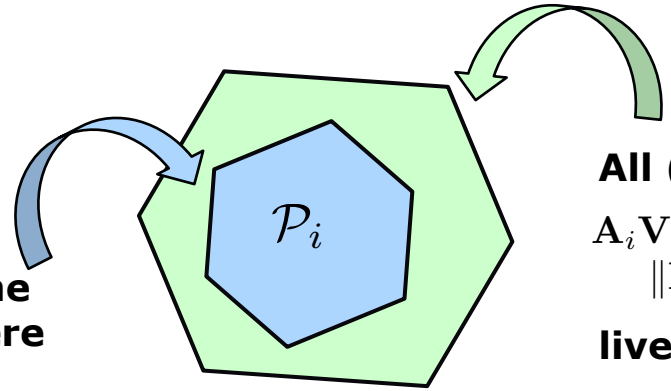
for all matrices in the polytopes $\{\mathcal{P}_i\}$

- **A non-convex min-max problem, but solvable using SoS (via Farkas' Lemma)**



All $(A_i, B_i, H_i,)$ such that

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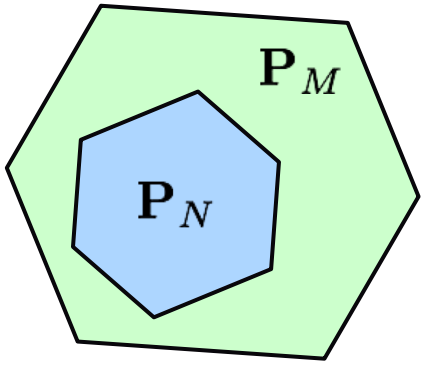
live here

All (A_i, B_i) compatible with the available information live here

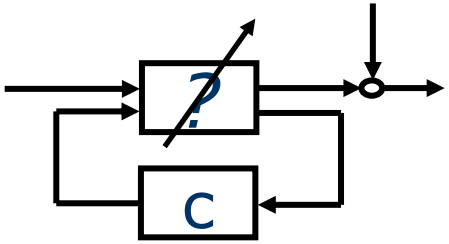
Extended Farkas Lemma

Lemma: Consider two polytopes of the form $\mathbf{P}_N \doteq \{x : \mathbf{N}x \leq n\}$ and $\mathbf{P}_M \doteq \{x : \mathbf{M}x \leq m\}$, where $\mathbf{M}, \mathbf{N}$ are matrices and m, n are vectors. Then $\mathbf{P}_N \subseteq \mathbf{P}_M$ if and only if there exists a matrix $\mathbf{Y}$ with non-negative entries such that

$$\mathbf{YN} = \mathbf{M}$$
$$\mathbf{Y}n \leq m$$



Data Driven Control:



- **Fact:**
  - **There exist a controller F_i that stabilizes iff there exist $\mathbf{Y}^i \geq 0, \mathcal{F}_i, \mathbf{V}$ such that:**

$$\mathbf{Y}^i \mathbf{M}(\mathbf{x^i}, \mathbf{u^i}) = \left[(\mathbf{S}(\mathbf{V}^{-T} \otimes \mathbf{V}) \quad \mathbf{S}(\mathcal{F}_i^T \otimes \mathbf{V}) \right]$$

$$\mathbf{Y}^i \begin{bmatrix} \mathbf{x}^i + \epsilon \mathbf{1} \\ -\mathbf{x}^i + \epsilon \mathbf{1} \end{bmatrix} \leq \mathbf{d}^i < \mathbf{1}$$

$$\forall\, \mathbf{S} \doteq \begin{bmatrix} \mathbf{s}_1^T & \mathbf{0}_{1\times n} & \dots & \mathbf{0} \\ \mathbf{0}_{1\times n} & \mathbf{s}_2^T & \dots & \mathbf{0} \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{0}_{1\times n} & \mathbf{0}_{1\times n} & \dots & \mathbf{s}_n^T \end{bmatrix}, \; s_j = \pm 1$$

Dai and S., CDC'18

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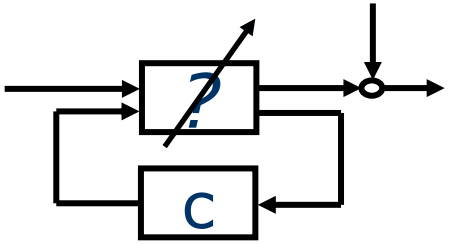

- **Q: how do we solve these:**

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bilinear

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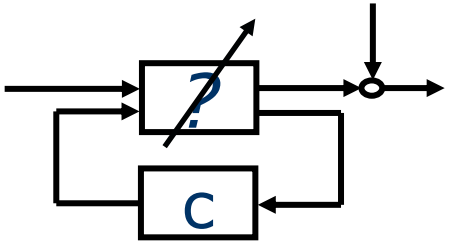

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- **A: equivalent to checking feasibility of a semi-algebraic set.**

Convex relaxations via SoS/Moments (P-satz)

Data Driven Control:

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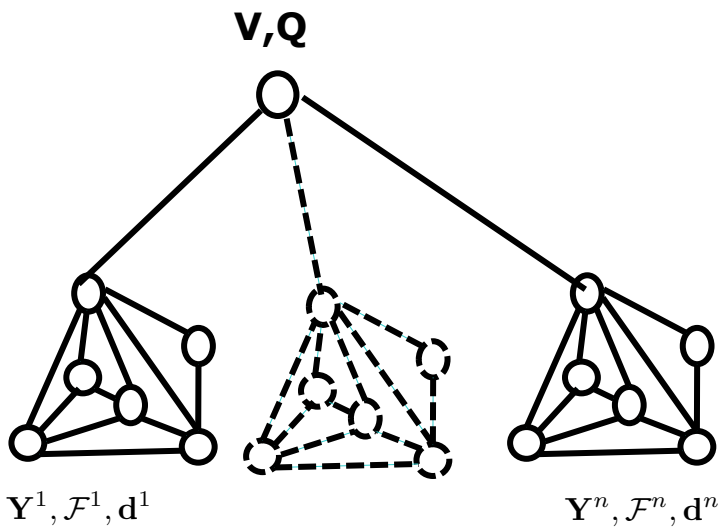
A ***vey large*** *SDP:* $\mathcal{O}\left([n^2 + N_s(n^2 + 3n + 1]^2 \right)$ *variables (!!)*

Exploiting sparsity:

- **The underlying CSG has chordal sparsity**
  - **Implication: the "big" SD and rank constraints decouple into a collection of smaller ones**



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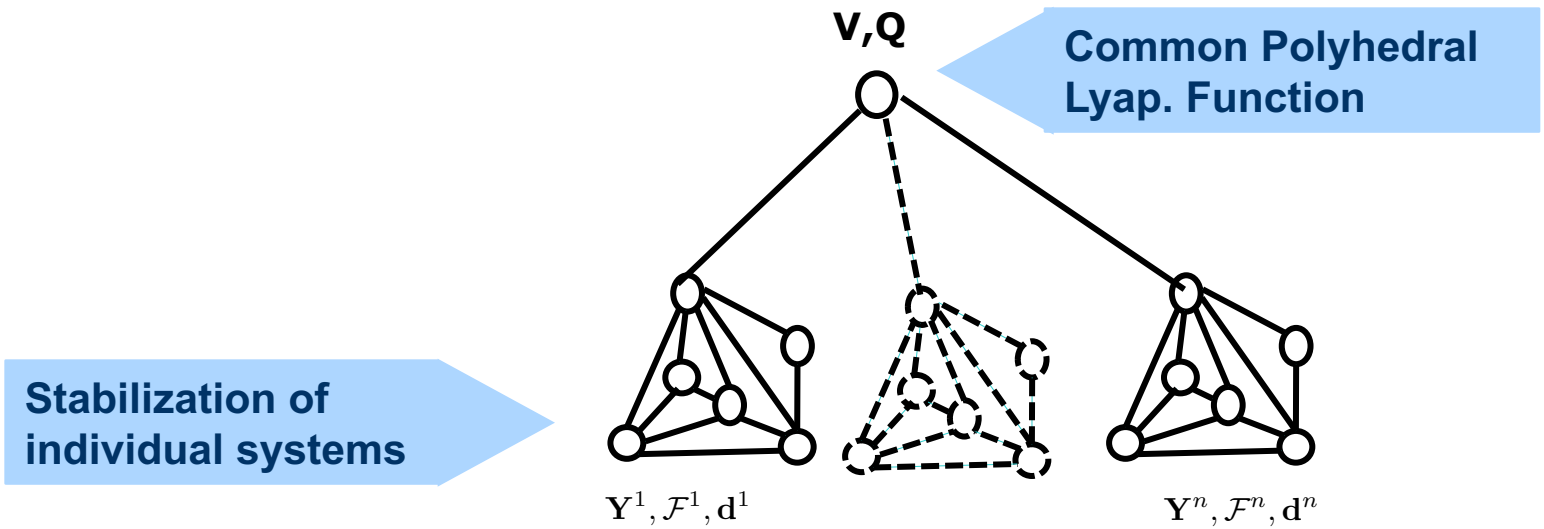

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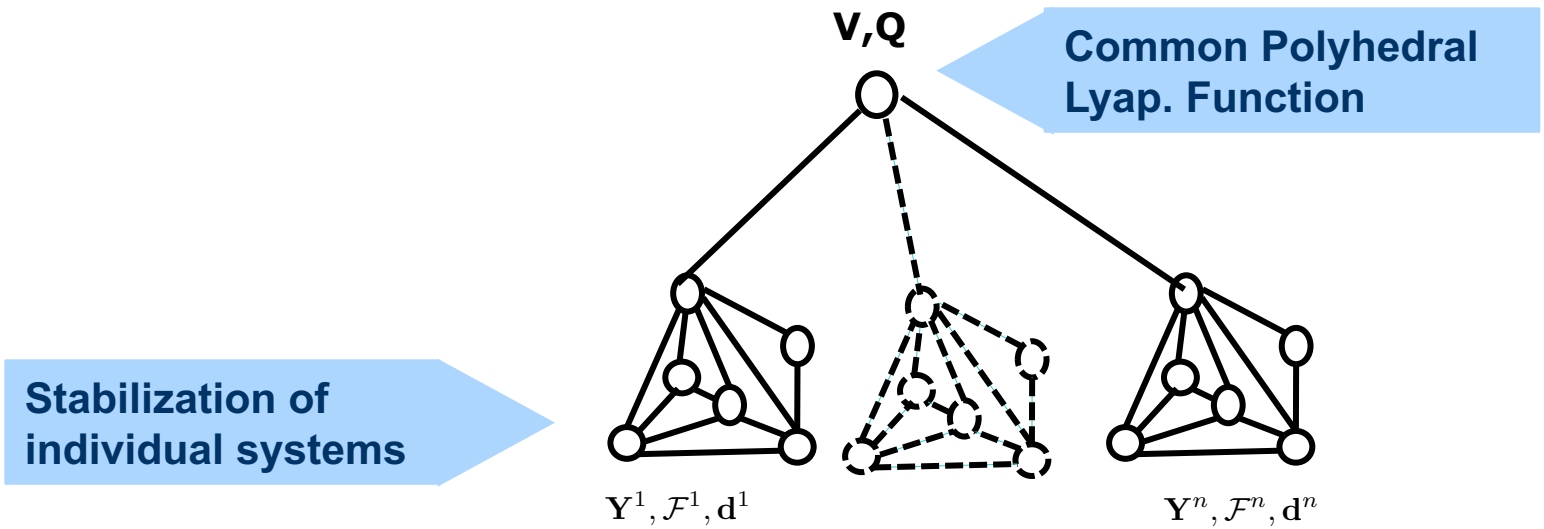


*For three third order systems: ~ **220** variables down from 4300*

Simple Example:

$$\begin{bmatrix} \dot{X} \\ \ddot{X} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{4K sin(\theta)}{M} \\ 0 & 0 & -\omega \end{bmatrix} \begin{bmatrix} X \\ \dot{X} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} u_Z$$

Quanser Qball-X4

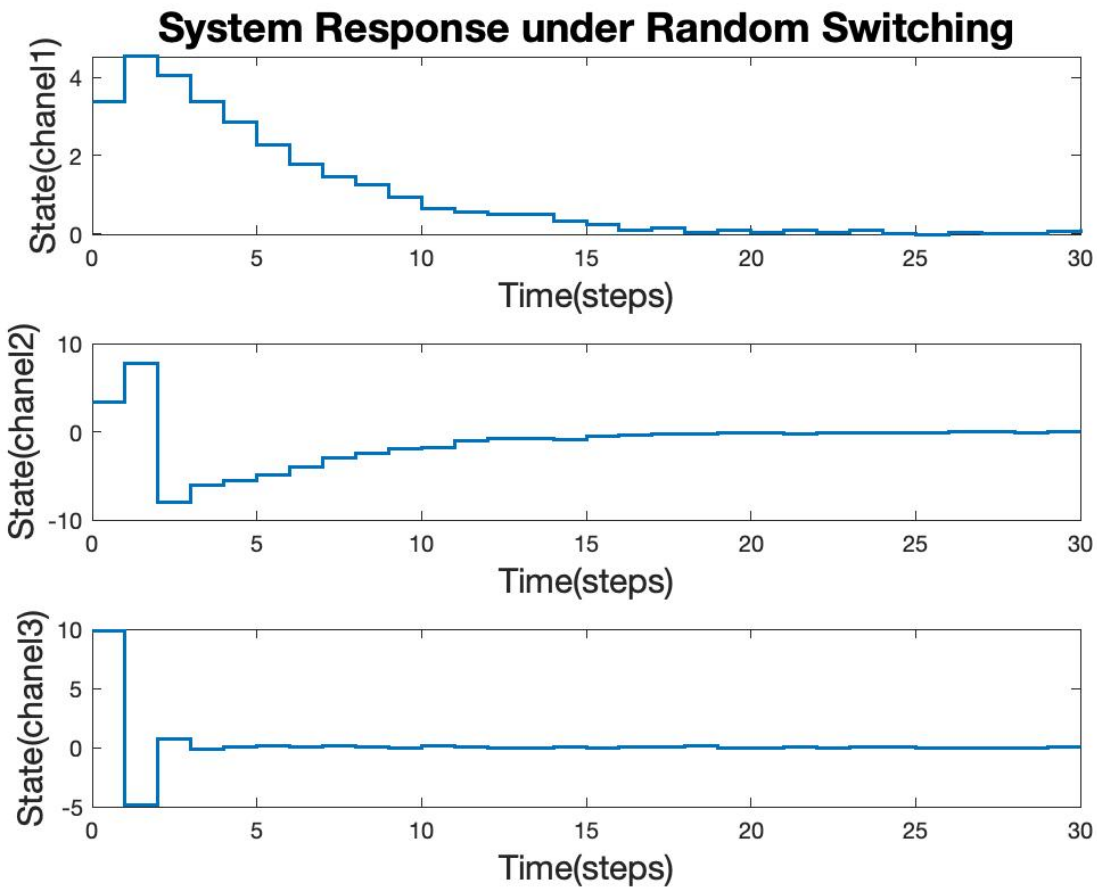
$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0.1 & 0.1913 \\ 0 & 1 & 3.083 \\ 0 & 0 & 0.2231 \end{bmatrix} \quad \mathbf{B}_1 = \begin{bmatrix} 0.1063 \\ 2.87 \\ 0.7769 \end{bmatrix} \qquad \text{(System 1)}$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0.1 & 0.3769 \\ 0 & 1 & 6.073 \\ 0 & 0 & 0.2231 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0.2094 \\ 5.653 \\ 0.7769 \end{bmatrix} \qquad \text{(System 2)}$$

$$\mathbf{F}_1 = \begin{bmatrix} -0.2580 & -0.2123 & -0.8111 \end{bmatrix} \qquad \text{(Gain 1)}$$

$$\mathbf{F}_2 = \begin{bmatrix} -0.1527 & -0.1238 & -0.9908 \end{bmatrix} \qquad \text{(Gain 2)}$$

Simple Example:



State trajectories under arbitrary switching

Future work:

- **Extension to non-linear dynamics**
  - **Alternatives: manifold embeddings or polynomial liftings**
  - **Non convex, but good relaxations available**

- **Combine with U. Topcu's Myopic Control**
  - **Use Rantzer's dual Lyapunov formulation to maintain viability until invariance can be certified**

- **All of these reduce to polynomial optimization**
  - **Use low complexity relaxations (A. Ahmadi)**
  - **Use randomized linear algebra**