Today’s Topics

• Any question before turning in the Flight 1 report?
  • Always wait the Airspeed to stabilize
  • Conversions: http://mdmetric.com/tech/lthmasscvt.htm
  • Fit a curve:
    » Excel = add trendline
    » Matlab = polyfit

⇒ Flight 1 Report Due

• Trim Flight Briefing - Memo Flight
• Derive equations of motion in vector form.

Trim Tabs

• The placement of fuel, passengers, baggage, …, sometimes results in the A/C being wing-heavy, nose-heavy, tail heavy…. To counteract such unbalanced conditions the pilot will find it necessary to continually exert pressure on the control stick or wheel, or on the rudder pedals. ⇒ Over a period of time this becomes annoying and fatiguing.

• There exist secondary flight control surfaces known as trim tabs on the elevator, rudders and ailerons.

Use of Trim

• This trim tab is used to create a “zero hinge moment” on the elevators, rudders or ailerons for a certain displacement.
Use of Trim (2)

• When the trim tab is not deployed a hinge moment $M_h$ acts upon the elevator.

Use of Trim (3)

• By deploying the trim tab, a counter moment is created that will create a ‘fingertip’ flying control requirement for the displaced elevator.

Use of Trim (4)

• Airplanes are trimmed for a specific airspeed. At this airspeed all moments are balanced. Even if the power setting is changed, the aircraft will return to the original velocity.

Balance Tabs and Servo Tabs

• Balance tabs:
  – Trim tabs can be used to reduce force on actuators by reducing the hinge moment.

• Servo tabs:
  – Trim tabs can be used to move primary control surfaces. Deflecting the servo tab creates a moment which then deflects the primary control surface.
The Memo – Due: 10/14

- Together, the two memos count as much as a single lab report.
- The memo should not be more than 2-3 pages long.
- Your goal for the memo is to convey as much information as possible in the space provided. Envision that your manager has five minutes to see what test you ran in the field. You wish to create a document that will thoroughly cover the field test in five minutes or less.
- Use a standard memo header including to, from, re, date, etc. (Templates in Word, Latex, etc.)

Aircraft EOM

- Static Stability: tendency of the A/C to return to its equilibrium position
- Dynamic Stability: After disturbed from its equilibrium flight condition, the following motion diminish with time: get amplitude, frequencies/period of the oscillation
- Importance:
  - Assessing the handling or flying qualities of an airplane. (depends on pilot’s experience and point of view)
  - Flight controls design (engineering point of view)
  - It is possible to design an A/C with excellent performance, but considered unsatisfactory by the pilot
- Derive the general EOM in 3D
- Reduce to EOM in flight in a vertical plane (Dr. Hull’s class)
Lecture #3

Aircraft EOM (2)

- Derive the general EOM in 3D
  - Several Coordinate Frames/ Axes Systems
    - Work in Inertial Frame (Newton’s Law) – fixed to the Earth
    - Work in a Body Frame – fixed to the airplane: Products of Inertia = zero
    - Stability axis
    - Wind axis

- Reduce to EOM in flight in a vertical plane (Dr. Hull’s class)

Body Fixed Coordinate Systems

The Local Horizon and Body Fixed Coordinate Systems

Define the local horizon coordinates as \([i_h, j_h, k_h]\) and the body fixed coordinates as \([i_b, j_b, k_b]\).

The local horizon frame remains parallel to a flat Earth and has its origin at the vehicle center of gravity.

The vehicle rotation and velocity can be expressed in terms of the body fixed coordinates as:

\[ V_b = U_i + V_j + W_k \]
\[ \omega_b = P_i + Q_j + R_k \]

Where:
\[ V_b = U_b + V_b + W_b \]
\[ \omega_b = \dot{P}_b + \dot{Q}_b + \dot{R}_b \]

The components of velocity represent the forward, side, and downward body motion and the components of the angular rate represent the roll, pitch, and yaw rates.
Axes Transformation: Euler Angles

- Theorem: Any two independent orthonormal coordinate frames can be related by a sequence of rotations (no more than 3) about a coordinate axes.
- Using rotation matrices we will be able to move from any one coordinate system to another using at most three consecutive rotations.
- The angles through which these rotations are performed are called Euler Angles and are dependent upon the type and order of rotation used.
- For example, an aircraft performing a 90° roll followed by a 90° pitch will be in a different orientation than if it had performed a 90° pitch followed by a 90° roll.

Aircraft Euler Angle Conventions

- We use the following conventions when dealing with aircraft (Aerospace Sequence)
  - Bank (or Roll) \( \phi \) describes rotations about \( i_b \)
  - Pitch \( \theta \) describes rotations about \( j_b \)
  - Yaw \( \psi \) describes rotations about \( k_b \)
- Be aware that
  \[ \omega_h = \dot{i}_b + Q \cdot j_b + R \cdot \dot{k}_b = \phi i_b + \theta j_b + \psi k_b \]

Coordinate Transformation Applications

- Velocity Vector Transformations
  - Given: velocity components in one reference frame
  - Find: velocity components in another reference frame

- Angular Rate Transformations:
  - Given: \( \omega \) in one frame
  - Find: \( \omega \) in another frame in terms of the Euler angles \( (\phi, \theta, \psi) \) and Euler rates \( (\dot{\phi}, \dot{\theta}, \dot{\psi}) \)

Sequencing Rotations to Create Transformation Matrices

- We transform a general vector, \( V \), from frame \( g \) to frame \( h \) by applying a transformation matrix \( [T_{hg}] \)
  \[ V_h = [T_{hg}] \cdot V_g \]
- \( [T_{hg}] \) may be constructed by performing a series of consecutive rotations
- We use the following terms when describing rotations
  - Type 1: rotation about an ‘x’ axis
  - Type 2: rotation about an ‘y’ axis
  - Type 3: rotation about an ‘z’ axis
Lecture #3

Type 1 Rotations

- Perform a type 1 rotation through $\beta$
- By inspection we can write
  
  $\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$

Type 2 and Type 3 Rotations

- Similarly, we may create type 2 and type 3 rotation matrices:
  
  $\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$

Notes on Rotation Matrices

- A simple method of remembering the form of the rotation matrices is to put a 1 in the diagonal element equivalent to the rotation type and then fill in the $\cos$ and $\sin$ relations on the diagonal and off diagonal (beware of signs on type 2).
- Rotation matrices are orthonormal so that $[R_i]^{-1} = [R_i]^T = [-\beta_i]$.  

3-2-1 Transformation from LH to Body Fixed

- To transform from the local horizontal to body fixed we can utilize a 3-2-1 sequence of rotations:
  - Rotate $\psi$ degrees about $k_h$ to $(i'j'k')$
  - Rotate $\theta$ degrees about $j'$ to $(i''j''k'')$
  - Rotate $\phi$ degrees about $i''$ to $(i''j''k''b)$
- Then
  
  $v_b = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} v_h$  
  
  $v_b = [T_{bh}] v_h$
Euler Angles Rates

- Compute the Euler Angles history (not available)
- Available \( \Rightarrow \) angular rates in the body axes (P, Q, R) (rate gyros, IMU, etc)
- Performing the transformations:

\[
\begin{bmatrix}
P \\
Q \\
R
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -\sin \phi \\
0 & \cos \phi & \sin \phi \cos \theta \\
0 & -\sin \phi & \cos \phi \cos \theta
\end{bmatrix}
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix}
\]

\[
\begin{bmatrix}
P \\
Q \\
R
\end{bmatrix} =
\begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix}
\]

Can have singularities

Quaternions

- It can be shown that any sequence of rotations can be represented by a single rotation about a unique vector and angle (4 pieces of information);
- Another approach to rotate coordinate systems is through quaternions. Quaternions are four dimensional objects with an algebra similar to complex numbers where:
  - Scalar part \( \Rightarrow \) real part \( \bar{q} = (q_0, q_1, q_2, q_3) \)
  - Vector part \( \Rightarrow \) imaginary part \( q + \bar{q} \)

where:

\[
q = \cos \frac{\beta}{2}, \quad |q| = \sin \frac{\beta}{2}
\]

Quaternions (2)

- For rotation:
  Assume we want to rotate the vector \( y \) around the unit vector \( u \) through an angle \( \beta \)

\[
\bar{q} = \cos \frac{\beta}{2} + u \sin \frac{\beta}{2}
\]

\[
y'_{\text{new}} = \bar{q}y \bar{q}^* 
\]

where \((\cdot)^*\) is the conjugate

Quaternions (3)

\[
y'_{\text{new}} = \bar{q}y \bar{q}^* 
\]

\[
y'_{\text{new}} = \left(\frac{\cos \frac{\beta}{2} + u \sin \frac{\beta}{2}}{2} \right) y \left(\frac{\cos \frac{\beta}{2} - u \sin \frac{\beta}{2}}{2} \right)
\]

which can be reduced to be

\[
y'_{\text{new}} = \left( y - u (u \cdot y) \right) \cos \frac{\beta}{2} + \left( u \times y \right) \sin \frac{\beta}{2} + u (u \cdot y)
\]
An Exam Type Problem

Given that an aircraft measures its velocity to be \( V_b = 197 i_b + 0 j_b + 34.7 k_b \) while experiencing straight and level flight.
- What is the aircraft attitude?
- Develop a rotation matrix between the body and local horizontal.

Assumptions:
- Straight and level \( \frac{dh}{dt} = 0 \)
- No sideslip (\( \psi = 0 \)) or roll (\( \phi = 0 \))

The attitude in this case is simply given by the pitch angle.
- \( \theta = \tan^{-1}(34.7/197) = 10^\circ \)
- Note that you must take quadrature in account

The transformation matrix in this case is simply a type 2 rotation by an amount (- \( \theta \)).

\[
V_i = [R_{\theta}] V_i = \begin{bmatrix} \cos(-\theta) & 0 & -\sin(-\theta) \\ 0 & 1 & 0 \\ \sin(-\theta) & 0 & \cos(-\theta) \end{bmatrix} V_i = [(-\theta)] V_i = 200 i_b
\]

Check results using quaternions

Using MATLAB:
>> b = 10 * pi/180;
>> u = [0 1 0];
>> y = [197 0 34.7];
>> ynew = (y-u*dot(u,y))*cos(b) + cross(u,y)*sin(b) + u*dot(u,y)
>> ynew =
   200.0327         0   -0.0359

Also check out QUATDEMO
Suppose you have a vector $\mathbf{V}_b$ in a reference frame rotating by $\omega_b$ relative to inertial space. How can we relate changes in $\mathbf{V}_b$ to changes in its transformed vector $\mathbf{V}_n$ in inertial space?

Take time derivatives of both vectors:

$$
\begin{align*}
\frac{d\mathbf{V}_n}{dt} &= \hat{x} \frac{d\mathbf{i}_b}{dt} + \hat{y} \frac{d\mathbf{j}_b}{dt} + \hat{z} \frac{d\mathbf{k}_b}{dt} \\
\frac{d\mathbf{V}_b}{dt} &= \hat{x} \frac{d\mathbf{i}_b}{dt} + \hat{y} \frac{d\mathbf{j}_b}{dt} + \hat{z} \frac{d\mathbf{k}_b}{dt} + \omega \times \mathbf{V}_b
\end{align*}
$$

The first three terms of the local derivative are

$$
\hat{x} \frac{d\mathbf{i}_b}{dt} + \hat{y} \frac{d\mathbf{j}_b}{dt} + \hat{z} \frac{d\mathbf{k}_b}{dt} = \omega \times \mathbf{V}_b
$$

The next three may be collected as follows since the rate of change of the local reference frame comes from its rotation.

$$
\frac{d}{dt} \mathbf{i}_b = \omega \times \mathbf{i}_b, \quad \frac{d}{dt} \mathbf{j}_b = \omega \times \mathbf{j}_b, \quad \frac{d}{dt} \mathbf{k}_b = \omega \times \mathbf{k}_b
$$

Collecting terms we have the transport theorem:

$$
\left( \frac{d\mathbf{V}_b}{dt} \right)_b = \left( \frac{d\mathbf{V}_n}{dt} \right)_n + \omega \times \mathbf{V}_b
$$

Which in general can be expressed as

$$
\left( \frac{d\mathbf{V}}{dt} \right)_b = \left( \frac{d\mathbf{V}}{dt} \right)_n + \omega \times \mathbf{V}
$$

We must make several assumptions before we discuss the forces that act upon the aircraft:

- Rigid airplane
- Constant mass
- Body axes origin at center of gravity
- Flat Earth with constant gravity
- Forces consist of:
  - Gravity
  - Aerodynamics
  - Thrust
Lecture #3

How to get EOM?

- Newton’s Law:
  \[ \sum F = ma_{cg} \]
  \[ \sum M = \frac{dV_{cg}}{dt} \]

Newton’s Second Law

- We can get the acceleration of the aircraft by applying the transport theorem to the time derivative of velocity:

  \[
  (a_{cg}) = \left( \frac{dV_{cg}}{dt} \right) = \left( \frac{dV_{cg}}{dt} \right)_a + \omega \times V_{cg}
  \]

  where

  \[
  (V_{cg})_b = (U_i + iV_j + kW)_b + (\omega \times V_{cg})(t_b)
  \]

  \[
  (\omega)(t_b) = (P_i + Q_j + R_k)_b
  \]

Aircraft Forces

- Evaluating the cross product and collecting terms:

  \[
  (a_{cg}) = \dot{U}\hat{i} + \dot{V}\hat{j} + \dot{W}\hat{k} + [P \atop Q \atop R] \times V
  \]

Angular Momentum

- The sum of moments about a point is the time derivative of the angular momentum about that point.

  \[
  L_{cg} = \lim_{t \to \infty} \int \sum_i r_i \times m_i \frac{dV_i}{dt}
  \]

  \[
  = \int r \times (V_{cg} + \omega \times r) \frac{dm}{dt}
  \]

  \[
  = -V_{cg} \int r \times \frac{dm}{dt} + \int r \times (\omega \times r) \frac{dm}{dt}
  \]

  \[
  \Rightarrow L_{cg} = \int r \times (\omega \times r) \frac{dm}{dt}
  \]
Angular Momentum (2)

From previous slide:

\[ \mathbf{L}_{\text{cg}} = \int_{m} \left( \mathbf{r} \times (\mathbf{\dot{r}} \times \mathbf{\hat{r}}) \right) \, dm \]

where

\[ \mathbf{\dot{r}} \times (\mathbf{\hat{r}} \times \mathbf{\hat{r}}) = (\omega_{z} - \omega_{x} y) \mathbf{i} + (\omega_{x} x - \omega_{z} z) \mathbf{j} + (\omega_{y} y - \omega_{x} x) \mathbf{k} \]

\[ \mathbf{\ddot{r}} \times (\mathbf{\hat{r}} \times \mathbf{\hat{r}}) = (\omega_{x} (r^2 + z^2) - \omega_{y} (xy) - \omega_{z} (xz)) \mathbf{i} + (-\omega_{x} (xy) + \omega_{y} (x^2 + z^2) - \omega_{z} (yz)) \mathbf{j} + (-\omega_{x} (xz) + \omega_{y} (yz) + \omega_{z} (x^2 + y^2)) \mathbf{k} \]

Angular Momentum (3)

Define the inertia of a mass:

\[ \mathbf{I} = \int_{m} \left[ \mathbf{r} \mathbf{r}' - m \mathbf{r} \mathbf{r}' \right] \, dm \]

And construct the inertia matrix \([\mathbf{I}]\):

\[ [\mathbf{I}] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \]

Angular Momentum (4)

- With a little bit of bookkeeping we can rewrite the angular momentum as:

\[ \left( \mathbf{L}_{\text{cg}} \right)_{b} = [\mathbf{I}] \mathbf{\omega}_{b} \]

And then the transport theorem allows us to move to inertial coordinates:

\[ \frac{d\mathbf{L}_{\text{cg}}}{dt} = [\mathbf{I}] \mathbf{\ddot{x}} + \mathbf{\dot{r}} \times [\mathbf{I}] \mathbf{\dot{x}} \]

Active Forces and Moments

- Gravity

\[ \mathbf{F}_{g} = mg \mathbf{k}_{z} \]

\[ \mathbf{F}_{g} = \begin{bmatrix} 0 \\ 0 \\ -mg \sin \theta \end{bmatrix} = \begin{bmatrix} mg \sin \phi \cos \theta \\ mg \cos \phi \cos \theta \end{bmatrix} \]

- Thrust

\[ \mathbf{F}_{T} = \begin{bmatrix} F_{Tx} \\ F_{Ty} \\ F_{Tz} \end{bmatrix} \]

\[ \mathbf{M}_{T} = \begin{bmatrix} L_{T} \\ M_{T} \\ N_{T} \end{bmatrix} \]

- Aero

\[ \mathbf{F}_{A} = \begin{bmatrix} F_{Ax} \\ F_{Ay} \\ F_{Az} \end{bmatrix} \]

\[ \mathbf{M}_{A} = \begin{bmatrix} L_{A} \\ M_{A} \\ N_{A} \end{bmatrix} \]
Lecture #3

Force Equations

$$\sum F = ma$$

$$m(\dot{U} - VR + WQ) = -mg \sin \theta + F_{Ax} + F_{rz}$$

$$m(\dot{V} - UR + WP) = -mg \sin \phi \cos \theta + F_{Ay} + F_{ry}$$

$$m(\dot{W} - UQ + VP) = -mg \cos \phi \cos \theta + F_{Az} + F_{rz}$$

Moment Equations

$$\frac{dL_z}{dt} = \sum M_z$$

$$I_{xx} \dot{\theta} - I_{yz} \dot{R} - I_{xz} P Q + (I_{xx} - I_{yy}) R Q = L_x + L_T$$

$$I_{yy} \dot{Q} + (I_{yy} - I_{zz}) R P + I_{zz} (P^2 - R^2) = M_x + M_T$$

$$I_{zz} \dot{R} - I_{xz} \dot{P} + (I_{yy} - I_{xx}) P Q + I_{xx} R Q = N_x + N_T$$

Angular Rate Relations

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Summary of Kinetic and Dynamic Equation

Where:

- X = F + F_a + F_r
- Y = F + F_a + F_r
- Z = F + F_a + F_r
- L = I_p + I_p^2 + I_p^2 + I_p^2
- M = I_p + I_p^2 + I_p^2 + I_p^2
- N = I_p + I_p^2 + I_p^2 + I_p^2

Body angular velocities in terms of Euler angles:

- $\dot{\phi} = \omega_x \sin \theta$
- $\dot{\theta} = \omega_x \cos \theta$
- $\dot{\psi} = \omega_x \tan \phi$

Euler rates in terms of Euler angles and body angular velocities:

- $\dot{\phi} = \omega_x \sin \theta$
- $\dot{\theta} = \omega_x \cos \theta$
- $\dot{\psi} = \omega_x \tan \phi$

Velocity of aircraft in the fixed frame in terms of Euler angles and body velocity components:
Conclusions about Forces and Moments + Demos

- Note that there are 9 nonlinear coupled 1st order differential equations with the following states:
  \[ U, V, W, P, Q, R, \phi, \theta, \psi \]
- These equations can be reduced to Dr. Hull's 367 base equations with appropriate assumptions and coordinate transformations
  - Flight in a vertical plane
  - Flat earth
  - Definitions
  - etc...

This Week / Next Week

This week:
- Flight 2 – Simulator Lab
- Memo Due: 02/24

Next week:
- Level Flight Envelope
- Newton-Raphson Nonlinear Equation Solution Method ⇒ Computer 2 Assignment
- Computer 1 Due (02/17)