Equations of Motion for an Inverted Pendulum

The experimental control system which will be used in the Test #2 consists of two subsystems, as shown in Fig 1. The first of these systems is the electromechanical plant, which consists of the inverted pendulum mechanism, its sensors and actuators. The other subsystem is a PC-based digital controller, which includes the data acquisition systems (DAQ), the digital signal processor (DSP Board), and the control program. The control algorithms are designed and implemented via LabVIEW.

Figure 1: Laboratory system overview.

The plant, shown in Fig 2, is the ECP model 505 Inverted Pendulum Apparatus. It consists of a pendulum rod which supports the sliding balance rod. The mechanism itself is open-loop unstable and non-minimum phase, thus closed-loop feedback control is essential for stability. The balance rod is driven via a belt and pulley which in turn is driven by a drive shaft connected to a dc servo motor below the pendulum rod. The pendulum rod angle is controlled by moving the sliding rod in the presence of gravity. The weights at the bottom may be adjusted to alter the inertia configurations of the pendulum rod, and as a result, the dynamics of the system. A brushed DC

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motor and encoders are used to drive the sliding rod through measurements of the angular position of the pendulum rod and linear position of the sliding rod. Therefore, the only input on the plant is the force applied at the sliding rod.

Figure 2: Inverted Pendulum Apparatus

In order to derive the equations of motion of the Inverted Pendulum system, the body diagram shown in Fig. 3 and data in Table 1 are given.
Figure 3: Inverted Pendulum Body Diagram

Table 1 - Parameters description and values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_0$</td>
<td>0.330(m)</td>
<td>Length of pendulum rod from pivot to the sliding rod T section</td>
</tr>
<tr>
<td>$m_1$</td>
<td>0.213(kg)</td>
<td>Mass of the complete sliding rod including all attached elements</td>
</tr>
<tr>
<td>$J_1$</td>
<td>(kg.m$^2$)</td>
<td>Inertia of the complete sliding rod including elements</td>
</tr>
<tr>
<td>$m_2$</td>
<td>1.785(kg)</td>
<td>Mass of the complete assembly minus $m_1$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>(kg.m$^2$)</td>
<td>Inertia of the complete balanced parts</td>
</tr>
<tr>
<td>$l_{m2}$</td>
<td>-0.0409(m)</td>
<td>Position (signed) of c.g.2 of the complete pendulum assembly</td>
</tr>
<tr>
<td>$J_0$</td>
<td>0.0644(kg.m$^2$)</td>
<td>Inertia evaluated at equilibrium point</td>
</tr>
<tr>
<td>$J^*$</td>
<td>0.0412(kg.m$^2$)</td>
<td>$J^* = J_0^* - m_1l_0^2$</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>(m)</td>
<td>Distance measured from the motor to the c.g.1 of the sliding rod</td>
</tr>
<tr>
<td>$F(t)$</td>
<td>(N)</td>
<td>Force applied by the DC motor on the sliding rod.</td>
</tr>
</tbody>
</table>
The equations of motion (EOM) for the inverted pendulum are derived using the Lagrange’s equations in the form:

$$\frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial \dot{q}_i} = Q_i,$$

where $T$ is the kinetic energy, $V$ is the potential energy, $q_i$ is the generalized coordinate and $Q_i$ is the associated generalized force.

With the Lagrangian defined as $L = T - V$, we can rewrite Eq. (1) as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i.$$ 

In the subsequent derivation, the friction is neglected.

Define the **generalized coordinates** as:

$q_1 := x$ where $x$ is the linear displacement,

$q_2 := \theta$ where $\theta$ is the angular displacement.

Consider the **kinetic energy**

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} J_2 \dot{\theta}^2$$

where $v_1$ is the magnitude of the velocity for body 1, and can be written as

$$v_1 = v_{1\text{ROT}} + v_{1\text{TRANS}}.$$ 

Therefore it follows that

$$v_1 = [-l_{m1} \dot{\theta} \sin \alpha u_2 + l_{m1} \dot{\theta} \cos \alpha u_1] + [\dot{x} u_1],$$

where $u_1$ and $u_2$ are the unit vectors in the direction of the sliding rod and perpendicular, respectively. Therefore, we can write

$$v_1 = (l_{m1} \dot{\theta} \cos \alpha + \dot{x}) u_1 + (-l_{m1} \dot{\theta} \sin \alpha) u_2$$

and $v_2$ is the magnitude of the velocity for body 2 and is given by

$$v_2 = l_{m2} \dot{\theta} u_1.$$ 

Combining the relationship for $u_1$ and $u_2$ into the kinetic energy yields

$$T = \frac{1}{2} m_1 \left[(l_{m1} \dot{\theta} \cos \alpha + \dot{x})^2 + (l_{m1} \dot{\theta} \sin \alpha)^2 \right] + \frac{1}{2} J_1 \dot{\theta}^2$$

$$+ \frac{1}{2} m_2 \left(l_{m2} \dot{\theta} \right)^2 + \frac{1}{2} J_2 \dot{\theta}^2.$$
Rearranging the terms into the above relationship yields

\[ T = \frac{1}{2} m_1 \left( l_{m1}^2 \dot{\theta}^2 + 2 l_{m1} \dot{\theta} \dot{x} \cos \alpha + \dot{x}^2 \right) + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} m_2 l_{m2}^2 \dot{\theta}^2 + \frac{1}{2} J_2 \dot{\theta}^2. \]

Since the system is constrained by

\[ l_{m1} \cos \alpha = l_0 \]
\[ l_{m1}^2 = l_0^2 + x^2 \]

we can write

\[ T = \frac{1}{2} m_1 \dot{x}^2 + m_1 l_0 \dot{\theta} \dot{x} + \frac{1}{2} \left[ m_1 \left( l_0^2 + x^2 \right) + m_2 l_{m2}^2 + J_1 + J_2 \right] \dot{\theta}^2. \]

Observing that the system moment of inertia about the point \( O \) (pivot) is

\[ J_0 = J_1 + m_1 \left( l_0^2 + x^2 \right) + J_2 + m_2 l_{m2}^2, \]

then the kinetic energy becomes

\[ T = \frac{1}{2} m_1 \dot{x}^2 + m_1 l_0 \dot{\theta} \dot{x} + \frac{1}{2} J_0 \dot{\theta}^2 \] (3)

Consider the **potential energy**. Taking the reference point as \( \theta = 0^\circ \) and \( x = 0 \), we have

\[ V = m_1 gl_{m1} \cos (\theta + \alpha) + m_2 gl_{m2} \cos \theta, \]

or

\[ V = m_1 gl_{m1} (\cos \theta \cos \alpha - \sin \theta \sin \alpha) + m_2 gl_{m2} \cos \theta. \]

Since \( l_{m1} \cos \alpha = l_0 \) and \( l_{m1} \sin \alpha = x \), we have the potential energy relationship

\[ V = m_1 g (l_0 \cos \theta - x \sin \theta) + m_2 gl_{m2} \cos \theta \] (4)

The Lagrangian, is thus given by

\[ L = T - V, \]

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\[
L = \frac{1}{2}m_1 \ddot{x}^2 + m_1 l_0 \ddot{\theta} \dot{x} + \frac{1}{2}J_0 \dot{\theta}^2 - m_1 g (l_0 \cos \theta - x \sin \theta) - m_2 g l_{m2} \cos \theta
\]

(5)

Recalling that \( J_0 = J_1 + m_1 (l_0^2 + x^2) + J_2 + m_2 l_{m2}^2 \), we have

(a) For \( q_1 = x \)

\[
\frac{\partial L}{\partial x} = \frac{1}{2} m_1 (2x) \ddot{\theta}^2 + m_1 g \sin \theta = m_1 x \ddot{\theta}^2 + m_1 g \sin \theta
\]

\[
\frac{\partial L}{\partial \dot{x}} = \frac{1}{2} m_1 2 \ddot{x}^2 + m_1 l_0 \ddot{\theta} = m_1 \ddot{x} + m_1 l_0 \ddot{\theta}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = m_1 \dddot{x} + m_1 l_0 \dddot{\theta}
\]

\[
Q_x = F(t)
\]

(b) For \( q_2 = \theta \)

\[
\frac{\partial L}{\partial \theta} = -m_1 g (-l_0 \sin \theta + x \cos \theta) + m_2 g l_{m2} \sin \theta =
\]

\[
= g (m_1 l_0 + m_2 l_{m2}) \sin \theta + x m g \cos \theta
\]

\[
\frac{\partial L}{\partial \dot{\theta}} = m_1 l_0 \ddot{x} + J_2 \ddot{\theta}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = m_1 l_0 \dddot{x} + J_2 \dddot{\theta}
\]

\[
Q_{\theta} = 0
\]

Therefore, using the Lagrange’s equation the **non-Linear, coupled EOM** are give by

\[
m_1 \dddot{x} + m_1 l_0 \dddot{\theta} - m_1 \ddot{\theta}^2 \dot{x} - m_1 g \sin \theta = F(t)
\]

(6)

\[
m_1 l_0 \dddot{x} + J_2 \dddot{\theta} - (m_1 g l_0 + m_2 g l_{m2}) \sin \theta - x m_1 g \cos \theta = 0
\]

(7)

From Eqs. (6) and (7) we have for a motionless system, i.e., equilibrium points, that \( \dot{\theta} = \ddot{\theta} = \dot{x} = \ddot{x} = 0 \), and \( F(t) = 0 \). Linearizing the equations with respect to the equilibrium points \((x, \theta) = (x_e, \theta_e)\), we have:

\[
m_1 g \sin \theta_e = 0 \rightarrow \theta_e = 0^\circ \rightarrow x_e = 0
\]
Using Taylor series expansions, about a small angle, we can neglect second order terms and write \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \). Then the linearized EOM can be written as:

\[
\begin{align*}
m_1\ddot{x} + m_1l_0\ddot{\theta} - m_1g\theta &= F(t) \quad (8) \\
m_1l_0\dddot{x} + J_0^*\ddot{\theta} - (m_1l_0 + m_2l_{m2})g\theta - m_1gx &= 0 \quad (9)
\end{align*}
\]

where \( J_0^* \) is the moment of inertia of the system about the pivot \( O \) at the equilibrium point.

Based on the fact that we can measure the angular position of the pendulum \( \theta(t) \) and the linear position of the sliding rod \( x(t) \) using two encoders attached at the axis, the task is to design a controller in order to keep the inverted pendulum at a desired angular position \( \theta_d \) through an input, i.e., a force \( u(t) = F(t) \), applied by a dc motor in the sliding rod. This can be visualized through the block diagram shown in Fig. 4.

![Control system block diagram](image)

**Figure 4: Control system block diagram**

Defining the state vector as

\[
z = [\theta \quad \dot{\theta} \quad x \quad \dot{x}]^T,
\]

we can get the state-space form

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) \\
y(t) &= Cz(t) + Du(t).
\end{align*}
\]

Setting

\[
\begin{align*}
J_0^* &= J_1 + J_2 + m_1l_0^2 + m_2l_{m2} \\
J^* &= J_0^* - m_1l_0^2.
\end{align*}
\]

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we can rewrite the linear model in Eqs. (8) and (9) to have the second-order terms isolated, so that,
\[
\ddot{x} + \frac{m_1g}{J^*}x + \frac{m_2l_0}{J^*} - \frac{J^*}{J^*}g\theta = \frac{J^*_0}{J^*_m}F(t) \tag{10}
\]
and
\[
\dot{\theta} - \frac{m_1g}{J^*}x - \frac{m_2gl_0}{J^*} = -\frac{l_0}{J^*}F(t). \tag{11}
\]
A state-space realization of the linearized plant is
\[
A = \frac{1}{J^*} \begin{bmatrix}
0 & J^* & 0 & 0 \\
-\frac{m_2l_0g}{J^*} & 0 & m_1g & 0 \\
(J^* - m_2l_0l_0g) & 0 & -m_1l_0g & 0 \\
\frac{l_0}{J^*_m} & 0 & 0 & 0
\end{bmatrix}
\]
\[
B = \frac{1}{J^*} \begin{bmatrix}
0 \\
-l_0 \\
0 \\
\frac{l_0}{m_1}
\end{bmatrix}
\]
If we measure \(\theta\), the output matrix is
\[
C_\theta = [1 \ 0 \ 0 \ 0].
\]
If we measure \(x\), the output matrix is
\[
C_x = [0 \ 0 \ 1 \ 0].
\]
In all cases, the matrix \(D\) is
\[
D = [0].
\]
By taking the Laplace transform of Eqs. (10) and (11), and assuming zero initial conditions, we get the following transfer functions:
\[
\frac{\Theta(s)}{X(s)} = \frac{m_1l_0}{J^*_0} \frac{s^2 - g/l_0}{s^2 - (m_1l_0 + m_2l_0)g/J^*_0}
\]
\[
\frac{\Theta(s)}{F(s)} = \frac{l_0}{J^*_0} \frac{s^2 - g/l_0}{s^2 - m_1g^2/J^*}
\]
\[
\frac{X(s)}{F(s)} = \frac{J^*_0}{m_1J^*} \frac{s^2 - (m_1l_0 + m_2l_0)g/J^*_0}{s^2 - m_1g^2/J^*}
\]
\[
\frac{\dot{\theta}}{F(s)} = \frac{l_0}{m_1J^*} \frac{s^4 + ((m_1l_0 - m_2l_0)g/J^*)s^2 - m_1g^2/J^*}{s^4 + ((m_1l_0 - m_2l_0)g/J^*)s^2 - m_1g^2/J^*}
\]