Frequency response at steady state

$$D + C\,(j\omega I - A)^{-1}\,B =: Me^{j\theta} \text{ (by proper choice of } M \text{ and } \theta)$$

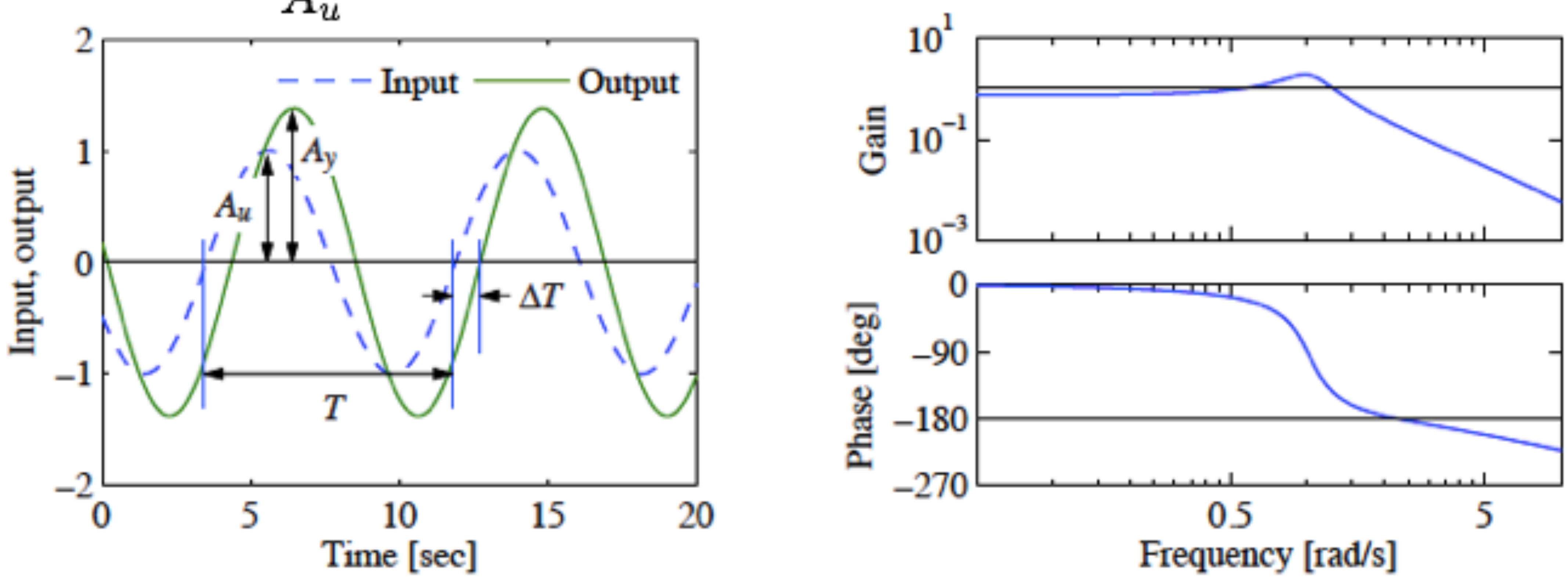
$$u = A_u \sin(\omega t + \psi) \;\rightarrow\; y = A_y \sin(\omega t + \phi)$$

gain of the system

$$M = \frac{A_y}{A_u}$$

phase of the system

$$\theta = \phi - \psi \qquad (= 2\pi/(\Delta T/T))$$



(a) Input/output response

(b) Frequency response

Frequency response

The frequency response of a linear system can be computed from its transfer function by setting s = iω.

$$u(t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t).$$

$$y(t) = G(i\omega)e^{i\omega t} = Me^{i(\omega t+\varphi)} = M\cos(\omega t+\varphi) + iM\sin(\omega t+\varphi),$$

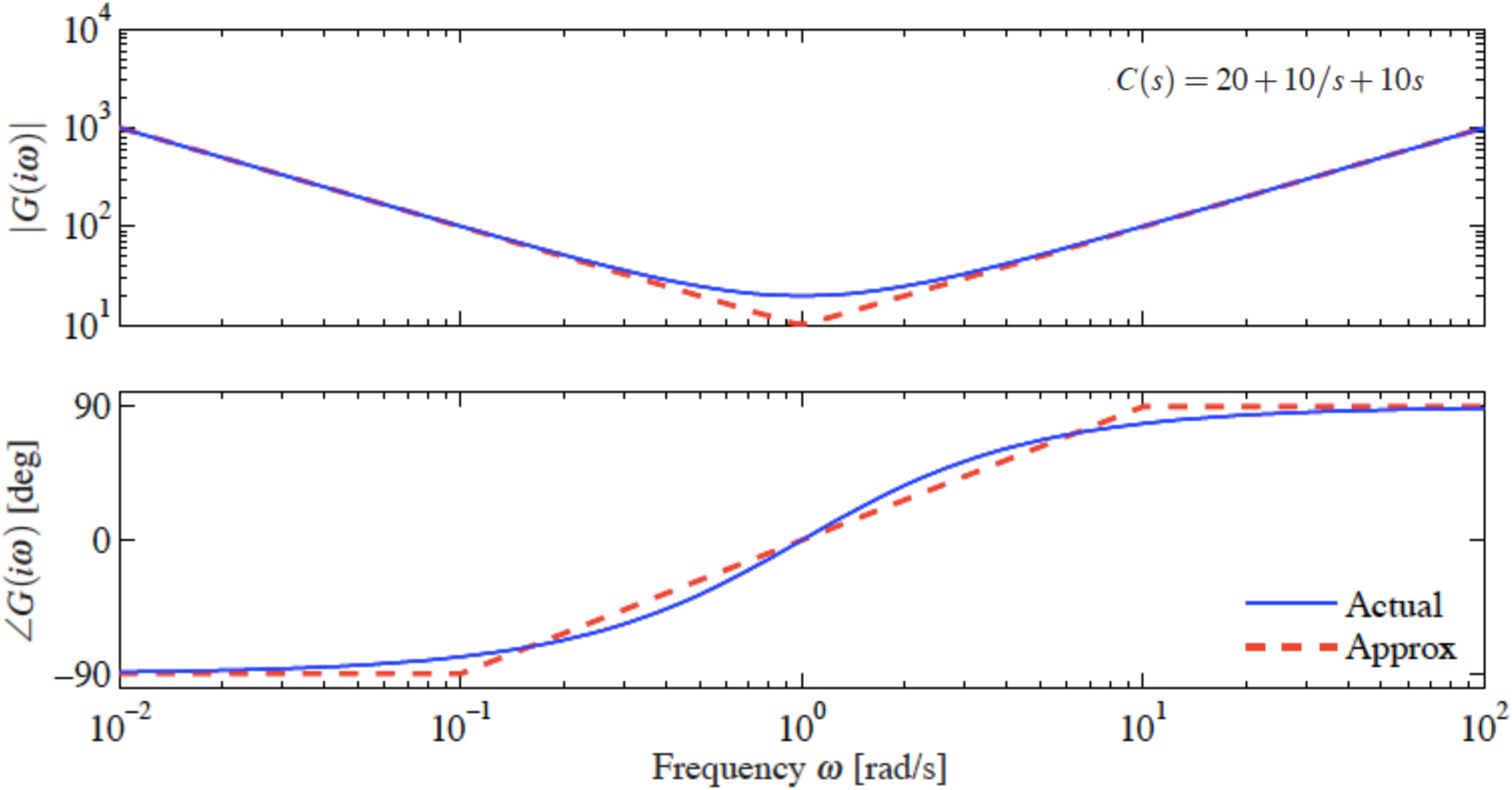
gain $M = |G(i\omega)|$

phase
(or argument [radian]) $\varphi = \arctan\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}.$

The Bode plot

gain $M = |G(i\omega)|$

phase (or argument [radian]) $\varphi = \arctan \frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}.$



Sketching Bode plots

$$G(s) = \frac{b_1(s)b_2(s)}{a_1(s)a_2(s)}$$

$$\log|G(s)| = \log|b_1(s)| + \log|b_2(s)| - \log|a_1(s)| - \log|a_2(s)|$$

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s)$$

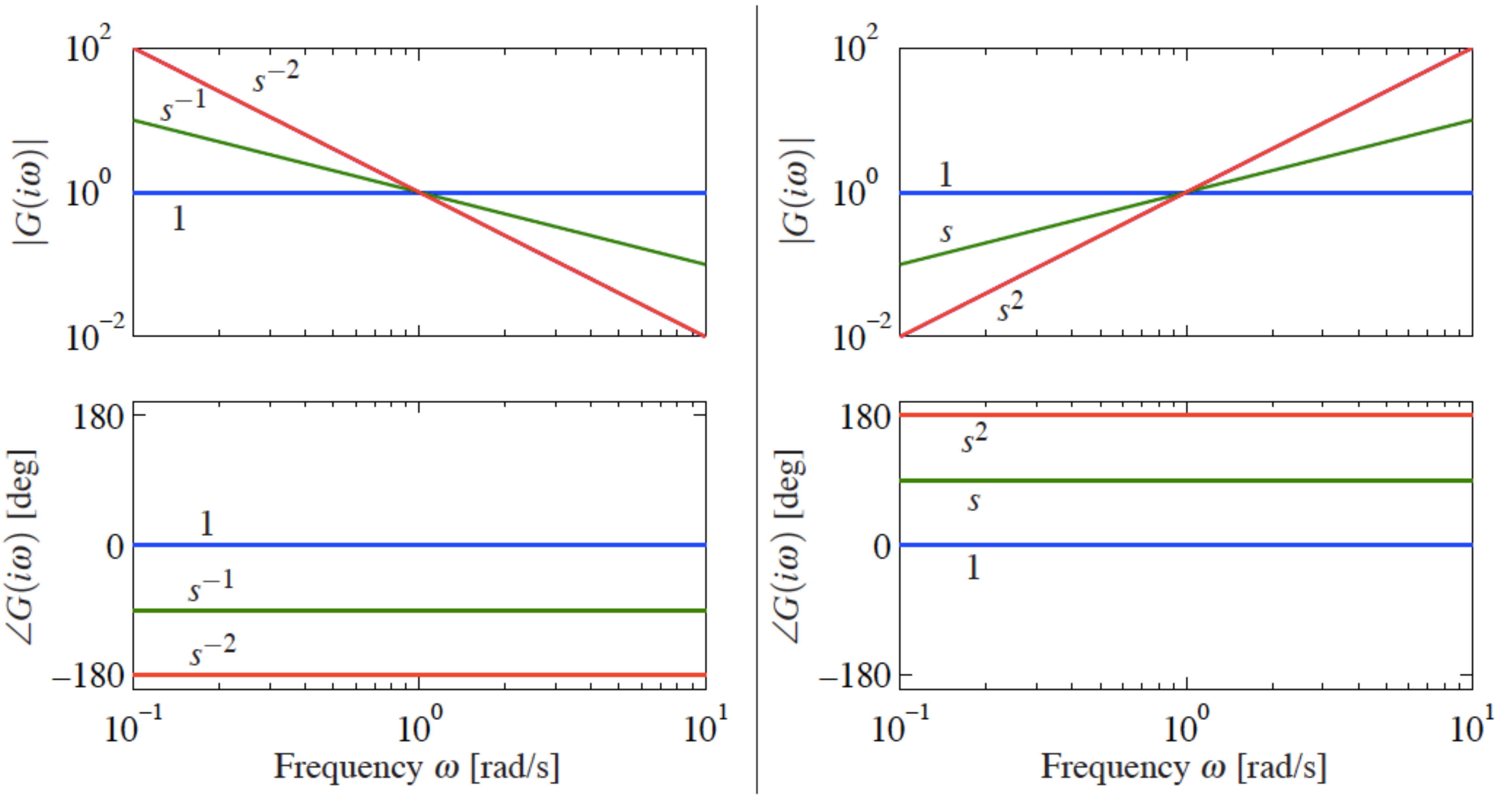
A polynomial can be written as the product of terms of the type

$$k, \quad s, \quad s+a, \quad s^2 + 2\zeta\omega_0 s + \omega_0^2$$

Sketch for low-order polynomials and add the gains and phases.

$G(s) = s^k$ for integer k

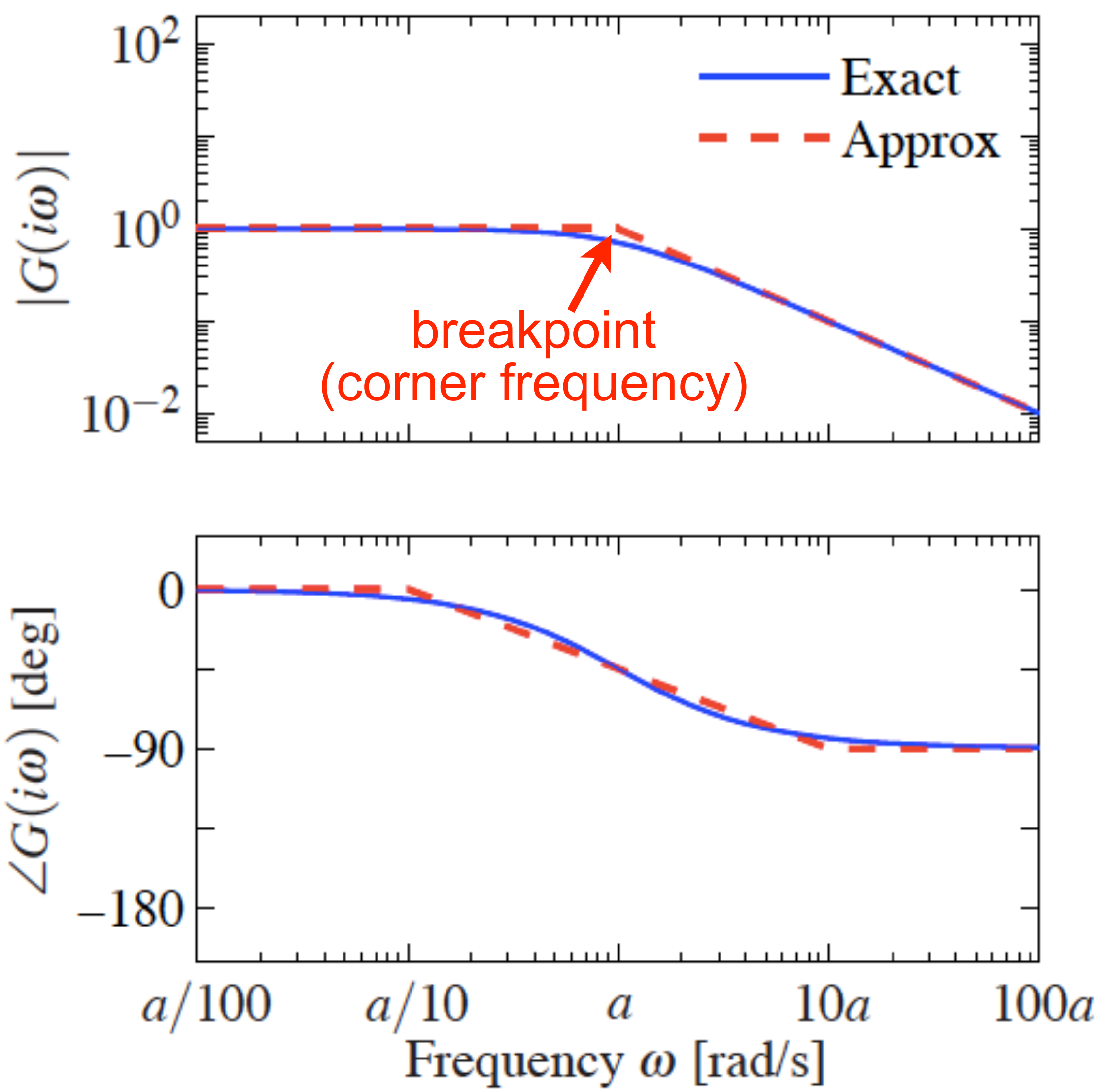
$$\log|G(i\omega)| = k\log\omega, \quad \angle G(i\omega) = 90k.$$



$G(s) = a/(s+a)$

$$|G(s)| = \frac{|a|}{|s+a|} \qquad \angle G(s) = \angle(a) - \angle(s+a)$$

$$\log|G(i\omega)| = \log a - \frac{1}{2}\log(\omega^2 + a^2) \qquad \angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{\omega}{a}$$



$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a \\ \log a - \log\omega & \text{if } \omega > a, \end{cases}$$

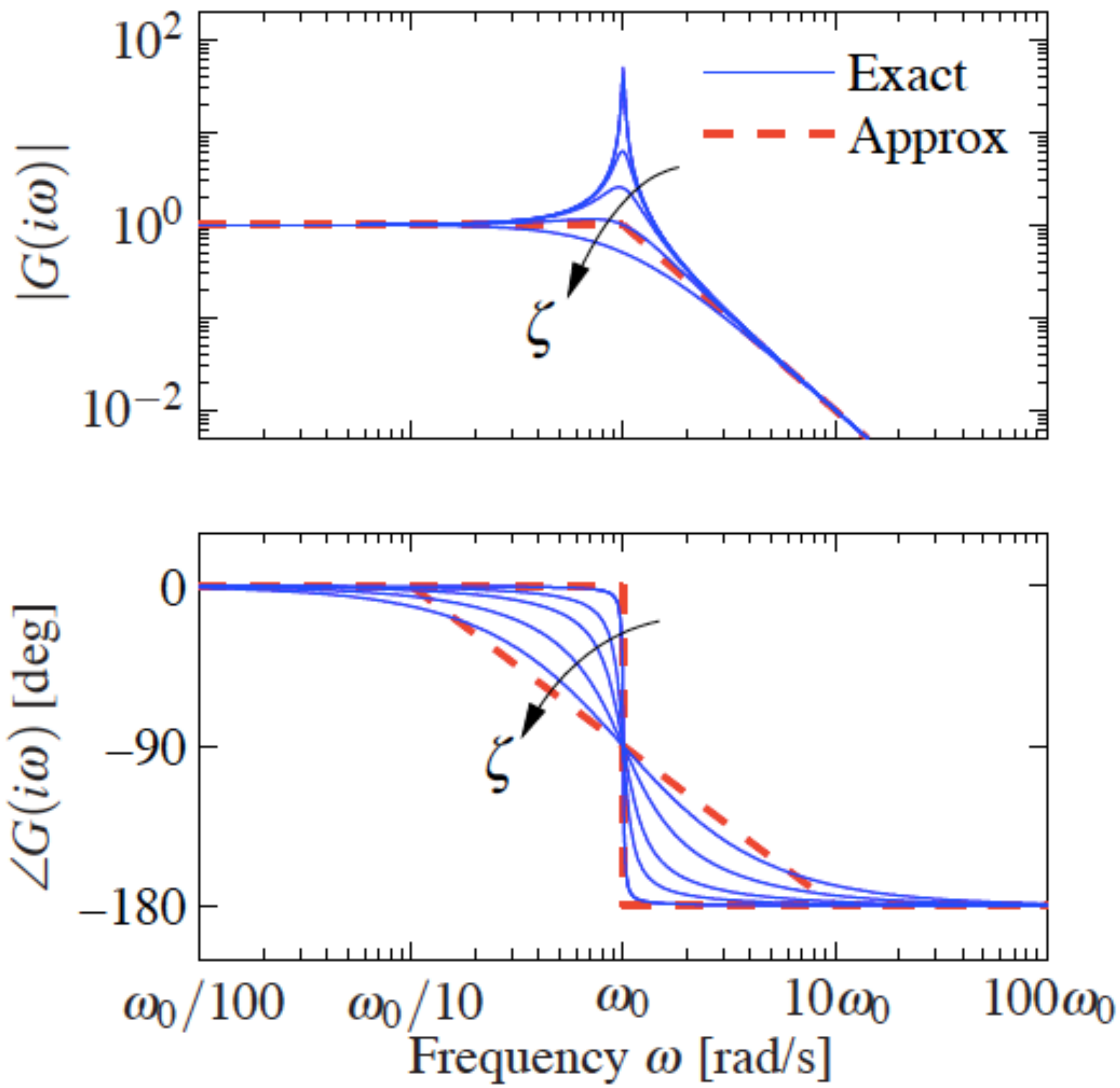
$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10 \\ -45 - 45(\log\omega - \log a) & a/10 < \omega < 10a \\ -90 & \text{if } \omega > 10a. \end{cases}$$

Like a constant at low frequencies and like an integrator at high frequencies.

$G(s) = \omega_0^2/(s^2+2a\zeta s+w_0^2)$

$$\log|G(i\omega)| = 2\log\omega_0 - \frac{1}{2}\log\left(\omega^4 + 2\omega_0^2\omega^2(2\zeta^2-1)+\omega_0^4\right),$$

$$\angle G(i\omega) = -\frac{180}{\pi}\arctan\frac{2\zeta\omega_0\omega}{\omega_0^2-\omega^2}.$$



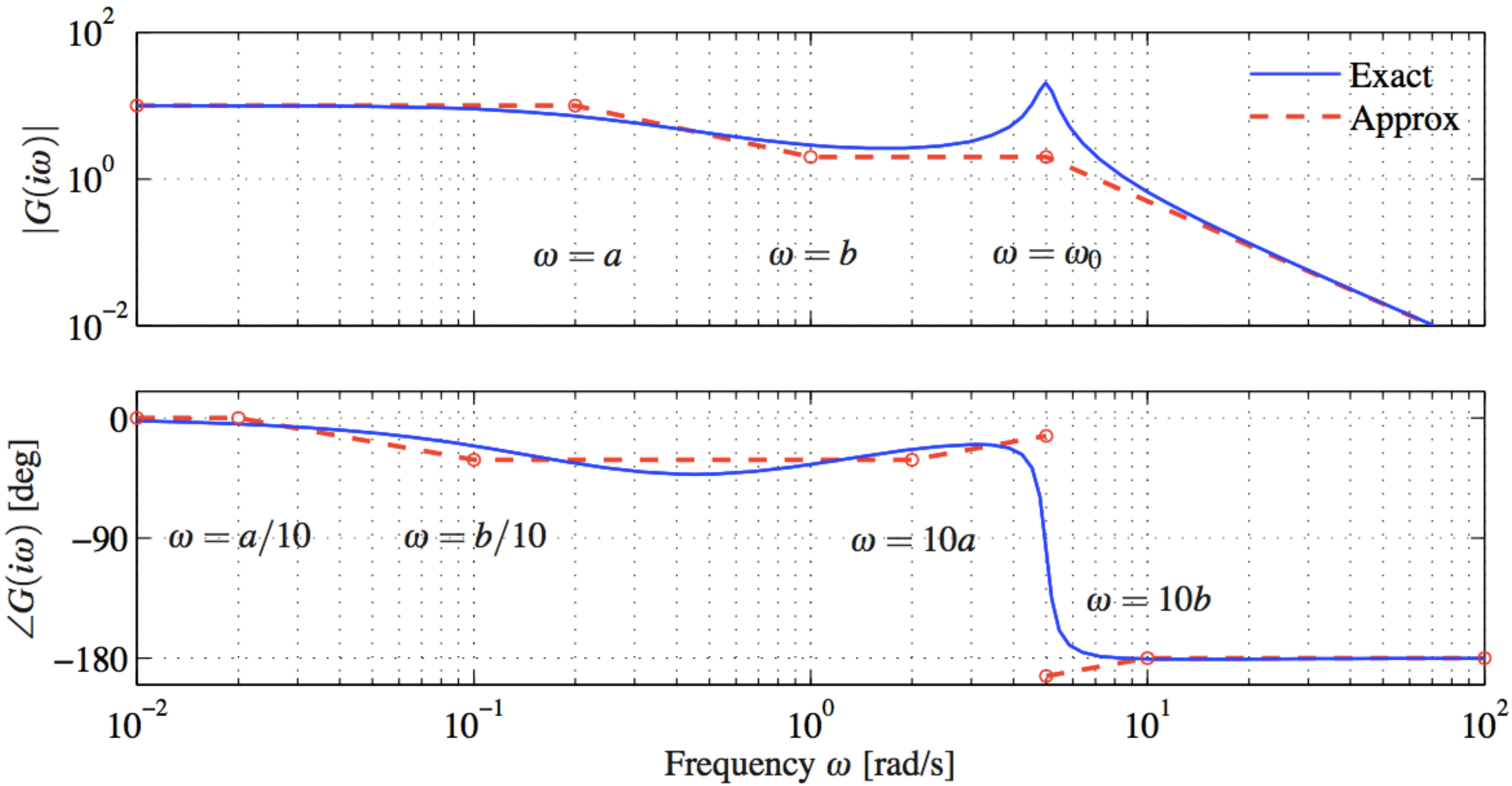
The largest gain $Q \approx 1/(2\zeta)$ is obtained around ω_0

$$\log|G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log\omega_0 - 2\log\omega & \text{if } \omega \gg \omega_0. \end{cases}$$

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0 \end{cases}$$

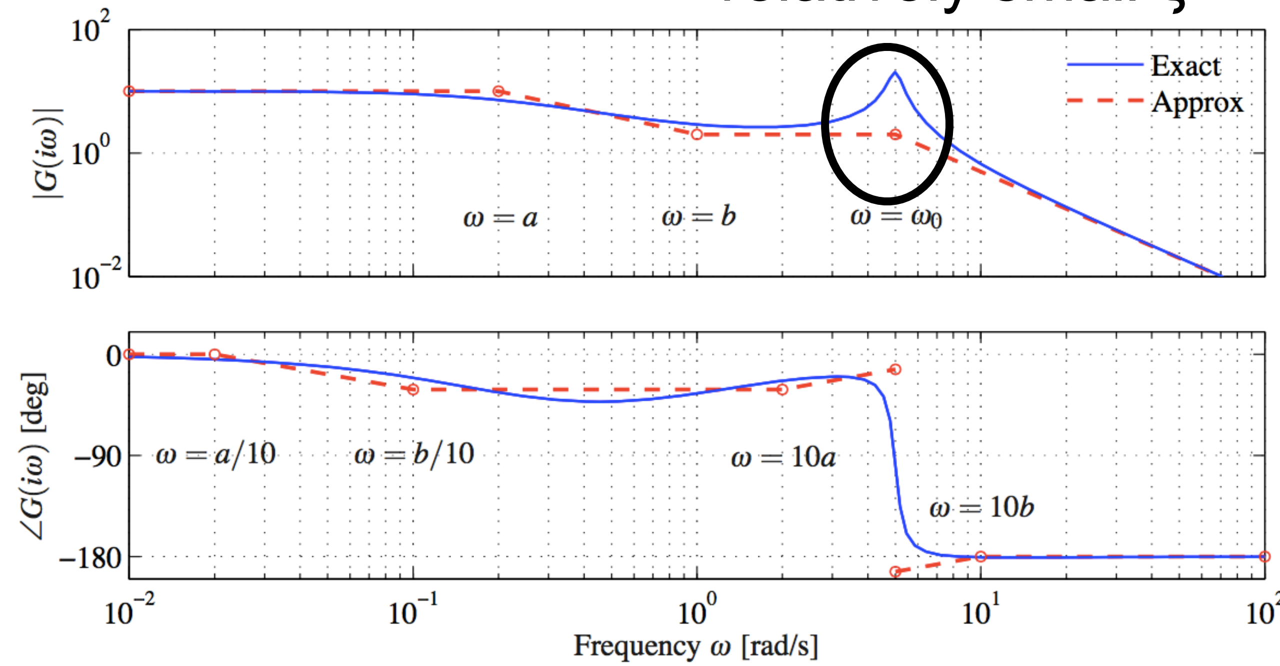
Example

$$G(s) = \frac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}, \qquad a \ll b \ll \omega_0.$$



Example

relatively small ζ



$$G(s) = \frac{k(s+b)}{(s+a)(s^2+2\zeta\omega_0 s+\omega_0^2)}, \qquad a \ll b \ll \omega_0.$$

Gain:

At low frequency: $G(0) = \dfrac{kb}{a\omega_0^2}.$

At s = a, the gain begins to decrease with slope -1.

At s = b, the decrease stops because of the zero (which increase the slope by 1).

When s = ω_0 is reached the second-order poles shift the slope to -2.

Phase:

The effect of the first-order pole begins at s = 1/10. It changes from 0 to a slope of -45°/decade.

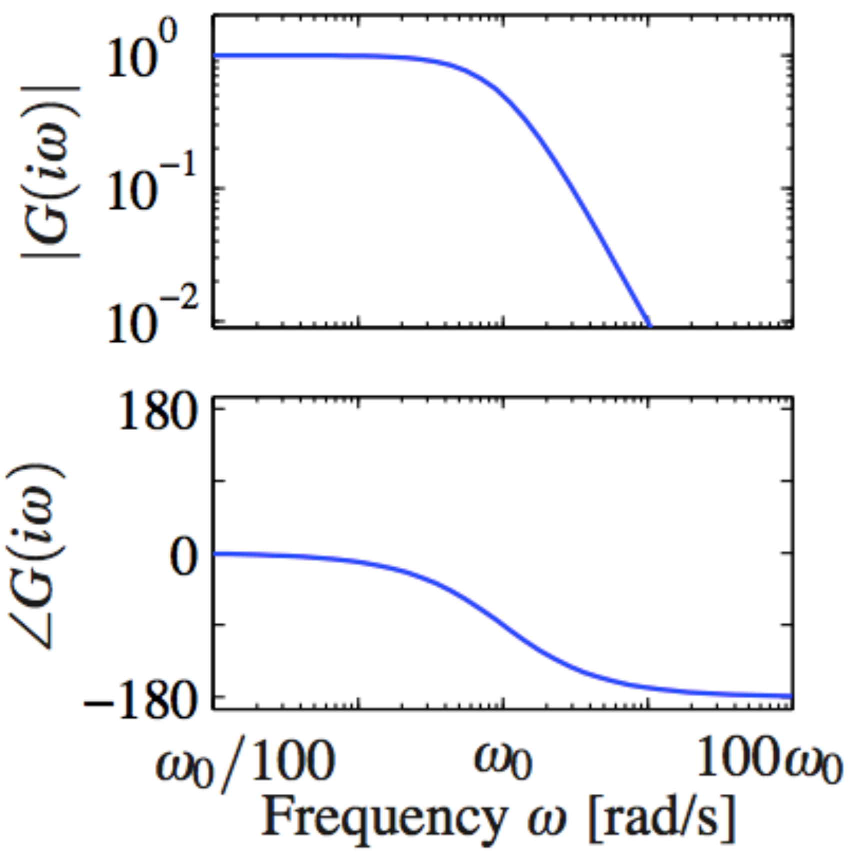
The zero begins to affect at s = b/10 by adding a 45°/decade slope (flat phase curve).

At s = 10a, the contribution from the pole ends. The slope becomes 45°/decade.

Around ω_0, there is a jump in phase of -180°

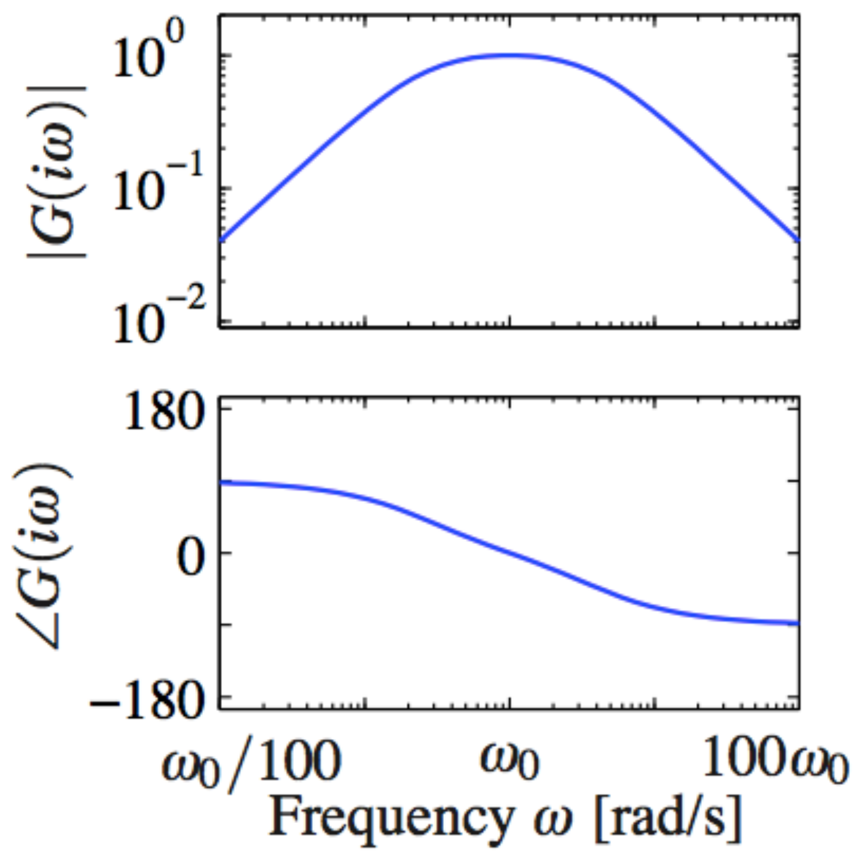
At s = 10b, the contribution from the zero ends. The slope becomes 0.

Bode plots for three common types of system elements



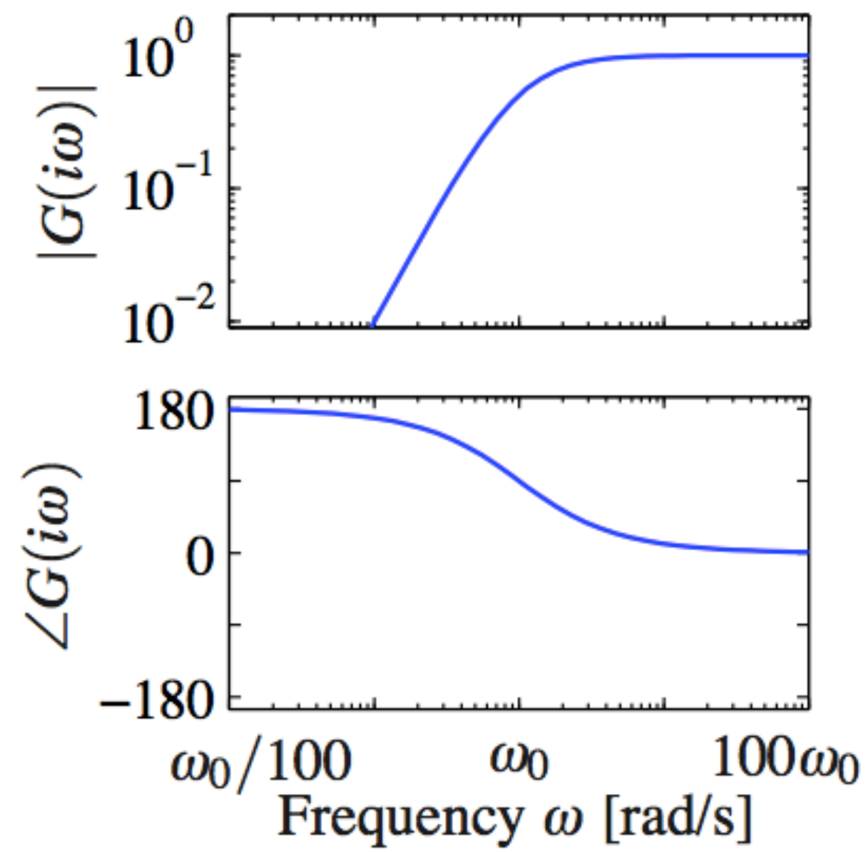
$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter



$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter



$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

Example

The response of the protein concentration to fluctuations in the mRNA dynamics:

$$\frac{dm}{dt} = \alpha(p) - \gamma m - v, \qquad \frac{dp}{dt} = \beta m - \delta p,$$

With no feedback: $\alpha(p) = \alpha_0$

Equilibrium: $m_e = \alpha_0/\gamma,\ p_e = \beta\alpha_0/(\delta\gamma)$

Open-loop transfer function:

$$G_{pv}^{\mathrm{ol}}(s) = \frac{-\beta}{(s+\gamma)(s+\delta)}$$

With negative feedback:

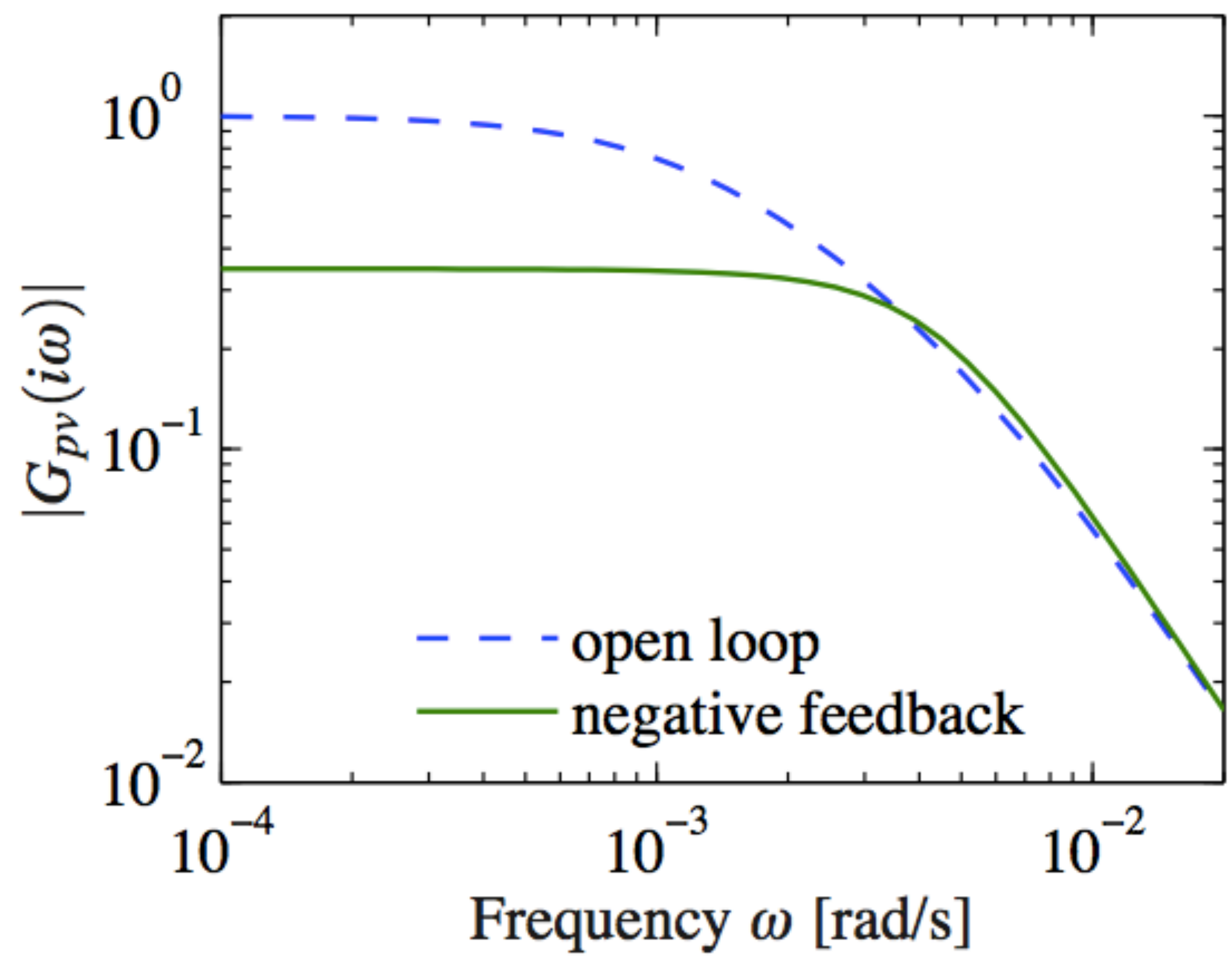
$$\alpha(p) = \frac{\alpha_1}{1+kp^n} + \alpha_0$$

The equilibrium satisfies:

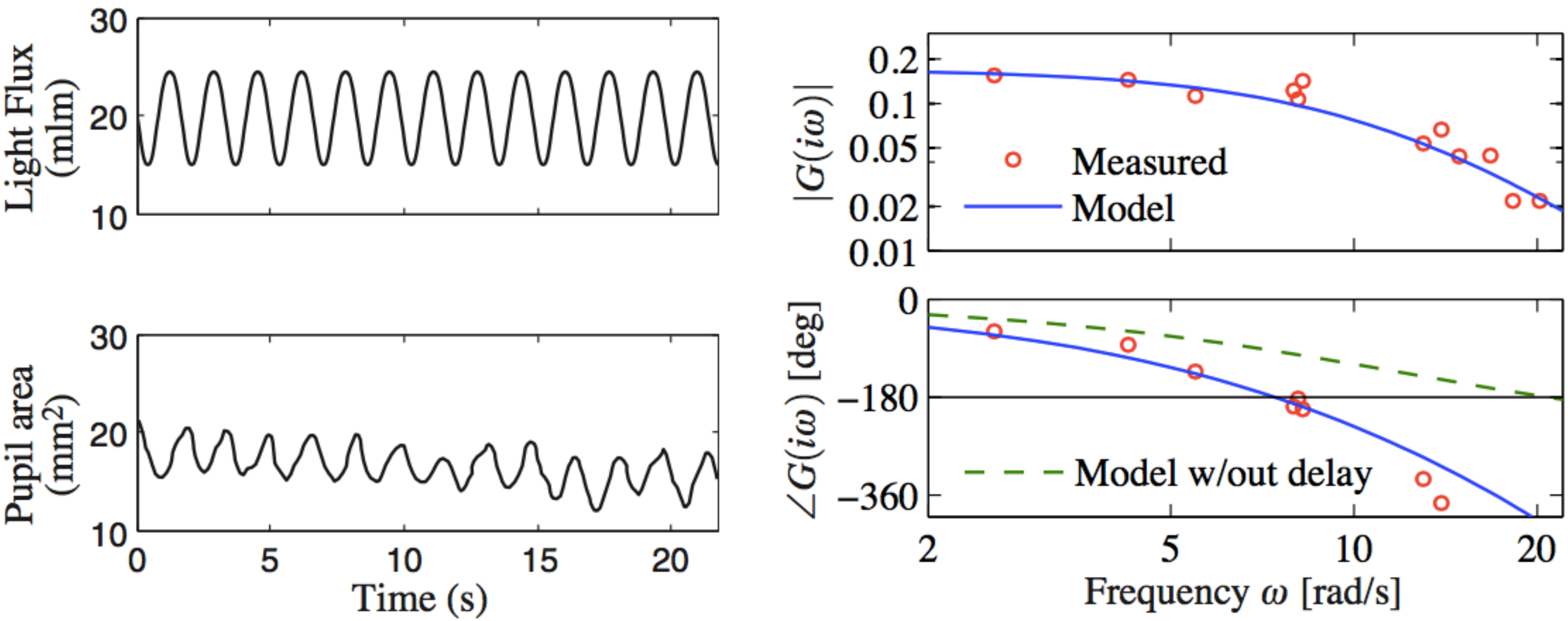
$$m_e = \frac{\delta}{\beta} p_e, \qquad \frac{\alpha}{1+kp_e^n} + \alpha_0 = \gamma m_e = \frac{\gamma\delta}{\beta} p$$

Closed-loop transfer function:

$$G_{pv}^{\mathrm{cl}}(s) = \frac{\beta}{(s+\gamma)(s+\delta)+\beta\sigma}, \qquad \sigma = \frac{n\alpha_1 k p_e^{n-1}}{(1+kp_e^n)^2}.$$



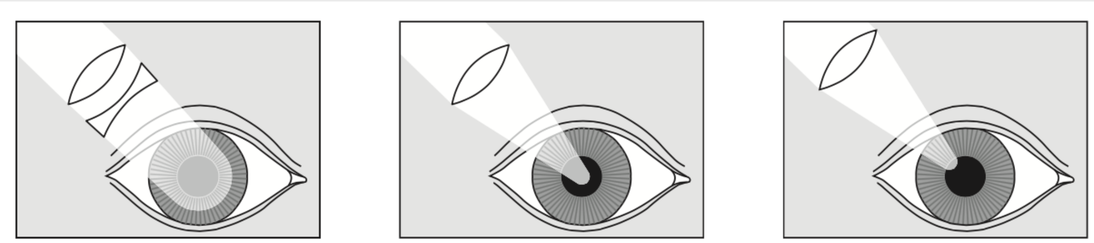
Transfer functions from experiments



Dashed curve: $0.17/(1+0.08s)^3$

Solid curve: $\dfrac{0.17}{(1+0.08s)^3}e^{-0.2s}$

The human eye has a control system that adjusts the pupil opening to regulate the light intensity at the retina.

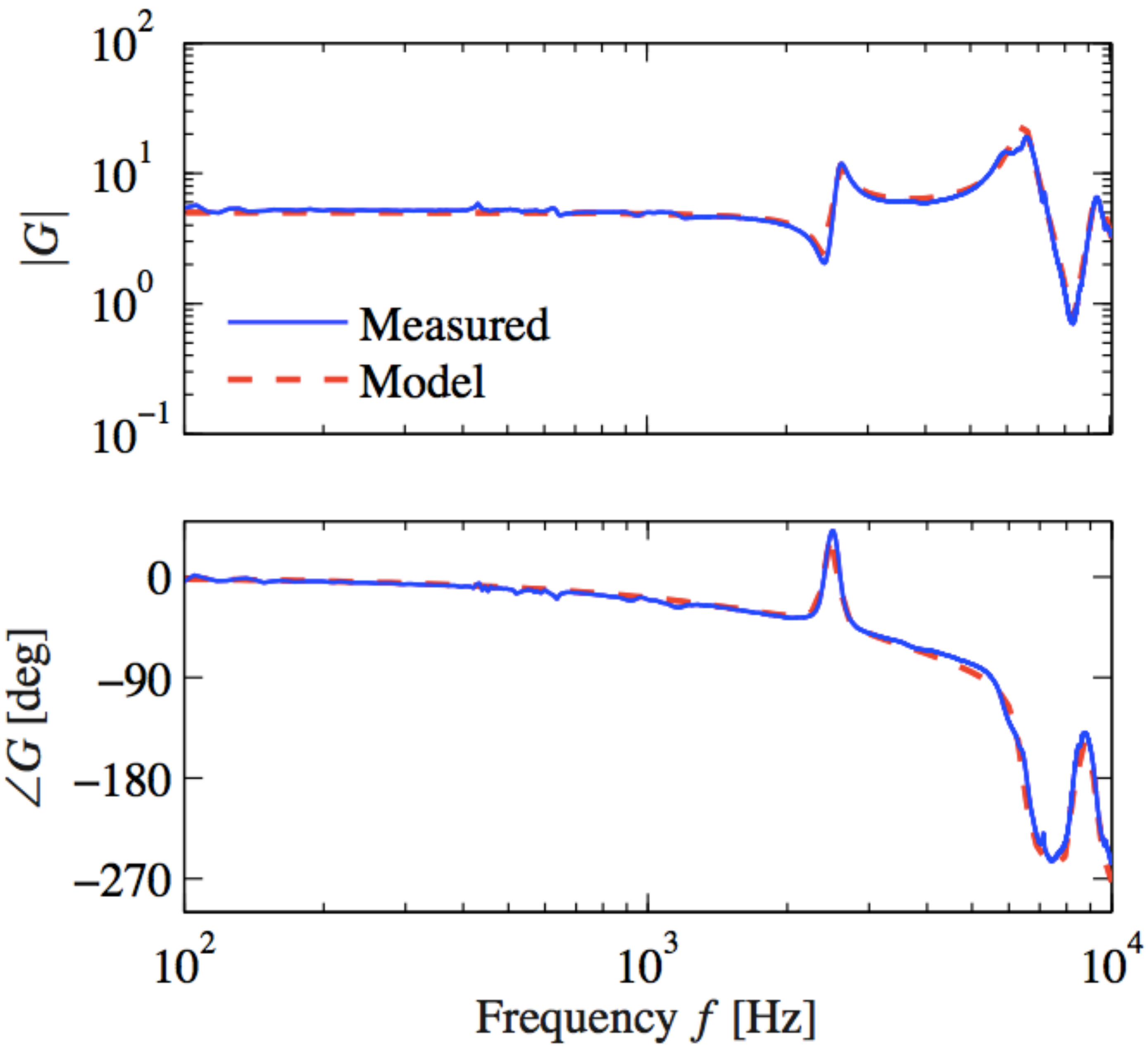


(a) Closed loop (b) Open loop (c) High gain

Transfer functions from experiments

The dynamics of an atomic force microscope

$$G(s) = \frac{k\omega_2^2\omega_3^2\omega_5^2(s^2+2\zeta_1\omega_1 s+\omega_1^2)(s^2+2\zeta_4\omega_4 s+\omega_4^2)e^{-s\tau}}{\omega_1^2\omega_4^2(s^2+2\zeta_2\omega_2 s+\omega_2^2)(s^2+2\zeta_3\omega_3 s+\omega_3^2)(s^2+2\zeta_5\omega_5 s+\omega_5^2)}$$



Zeros at local minima of the gain curve.

Poles at local maxima of the gain curve.

Damping ratios to give good fit at the minima and maxima of the gain curve.

Delay to fit the phase curve.