Appendix B. Coherence Theory

The following section presents the assumptions and theory for the Coherence Function. The information is summarized from the information available in [14].

The FFT produces frequency functions that contain real and imaginary information. The coherence function uses this information to assemble a relationship between the output power due to the input. The relationships are based on the auto-power spectrum and cross-power spectrum of the signals.

The auto-power spectrum is defined as

\[ G_{XX} = X(f) \cdot X^*(f) \]  

(Eq. B.1)

where

- \( G_{XX} \equiv \) Auto-power spectrum for input signal \( X \)
- \( f \equiv \) Frequency
- \( X(f) \equiv \) FFT of input signal
- \( X^*(f) \equiv \) Complex conjugate of \( X(f) \).

An equation for the auto-power spectrum of the output, \( Y \), can be similarly defined as equation 1 by simply replacing \( X \) with \( Y \).

The cross-power spectrum is a type of complex relationship between two frequency signals and is defined similarly to the auto-power spectrum in Eq. B.1:

\[ G_{YX} = Y(f) \cdot X^*(f) \]  

(Eq. B.2)

where

- \( G_{YX} \equiv \) Cross-power spectrum between input signal \( X \) and output signal \( Y \)
- \( Y(f) \equiv \) FFT of output signal.

In order to incorporate Eqs. B.1 and B.2 into the coherence function, coherence must first be defined. Coherence is defined as the ratio of the cross-power spectrum to the auto-power spectra of both the input and output:

\[ \gamma^2 = \frac{|G_{YX}(f)|^2}{G_{XX}(f) \cdot G_{YY}(f)} \]  

(Eq. B.3)

where

- \( \gamma \equiv \) Coherence.

In a linear system, the output \( Y(f) \) is related to the input \( X(f) \) through the system transfer function \( H(f) \) (Fig. B1).
Figure B1. Transfer Function Relationship of \( Y(f) \) to \( X(f) \)

Figure B1 also provides a basic description of how noise enters a system [14]. The introduction of noise makes the relationship between \( Y(f) \) and \( X(f) \) equal to

\[
Y(f) = X(f) \cdot H(f) + N(f)
\]  
(Eq. B.4)

where

\[
N(f) \equiv \text{Noise}
\]

From this point forward, the domain parameter, \( f \), will be dropped from the equations in order to simplify their presentation. Substitution and manipulation of Eq. B.4 in the auto-power spectrum for \( Y(f) \) (Eq. B.1) and the cross-power spectrum in Eq. B.2 produces the following relationships:

\[
G_{YY} = (HX + N)(HX + N)^* = |H|^2 G_{XX} + G_{NN} + H G_{NX} + H^* G_{NX}
\]  
(Eq. B.5)

\[
G_{YX} = (HX + N)X^* = HG_{XX} + G_{NX}
\]  
(Eq. B.6)

The assumption that noise is completely uncorrelated with the response, \( X(f) \), allows the cross-power spectrum between the signals to go to zero when the signals are averaged from multiple data samples:

\[
G_{NX} = 0
\]  
(Eq. B.7)

Substitution and manipulation of Eqs. B.5, B.6, and B.7 in Eq. B.3 produces the final form of the coherence equation:

\[
\gamma^2 = \frac{|H|^2 G_{XX}}{|H|^2 G_{XX} + G_{NN}}
\]  
(Eq. B.8)

The information in B.8 represents the fraction of the output power (in the numerator) that is attributed to the input power (in the denominator). In order to obtain the coherence as presented in B.8, the coherence parameters are averaged over several data sets. If only one set of data is taken, the coherence assumes that the noise is part of the input power and gives a false value of 1 for coherence.