Appendix A. Discrete Sampling Theory

The following section presents the assumptions and theory for the Fast Fourier Transform, as well as the selection of sampling frequency and block size for digital signal processing. The assumptions presented in the first section are used for both the formulation of the Fast Fourier Transform and the requirements for discrete sampling of a complex signal. Additional information regarding the theory and equations presented in this section is available in [23].

A.1 Assumptions

The formulation of the FFT (Fast Fourier Transform) is based on the characteristics of a discrete complex signal. In order to produce accurate discrete sampling, the following assumptions are made:

- The complex signal is periodic in the sampling period $T$
- The signal has been sampled in $N$ intervals of time
- The intervals of time are uniform and equal to $\Delta t$

These assumptions provide a means of accurately resolving the frequency content of a complex signal. If the assumptions are violated, the discrete data does not accurately reflect the actual harmonic signal, and the FFT of the signal is invalid.

A.2 Fast Fourier Transform

According to Fourier, a complex signal can be broken into several components made up of simple harmonic signals of varying amplitude and phase [23]. The general form of the Fourier Series, which represents a complex signal, is

$$y(t) = \frac{A_o}{2} + \sum_{n=1}^{\infty} \left( A_n \cos(n\omega t) + B_n \sin(n\omega t) \right)$$  \hspace{1cm} (Eq. A.2.1)

where

- $y(t)$ = Fourier Series representation of a complex signal
- $A_o, A_n, B_n$ = harmonic coefficients
- $n$ = harmonic order ($n = 1, 2, ..., \infty$)
- $\omega$ = fundamental circular frequency
- $t$ = time

The harmonic coefficients are defined by Fourier Sine and Cosine Transforms:

$$A_n = \frac{2\pi}{\omega} \int_0^{\omega} y(t) \cos(n\omega t) dt \quad (n = 0, 1, 2, ..., \infty)$$  \hspace{1cm} (Eq. A.2.2a)
The absolute value of the magnitudes of the harmonic coefficients provides the amplitude at each harmonic order in the Fourier Series (Eq. A.2.1):

\[ C_n = \sqrt{A_n^2 + B_n^2} \]  

(Eq. A.2.2c)

The FFT (Fast Fourier Transform) is based on conversion of Eqs. A.2.2a and A.2.2b into the discrete domain. For a set of discrete data, recorded in time increments of \( \Delta t \) with a block size of \( N \) points, the sampling period of the discrete signal is

\[ T = N \Delta t \]  

(Eq. A.2.3)

The actual time for each discrete point is related to the time increment (\( \Delta t \)) by

\[ t_r = r \Delta t \quad (r = 1, 2, \ldots, N) \]  

(Eq. A.2.4)

where

\( r \) = scalar multiple of the time increment

Figure A1 presents a typical complex signal that has been discretely sampled with time interval \( \Delta t \), block size \( N \), and period \( T \).
\[
A_n = \frac{2}{N} \sum_{r=1}^{N} y(r \Delta t) \cos \left( \frac{2 \pi r n}{N} \right) \quad \left( n = 1, 2, \ldots, \frac{N}{2} - 1 \right) \\
B_n = \frac{2}{N} \sum_{r=1}^{N} y(r \Delta t) \sin \left( \frac{2 \pi r n}{N} \right) \quad \left( n = 1, 2, \ldots, \frac{N}{2} - 1 \right)
\]
(Eq. A.2.5a)

In both Eqs. A.2.5a and A.2.5b, the harmonic order \( n \) in the discrete domain no longer has an infinite interval, as in Eq. A.2.1. The reduction in order from an infinite series to a finite interval is due to the Nyquist criterion, which will be explained in the next section.

### A.3 Discrete Sampling of a Complex Signal

Both the time interval and the block size of a digital sample play an important role in the accuracy of discrete data. Improper selection of these parameters can cause aliasing or leakage of the discrete signals.

#### A.3.1 Avoiding Aliasing in a Discrete Signal

In order to provide an accurate discrete signal, the sampling frequency

\[
f_s = \frac{1}{\Delta t}
\]

must satisfy the Nyquist criterion, which states that sampling frequency must be greater than twice the highest frequency (Nyquist frequency) present in the complex signal:

\[
f_s > 2f_{Nyq}
\]

(Eq. A.3.2)

Equation A.3.1 is a mathematical representation that implies a harmonic signal must be sampled more than twice per period in order to obtain enough data to represent the original signal [23]. When the sampling frequency obeys the Nyquist criterion (Eq. A.3.2), sampling is sufficient to accurately resolve the highest frequency content in the complex wave. The discrete sampling of Fig. A2 has been performed in accordance with the Nyquist criterion, and the resulting discrete signal can be curve fit to accurately reproduce the original signal.
If the sampling frequency violates the Nyquist criterion, insufficient discrete data is obtained from the original signal. When insufficiently sampled signals are plot and curve fit, the resultant output signal is an alias of the original signal (Fig. A3).
The Nyquist frequency is responsible for the change in order \((n)\) of the FFT (Eqs. A.2.5a and A.2.5b). Since the sampling frequency is more than twice the highest resolvable frequency (Eq. A.3.2), the maximum frequency that can be accurately represented by the FFT occurs when \(n < N/2\). This phenomena is most easily seen by manipulating the variables of the cosine and sine terms of the FFT equations (Eqs. A.2.5a and A.2.5b) with Eqs. A.2.4 and A.3.1 and \(n = N/2\):

\[
\frac{2\pi n}{N} = \frac{2\pi n}{\Delta t N} \frac{\Delta t}{n} = \frac{2\pi f_s}{N} \frac{n}{t_r}
\]

(Eq. A.3.4a)

Substitution of \(n = N/2\) into Eq. A.3.4a produces

\[
\frac{2\pi f_s}{N} \frac{n}{t_r} = \frac{2\pi f_s}{N} \frac{n}{t}
\]

(Eq. A.3.4b)

Equation A.3.4b shows that the resulting frequency in the FFT is \(f_{Nyq}\) for \(n = N/2\). For \(n\) larger than \(N/2\), the frequency of the FFT is above the Nyquist frequency and the discrete signal does not accurately represent the actual signal. Since the sampling frequency must be higher than twice the Nyquist frequency (Eq. A.3.2), \(n\) must be smaller than \(N/2\). The maximum value of \(n\) is generally set to

\[
n = \frac{N}{2} - 1
\]

(Eq. A.3.5)

to avoid phase ambiguity from \(f_s = f_{Nyq}\) or \(n = N/2\) [23].

A.3.2 Avoiding Leakage in a Discrete Signal

The equations for the FFT (Eqs. A.2.5a and A.2.5b) are based on a sampled signal with period \(T\). If the discrete signal is not periodic in the period \(T\), leakage of the discrete signal occurs in the frequency spectrum. The leakage phenomena is most apparent in an analysis of the equations for the continuous Fourier Transform (Eqs. A.2.2a and A.2.2b). The upper bound of the integrals of Eqs. A.2.2a and A.2.2b represents the period of a continuous harmonic input:

\[
T = \frac{2\pi}{\omega}
\]

(Eq. A.3.6)

The Fourier Transform is based on the signal \(y(t)\) that is a harmonic signal of period \(T\). When the sampled period of \(y(t)\) is \(T + \Delta T\) instead of \(T\), the \(\Delta T\) causes additional frequencies, which are not in the original signal, to appear in the Fourier Transform. These additional harmonics produce the effect known as leakage in the frequency spectrum. To illustrate leakage, a Fourier Transform will be determined for a function

\[
y(t) = \sin(\omega t)
\]

(Eq. A.3.7)
Since $y(t)$ in Eq. A.3.7 is an odd function, the value for $A_n$ is 0 due to the orthogonality of sine and cosine. The value of $B_n$, however, is non-zero and comes from evaluation of the integral in Eq. A.2.2b with $y(t)$ from Eq. A.3.7 and the upper value of integration equal to $T + \Delta T$. Integration is simplified through the use of the orthogonality property:

$$B_n = \frac{\omega}{\pi} \int_0^{T+\Delta T} \sin(\omega t)\sin(n\omega t) dt = \frac{\omega}{\pi} \int_0^T \sin(\omega t)\sin(n\omega t) dt + \frac{\omega}{\pi} \int_0^{\Delta T} \sin(\omega t)\sin(n\omega) dt = B_n^{(1)} + B_n^{(2)}$$

$$B_n^{(1)} = \frac{\omega}{\pi} \int_0^T \sin(\omega t)\sin(n\omega t) dt = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

$$B_n^{(2)} = \frac{\omega}{\pi} \int_0^{\Delta T} \sin(\omega t)\sin(n\omega) dt = \frac{1}{\pi(n-1)} \left[ -\sin(\omega\Delta T)\cos(n\omega\Delta T) + \frac{1}{n} \sin(n\omega\Delta T)\cos(\omega\Delta T) \right]$$

$$B_n^{(2)} = f(n, \Delta T)$$

$$B_n = B_n^{(1)} + B_n^{(2)} = 1 + f(n, \Delta T) \quad \text{(Eq. A.3.8)}$$

According to theory, a plot of the harmonic-coefficient magnitude from Eq. A.2.2c versus frequency, with $y(t)$ from Eq. A.3.7, produces a single vertical line at $\omega$. The magnitude is 1, as given by $B_n^{(1)}$ in Eq. A.3.8. When a signal is not periodic in the sampled period $T$, additional frequencies appear in the frequency spectrum. These additional frequencies are governed by $B_n^{(2)}$ and occur at scalar multiples of the fundamental frequency. Figure A4 presents a plot of the frequency spectrum for a Fourier Transform of the original signal with the correct period $T$; a general representation of the leakage caused by an improper sampling period of $T + \Delta T$ is shown along side the correct frequency data.
In order to ensure proper discrete sampling to avoid leakage, the period of the FFT must contain both the highest and lowest frequencies present in the complex signal. The highest frequency is resolved by setting the sampling frequency in accordance with the Nyquist criterion. The lowest frequency is found from careful selection of the block size $N$. The fundamental frequency of the discrete signal is based on the period of the sample, which in turn is related to the block size $N$ and the time interval $\Delta t$:

$$\Delta f = \frac{1}{T} = \frac{1}{N\Delta t} \quad \text{(Eq. A.3.9)}$$

where

$$\Delta f = \text{fundamental frequency}$$

In order to accurately obtain the lowest frequency, the block size of the sample must be large enough to make the fundamental frequency equal to the lowest frequency of the actual complex signal. Accurate selection of the block size allows the discrete data to represent the period of the complex signal, and the FFT accurately transforms the data to the frequency domain without leakage.