Verifiable Safe Data Driven Control

T. Dai, J. Miller, M. Sznaier

Robust Systems Lab
Electrical and Computer Engineering
Northeastern University

Learning Dynamical Systems
Given:

- Prior: the plant admits a model of the form
  \[ \dot{x} = F\phi(x) + G\gamma(x)u + \eta \quad F, G \text{ unknown} \]
  \[ \phi(.), \gamma(.) \text{ dictionaries (Rachel’s, Koopman, ...)} \]

- Experimental data:
  \[ (u(t_k), x(t_k), \dot{x}(t_k)), 0 \leq t_k \leq T \]

Find a control law that avoids an unsafe set

\textbf{for all plants that could have generated the experimental data}
Koopman operator
- Given the nonlinear system:

\[ \xi_{k+1} = f(\xi_k) \quad \text{where} \quad \xi_k = [x_{k-r+1}^T \ldots x_k^T]^T, \quad x_j \in \mathbb{R}^n \]

\[ y_k = \psi(\xi_k) \quad \text{(the observables)} \]

- The Koopman operator propagates the observables:

\[ (\mathcal{K} \circ \psi)(\xi_k) = (\psi \circ f)(\xi_k) = \psi(\xi_{k+1}) \]

Linear, but may be infinite dimensional

However, sometimes good finite representations exist.
Koopman based learning:

- Poor man’s nonlinear SysId
  - Very successful in fluid flow control

- Inspiration for recent NN architectures
  - Lusch et. al., Otto and Rowley

- Works, but many ad-hoc choices
  - Dictionaries, dimension of embedding manifold
  - No guarantees on the approximation error

All of these issues can be addressed by using Hankel/Loewner tools to recast the problem into 2 convex SDPs
(bonus: SDPs have chordal sparsity)
Background: Hankel matrices.

\[
H = \begin{bmatrix}
  y_k & y_{k+1} & y_{k+2} & \cdots & y_{k+n} \\
  y_{k+1} & y_{k+2} & \cdots & \cdots & y_{k+n+1} \\
  y_{k+2} & \cdots & \cdots & \cdots & y_{k+n+2} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  y_{k+n} & y_{k+n+1} & y_{k+n+2} & \cdots & y_{k+2n}
\end{bmatrix}
\]

model: \[ y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots + a_n y_{k-n} \]

Rank of the Hankel matrix measures the complexity of the dynamics.
Background: Loewner interpolation.

Loewner interpolation.

The Loewner matrix gives the order of the simplest rational interpolant:

\[
L = \begin{bmatrix}
\frac{y_1 - y_{p+1}}{x_1 - x_{p+1}} & \frac{y_1 - y_{p+2}}{x_1 - x_{p+2}} & \cdots & \frac{y_1 - y_{p+n}}{x_1 - x_{p+n}} \\
\frac{y_p - y_{p+1}}{x_p - x_{p+1}} & \frac{y_p - y_{p+2}}{x_p - x_{p+2}} & \cdots & \frac{y_p - y_{p+n}}{x_p - x_{p+n}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{y_{p+1} - y_{p+2}}{x_{p+1} - x_{p+2}} & \frac{y_{p+1} - y_{p+2}}{x_{p+1} - x_{p+2}} & \cdots & \frac{y_{p+1} - y_{p+n}}{x_{p+1} - x_{p+n}} \\
\end{bmatrix}
\]

Interpolant:

\[
y = \frac{\sum a_i y_{p+i}}{\sum a_i (x - x_{p+i})}, \quad [a_1 \ldots a_n]^T \in \mathcal{N}(L)
\]

Rank of the Loewner matrix gives the order of the simplest rational interpolant.
Finding the latent manifold
Finding the latent manifold:

- Find latent variables with low rank Gramian:

\[
G = H_y^T H_y = \begin{bmatrix}
  y_i^T y_i & y_i^T y_{i+1} & \cdots & y_i^T y_{i+j} \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{i+j}^T y_i & y_{i+j}^T y_{i+1} & \cdots & y_{i+j}^T y_{i+j}
\end{bmatrix}
\]

- Need to preserve local geometry or bound Lipschitz constants to get well posed problems.
A Kernel Formulation:

\[
\min_{K \succeq 0} \ rank(G) - \lambda \text{trace}(K)
\]

subject to:

\[
K_{ii} - 2K_{ij} + K_{jj} = \|x_i - x_j\|^2_2 \ \forall \eta_{ij} = 1
\]

where

\[
G = H_y^T H_y = \sum_{i,j} K_{i,j}
\]

\[
K_{i,n-1} = \begin{bmatrix}
    y_i^T & y_i & y_i^T & y_i+1 & \cdots & y_i^T & y_{i+n-1} \\
    y_{i+1}^T & y_i & y_{i+1}^T & y_{i+1} & \cdots & y_{i+1}^T & y_i \\
    \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    y_{i+n-1}^T & y_i & y_{i+n-1}^T & y_{i+n-1} & \cdots & y_{i+n-1}^T & y_{i+n-1}
\end{bmatrix}
\]
A Kernel Formulation:

\[ \min_{K \geq 0} \ rank(G) - \chi \text{trace}(K) \]

subject to:

\[ K_{ii} - 2K_{ij} + K_{jj} = \|x_i - x_j\|^2_2 \quad \forall \eta_{ij} = 1 \]

where

\[ G = H_y^T H_y = \sum_{i,j} K_{i,j} \]

\[ K_{i,n-1} = \begin{bmatrix} y_i^T & y_i^T & y_{i+1}^T & \cdots & y_i^T & y_{i+n-1}^T \\ y_{i+1}^T & y_i^T & y_{i+1}^T & \cdots & y_{i+1}^T & y_i^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{i+n-1}^T & y_i^T & y_{i+n-1}^T & \cdots & y_{i+n-1}^T & y_{i+n-1}^T \end{bmatrix} \]
A Kernel Formulation:

\[
\min_{\mathbf{K} \succeq 0} \ rank(\mathbf{G}) - \lambda \text{trace}(\mathbf{K})
\]

subject to:

\[
\mathbf{K}_{ii} - 2\mathbf{K}_{ij} + \mathbf{K}_{jj} = \|\mathbf{x}_i - \mathbf{x}_j\|^2_2 \quad \forall i, j = 1
\]

Promotes low dimensional embeddings
A Kernel Formulation:

\[
\min_{K \geq 0} \quad \text{rank}(G) - \lambda \text{trace}(K)
\]

subject to:

\[
K_{ii} - 2K_{ij} + K_{jj} = \|x_i - x_j\|^2_2 \quad \forall i,j
\]

Promotes low dimensional embeddings
A Kernel Formulation:

$$\min_{K \succeq 0} \text{rank}(G) - \lambda \text{trace}(K)$$

subject to:

$$K_{ii} - 2K_{ij} + K_{jj} = \|x_i - x_j\|_2^2 \quad \forall i, j = 1$$

where

$$G = H_Y^T H_Y$$

$$K_{i,n-1} = \begin{bmatrix}
    y_i^T & y_i & y_i^T & y_{i+1} & \cdots & y_i^T & y_{i+n-1} \\
    y_i^T & y_i & y_i^T & y_{i+1} & \cdots & y_i^T & y_{i+n-1} \\
    y_{i+1}^T & y_i & y_{i+1}^T & y_{i+1} & \cdots & y_{i+1}^T & y_i \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    y_{i+n-1}^T & y_i & y_{i+n-1}^T & y_{i+1} & \cdots & y_{i+n-1}^T & y_{i+n-1}
\end{bmatrix}$$

**Low order dynamics**
A Kernel Formulation (convex relaxation):

\[
\min_{K \succeq 0} \ trace(G) - \lambda \ trace(K)
\]

subject to:
\[
K_{ii} - 2K_{ij} + K_{jj} = \|x_i - x_j\|^2_2 \forall i,j = 1
\]

where

\[
G = H^T_Y H_Y
\]

\[
K_{i,n-1} = \begin{bmatrix}
y^T_i y_i & y^T_i y_{i+1} & \cdots & y^T_i y_{i+n-1} \\
y^T_{i+1} y_i & y^T_{i+1} y_{i+1} & \cdots & y^T_{i+1} y_i \\
: & : & \ddots & : \\
y^T_{i+n-1} y_i & y^T_{i+n-1} y_{i+1} & \cdots & y^T_{i+n-1} y_{i+n-1}
\end{bmatrix}
\]
Exploting Sparsity in Learning

- **Correlative Sparsity Graph:**
  - $n^2$ vertices, each corresponds to $\langle y_i, y_j \rangle$

**Complexity dominated by spatio/temporal connectivity, not the number of points**

*(via Grone’s Theorem on PSD matrix completions)*
Finding the mapping

Loewner Embedding

Loewner Reconstruct

Non Linear Dynamics

Propagated

Linear Dynamics K
Finding the mapping:

- Since the algorithm forces Lipschitz continuity we can assume that \( y_k = \psi(\xi_k) \) is rational (Stone Weierstrass)

- We can find \( \psi(.) \) from the null space of \( L \)

\[
L = \begin{bmatrix}
\frac{\kappa y_1 - \kappa y_{p+1}}{\kappa x_1 - \kappa x_{p+1}} & \frac{\kappa y_1 - \kappa y_{p+2}}{\kappa x_1 - \kappa x_{p+2}} & \ldots & \frac{\kappa y_1 - \kappa y_q}{\kappa x_1 - \kappa x_q} \\
\frac{\kappa y_p - \kappa y_{p+1}}{\kappa x_p - \kappa x_{p+1}} & \frac{\kappa y_1 - \kappa y_{p+2}}{\kappa x_1 - \kappa x_{p+2}} & \ldots & \frac{\kappa y_p - \kappa y_q}{\kappa x_p - \kappa x_q} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\kappa y_p - \kappa y_{p+1}}{\kappa x_p - \kappa x_{p+1}} & \frac{\kappa y_1 - \kappa y_{p+2}}{\kappa x_1 - \kappa x_{p+2}} & \ldots & \frac{\kappa y_p - \kappa y_q}{\kappa x_p - \kappa x_q}
\end{bmatrix}
\]

- Rank of \( L \) gives the order of the interpolant

- Minimize rank(\( L \)) to obtain low order interpolants
Low order dynamics and interpolants:

\[
\min_{K, y \geq 0} \text{rank}(G) + \mu \text{rank}(L) - \lambda \text{trace}(K_y)
\]

subject to affine constraints in \( G, L, K_y \)

- Can relax to a structured SDP
- Min singular value of \( G \) gives the (linear) approximation error
- Overall architecture:
Examples:

Lorentz attractor example: one step ahead prediction of new data

$7^{th}$ order linear dynamics (vs. 14 for Havoc)
Example: using manifold dynamics for analysis

Original data: 2000 promoters

Only need the manifold dynamics for inference
Application: Video inpainting

Project onto a suitable manifold
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Project onto a suitable manifold

Interpolate manifold trajectories and back-project.
Conclusions:

- Koopman operators provide tractable alternatives for learning non-linear dynamics
- Finding Koopman operators reduces to a structured SDP
- Computational complexity related to spatio/temporal connectivity
- In many cases finding the linear dynamics is enough for inference
- Work in progress: extension to piece-wise linear dynamics