On Local Minimality in Cubic Optimization

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Deciding Local Minimality

Consider the optimization problem

\[
\inf_{x \in \mathbb{R}^n} f(x)
\]

s.t. \( x \in \Omega \)

Given a point \( \bar{x} \), decide if \( \bar{x} \) is a local minimizer of \( f \).

Why local minima?

- Global minima are often intractable
- Recent interest in local minima, particularly in machine learning applications
- Existing notions that local minima are “easier” to find or are “sufficient for applications”
- Formal understanding of local minima is desirable
- Polynomial optimization gives a rigorous framework for analyzing complexity
Our focus: polynomial optimization problems

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \geq 0, \quad i = 1, \ldots, m
\end{align*}
\]

Deciding local minimality of a given point?

**Known polynomial-time cases**
- Unconstrained quadratics
- Linear programming

**Known NP-hard cases**
- Unconstrained quartics
  - Murty, Kabadi (1987)
- Quadratic programming
  - Pardalos, Schnitger (1988)

**One open case**
- Unconstrained cubics
Theorem (Ahmadi, Zhang)

Let $f$ be a cubic polynomial. A point \( \hat{x} \) is a local minimizer of $f$ if and only if \( \hat{x} \) satisfies

- \( \nabla f(\hat{x}) = 0 \)
- \( \nabla^2 f(\hat{x}) \succeq 0 \)
- \( d^T \nabla^2 f(\hat{x}) d = 0 \Rightarrow \nabla f_3(d) = 0 \)
Example: no local minimum

\[ f(x, y) = y(y - x^2) \]

\[ \nabla f(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \]

\[ d^T \nabla^2 f(0,0) d = 0 \Rightarrow \nabla f_3(d) = 0? \]
Example: local minimum

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\[ d^T \nabla^2 f(0,0)d = 0 \Rightarrow \nabla f_3(d) = 0? \]
Checking local minimality in polynomial time

Theorem (Ahmadi, Zhang)

Let $f$ be a cubic polynomial. Checking whether a point $\hat{x}$ is a local minimizer of $f$ can be done in polynomial time.

- Compute Hessian of $f$ at $\hat{x}$ and check that it is psd
- Compute a basis $v_1, v_2, \ldots$ for the null space of Hessian (e.g. Bareiss)
- Compute $\nabla f_3(\sum a_i v_i)$ and check that all entries are zero

Some geometric properties of local minima

- The set of $x$ where its Hessian is PSD is convex and semidefinite representable
- The local minimizers of cubics form a convex set
SDP for finding local minima

**Theorem (Ahmadi, Zhang)**

Let $f$ be a cubic polynomial that has a local minimum. Then the optimal value of the following SDP is the value of the local minimum:

$$\max_{\sigma(x) \text{sos}, S(x) \text{sos}} \gamma$$

$$f - \gamma = \sigma + Tr(\nabla^2 f(x)S(x))$$

$$\min_{H(x) \succeq 0} f(x)$$

$$sos \text{ relaxation}$$

$$(x - \hat{x})^T \nabla^2 f(\hat{x})(x - \hat{x}) \quad (x - \hat{x})(x - \hat{x})^T$$

There is always a *quadratic* $\sigma$ and a quadratic sos-matrix $S$ such that this holds!
Future questions

• Is deciding if a cubic polynomial has a local minimum in P?
  – Difficulties: irrational numbers, sizes of minima

• Are local minima of cubics polynomially sized?
  – Implications: Deciding if there is a local minimum is in P, and finding them is tractable