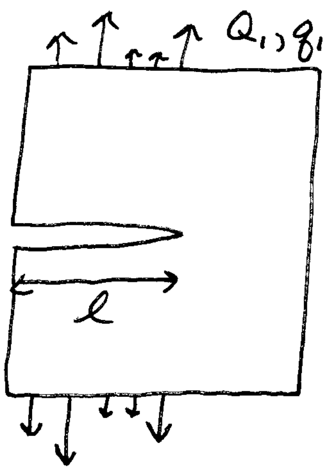
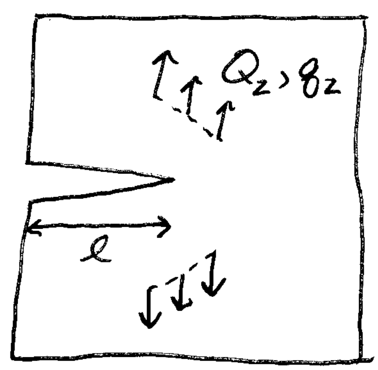
Weight Function Methods (Ref. Rice, IJSS, 1972, p.751)

Consider a linear elastic body with a crack on the x-axis and symmetric loading and elastic properties about the x-axis (it can be anisotropic but orthotropic with material directions aligned with the x and y axes). We are only considering mode I but the method can be generalized to other modes and more general anisotropy.

Now consider two different load systems





Q_1 and Q_2 are generalized forces such that the tractions and body forces are

$$t_i = Q_1 t_i^{(1)} \quad \text{and} \quad b_i = Q_1 b_i^{(1)} \quad \text{in problem 1}$$

and $$t_i = Q_2 t_i^{(2)} \quad \text{and} \quad b_i = Q_2 b_i^{(2)} \quad \text{in problem 2}$$

(You can think of the Q_i's as scaling factors) then the q_i are generalized displacements such that Q_i and q_i form a work-conjugate pair, i.e. for u_i^* = any displacement field

$$Q_1 q_1 = \int_S t_i u_i^* \, dS + \int_V b_i u_i^* \, dV$$

$$= Q_1 \int_S t_i^{(1)} u_i^* \, dS + Q_1 \int_V b_i^{(1)} u_i^* \, dV$$

$$\therefore \; q_1 = \int_S t_i^{(1)} u_i^* \, dS + \int_V b_i^{(1)} u_i^* \, dV$$

and similarly $q_2 = \int_S t_i^{(2)} u_i^* \, dS + \int_V b_i^{(2)} u_i^* \, dV$

If both load systems act simultaneously then we will write the total displacement as

$$u_i = Q_1 u_i^{(1)} + Q_2 u_i^{(2)}$$

where $u_i^{(1)}$ is the displacement field due to a unit Q_1 and $u_i^{(2)}$ is the displacement field due to a unit Q_2.

$$\therefore \; q_1 = \int_S t_i^{(1)} \left[Q_1 u_i^{(1)} + Q_2 u_i^{(2)} \right] dS$$

$$+ \int_V b_i^{(1)} \left[Q_1 u_i^{(1)} + Q_2 u_i^{(2)} \right] dV.$$

$$= \left[\int_S t_i^{(1)} u_i^{(1)} \, dS + \int_V b_i^{(1)} u_i^{(1)} \, dV \right] Q_1$$

$$+ \left[\int_S t_i^{(1)} u_i^{(2)} \, dS + \int_V b_i^{(1)} u_i^{(2)} \, dV \right] Q_2$$

Similarly: $q_2 = \left[\int_S t_i^{(2)} u_i^{(1)} dS + \int_V b_i^{(2)} u_i^{(1)} dV \right] Q_1$
$\qquad + \left[\int_S t_i^{(2)} u_i^{(2)} dS + \int_V b_i^{(2)} u_i^{(2)} dV \right] Q_2$

or $\quad q_1 = C_{11} Q_1 + C_{12} Q_2$

$\qquad q_2 = C_{21} Q_1 + C_{22} Q_2$

note that by the Rayleigh-Betti reciprocal theorem $C_{12} = C_{21}$.

Furthermore, note that the C_{ij} are structural compliances <u>and</u> depend on the crack length l, i.e. $C_{ij} = C_{ij}(l)$.

Now the stored strain energy in the body at fixed generalized displacement is

$$U(q_1, q_2, l) = \tfrac{1}{2} Q_i q_i = \tfrac{1}{2} C_{ij}^{-1}(l)\, q_i q_j$$

where $\quad \dfrac{\partial U}{\partial q_1} = C_{11}^{-1} q_1 + C_{12}^{-1} q_2 = Q_1$

$$\frac{\partial U}{\partial q_2} = C_{21}^{-1} q_1 + C_{22}^{-1} q_2 = Q_2$$

$$\frac{\partial U}{\partial l} = \frac{1}{2} \frac{\partial C_{ij}^{-1}}{\partial l} q_i q_j = -\mathcal{G} t \quad (\text{fixed } q_i)$$

thickness in out of plane direction

Now, the potential energy of the system is a function of the Q_i and ℓ and is the strain energy minus the work done by the loads.

$$PE = \psi = U - Q_i q_i = U - Q_1 q_1 - Q_2 q_2$$

$$d\psi = dU - dQ_1\, q_1 - Q_1\, dq_1 - dQ_2\, q_2 - Q_2\, dq_2$$

$$= \underbrace{\frac{\partial U}{\partial q_1}}_{Q_1} dq_1 + \underbrace{\frac{\partial U}{\partial q_2}}_{Q_2} dq_2 + \underbrace{\frac{\partial U}{\partial \ell}}_{-\mathcal{G}t} d\ell - dQ_1 q_1 - Q_1 dq_1 - dQ_2 q_2 - Q_2 dq_2$$

$$= \underbrace{-\mathcal{G}t\, d\ell}_{\frac{\partial \psi}{\partial \ell}} \underbrace{- q_1\, dQ_1}_{\frac{\partial \psi}{\partial Q_1}} \underbrace{- q_2\, dQ_2}_{\frac{\partial \psi}{\partial Q_2}}$$

$$\therefore \quad q_1 = \frac{-\partial \psi}{\partial Q_1}, \quad q_2 = \frac{-\partial \psi}{\partial Q_2}, \quad \mathcal{G} = -\frac{1}{t}\frac{\partial \psi}{\partial \ell}$$

(fixed Q_i)

Now, define k_1 to be the stress intensity factor for problem 1 when $Q_1 = 1$ and k_2 to be the SIF for problem 2 when $Q_2 = 1$. Then, due to linear superposition,

$K = k_1 Q_1 + k_2 Q_2$ for the combined problem.

also $\mathcal{G} = \dfrac{(k_1 Q_1 + k_2 Q_2)^2}{H}$ where H is an elastic modulus

$H = E'$ for isotropic materials

Our goal in this business is to determine k_2 given that we have a complete solution for problem $\underline{\underline{1}}$.

Note: $\dfrac{\partial(\mathcal{G}t)}{\partial Q_i} = -\dfrac{\partial^2 \psi}{\partial l \partial Q_i} = -\dfrac{\partial}{\partial l}\underbrace{\left(\dfrac{\partial \psi}{\partial Q_i}\right)}_{-q_i} = +\dfrac{\partial q_i}{\partial l}$

recall: $q_i = C_{ij}(l) Q_j$

$$\rightarrow \quad +\frac{\partial q_i}{\partial l} = +\frac{\partial C_{ij}}{\partial l} Q_j$$

$$\frac{\partial(\mathcal{G}t)}{\partial Q_i} = +\frac{\partial C_{ij}}{\partial l} Q_j$$

$$\frac{\partial}{\partial Q_i}\left[t \frac{(k_i Q_i)^2}{H}\right] = +\frac{\partial C_{ij}}{\partial l} Q_j$$

$$\frac{2t}{H}(k_j Q_j)(k_j \delta_{ij}) = +\frac{\partial C_{ij}}{\partial l} Q_j$$

$$\frac{2t}{H} k_i k_j Q_j = +\frac{\partial C_{ij}}{\partial l} Q_j$$

but we can take $Q_1 = 1, Q_2 = 0$ or $Q_1 = 0, Q_2 = 1$ or any other combination and this relationship holds

$$\therefore \quad \frac{\partial C_{ij}}{\partial l} = \frac{2t}{H} k_i k_j$$

Now, consider the cross term,

$$\frac{2t}{H} k_1 k_2 = \frac{\partial C_{12}}{\partial \ell} = \frac{\partial C_{21}}{\partial \ell}$$

$$\rightarrow k_2 = \frac{H}{2t} \underbrace{\frac{1}{k_1 Q_1}}_{\substack{K \text{ for} \\ \text{problem 1}}} \frac{\partial}{\partial \ell} \big(\underbrace{C_{21} Q_1}_{q_2(\ell, Q_1, Q_2 = 0)} \big)$$

$\therefore$ K due to problem 2 is

$$K^{(2)} = k_2 Q_2 = \frac{H}{2t} \frac{1}{K^{(1)}} \frac{\partial (C_{21} Q_1)}{\partial \ell} Q_2$$

$$= \frac{H}{2t} \frac{1}{K^{(1)}} Q_2 \left[\frac{\partial}{\partial \ell} \int_S t_i^{(2)} u_i^{(1)} dS + \frac{\partial}{\partial \ell} \int_V b_i^{(2)} u_i^{(1)} dV \right]$$

but $t_i^{(2)}$ does not depend on ℓ

$$\therefore \; K^{(2)} = Q_2 \frac{H}{2t} \frac{1}{K^{(1)}} \left[\int_S t_i^{(2)} \frac{\partial u_i^{(1)}}{\partial \ell} dS + \int_V b_i^{(2)} \frac{\partial u_i^{(1)}}{\partial \ell} dV \right]$$

but $K^{(2)}$ should not depend on how the loading in problem 1 is specified. Therefore, to within rigid body motions, the quantity $\frac{1}{K^{(1)}} \frac{\partial u_i^{(1)}}{\partial \ell}$ should be universal for the given geometry. Define the weight function h_i as

$$h_i = \frac{H}{2t} \frac{1}{K^{(1)}} \frac{\partial u_i^{(1)}(x, y, \ell)}{\partial \ell} = \frac{H}{2t} \frac{1}{K} \frac{\partial u_i}{\partial \ell}$$

Here the superscript is dropped to denote that h_i can be determined from any problem.

then $K^{(2)} = Q_2 \int_S t_i^{(2)} h_i \, dS + \int_V b_i^{(2)} h_i \, dV$

or $$\boxed{K = \int_S t_i h_i \, dS + \int_V b_i h_i \, dV}$$

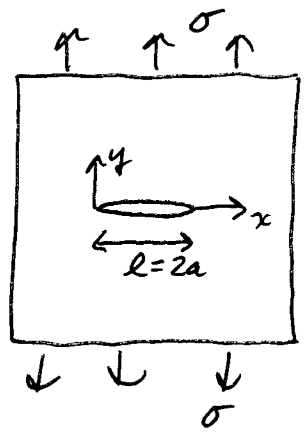
Note for 2-D problems it is common to do the surface integral over the boundary <u>line</u> and the volume integral over the <u>area</u> in which case the t is dropped and we have

$$\boxed{K = \int_\Gamma t_i h_i \, d\Gamma + \int_A b_i h_i \, dA \quad \text{where} \quad h_i = \frac{H}{2} \frac{1}{K} \frac{\partial u_i}{\partial l}}$$

Usually the most useful solution to have is that for a pair of point loads opening the crack. This solution can then be used as a Green's function to generate all other solutions using superposition.

Let's use weight functions to get such a solution for the center crack.

We know the following solution:



$$K^{(1)} = \sigma\sqrt{\pi a} = \sigma\sqrt{\frac{\pi \ell}{2}}$$

$$u_y^{(1)}(x, \pm 0; \sigma, P=0, \ell) = \pm \frac{2\sigma}{H}\sqrt{x(\ell - x)}$$

on crack surfaces.

on crack surfaces:
$$\frac{\partial u_y^{(1)}}{\partial \ell} = \pm \frac{2\sigma}{H}\frac{\partial}{\partial \ell}\sqrt{x(\ell-x)}$$

$$= \pm \frac{2\sigma}{H}\frac{1}{2}\left[x(\ell-x)\right]^{-1/2} x$$

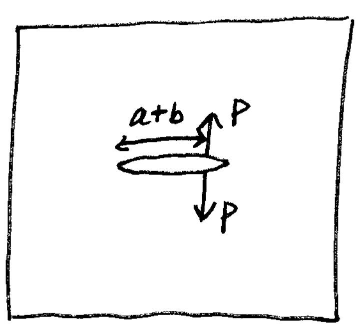
$$= \pm \frac{\sigma}{H}\sqrt{\frac{x}{\ell - x}}$$

$\therefore$ on crack surfaces:
$$h_y = \frac{H}{2}\frac{1}{K^{(1)}}\frac{\partial u_y^{(1)}}{\partial \ell}$$

$$= \pm \frac{\cancel{H}}{2}\sqrt{\frac{2}{\pi \ell}}\frac{1}{\cancel{\sigma}}\frac{\cancel{\sigma}}{\cancel{H}}\sqrt{\frac{x}{\ell - x}}$$

$$= \pm \frac{1}{\sqrt{2\pi \ell}}\sqrt{\frac{x}{\ell - x}}$$

Now consider:



$$K^{(2)} = \int_\Gamma t_i^{(2)} h_i \, d\Gamma + \int_A b_i^{(2)} h_i \, dA$$

$$b_i^{(2)} = 0 \quad , \quad t_i^{(2)} = \pm P \, \delta(x = a+b, y = \pm 0) \, \delta_{i2}$$

Dirac delta

Kronecker delta

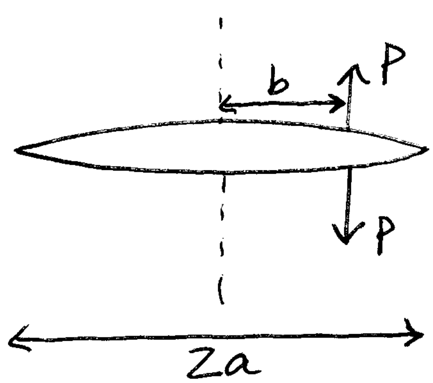
$$\therefore \quad K^{(2)} = 2P \frac{1}{\sqrt{2\pi \ell}} \sqrt{\frac{a+b}{\ell-(a+b)}}$$

$$\boxed{K^{(2)} = P \frac{1}{\sqrt{\pi a}} \sqrt{\frac{a+b}{a-b}}}$$

But for which crack tip?

Answer: For the crack tip that appears to grow with increasing ℓ, i.e. the right one.

So for:



$$K^{right} = \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a+b}{a-b}}$$

$$K^{left} = \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a-b}{a+b}}$$

Hints for Homework #2

- For 3-D crack problems crack tips are always in a state of generalized plane strain.

Generalized plane strain implies $\varepsilon_{zz} = \text{Constant} = C$

$$\therefore \quad \varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = C$$

$$\sigma_{zz} = EC + \nu(\sigma_{xx} + \sigma_{yy})$$

$$\rightarrow \quad \varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz})$$

$$= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{yy} + EC + \nu\sigma_{xx} + \nu\sigma_{yy})$$

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} - \nu C$$

Similarly $$\varepsilon_{yy} = \underbrace{\frac{1-\nu^2}{E}}_{\frac{1}{E'}}\sigma_{yy} - \underbrace{\frac{\nu(1+\nu)}{E}}_{\frac{\nu'}{E'}}\sigma_{xx} - \nu C$$

Why are 3-D cracks (and actually all cracks) in a state of generalized plane strain?

Consider the crack tip stress fields

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}}\tilde{\sigma}_{ij}(\theta)$$

$$\rightarrow \varepsilon_{ij} = \frac{K}{\sqrt{2\pi r}} S_{ijkl}\,\tilde{\sigma}_{kl}(\theta)$$

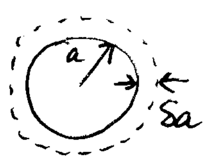
So $\quad \varepsilon_{zz} = \frac{K}{\sqrt{2\pi r}} s_{zzkl} \tilde{\sigma}_{kl}(\theta)$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \rightarrow u_z = \frac{K z}{\sqrt{2\pi r}} s_{zzkl} \tilde{\sigma}_{kl}(\theta) + \text{constant}$$

But for a finite z $|u_z| \to \infty$ as $r \to 0$. This must be ruled out.
For isotropic materials this is ruled out if $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) + Ec$, and this leads to the use of plane strain moduli near the crack tip. The exception to this is at a free surface that intersects the crack front.

- When using energy methods to compute the energy release rate we are computing the <u>average</u> energy release rate.

Ex.



$$\mathcal{G}_{Avg} \delta A = -\delta PE$$
$$A = \pi a^2 \rightarrow \delta A = 2\pi a \, \delta a$$

The $\mathcal{G}$-K relationship is for <u>points</u> along the crack front.

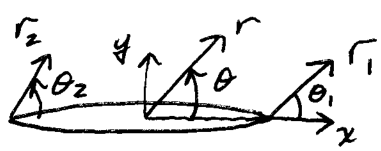
$$\rightarrow \quad \mathcal{G}_{Avg} = \frac{1}{L} \int_{\Gamma} \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \, d\Gamma$$

$L = \int_{\Gamma} d\Gamma$ and Γ is the contour of the crack front.

– Branch cuts and complex potentials

To be honest, I have been slopply when analyzing the real and imaginary parts of the Westergaard stress functions. I have gotten away with it because I have always analyzed components on the positive x-axis. To analyze any point for two special case we define the following polar coordinate systems.

Center Crack



$$-\pi \le \theta \le \pi$$
$$-\pi \le \theta_1 \le \pi$$
$$-\pi \le \theta_2 \le \pi$$

Semi-Infinite Crack

$$-\pi \le \theta \le \pi$$
$$z \to r e^{i\theta}$$
$$\sqrt{z} \to \sqrt{r}\, e^{i\frac{\theta}{2}}$$

$$z \to r e^{i\theta}$$
$$z-a \to r_1 e^{i\theta_1} \to \sqrt{z-a} = \sqrt{r_1}\, e^{i\frac{\theta_1}{2}}$$
$$z+a \to r_2 e^{i\theta_2} \to \sqrt{z+a} = \sqrt{r_2}\, e^{i\frac{\theta_2}{2}}$$
$$\therefore \sqrt{z^2-a^2} = \sqrt{(z-a)(z+a)} = \sqrt{r_1 r_2}\, e^{i\frac{\theta_1+\theta_2}{2}}$$

To be more specific, if you try to use $z=x$ for $x<-a, y=0$ to evaluate $Z(z)$ you will end up with a sign error.

This has not been an exhaustive study of linear elastic fracture mechanics (LEFM), but it concludes our coverage of the topic.

Under what conditions can LEFM be applied? As we alluded to earlier, when a K-annulus exists LEFM is applicable. This implies that the non-linear material process zone near the crack tip is small compared to all other relevant length scales in the problem/structure. This situation is referred to as small scale yielding. (SSY)

What are the conditions required for fracture?

<u>Mode I monotonic loading</u> (slow loading)

Under Mode I conditions there is an equivalence between K_I and $\mathcal{G}$.

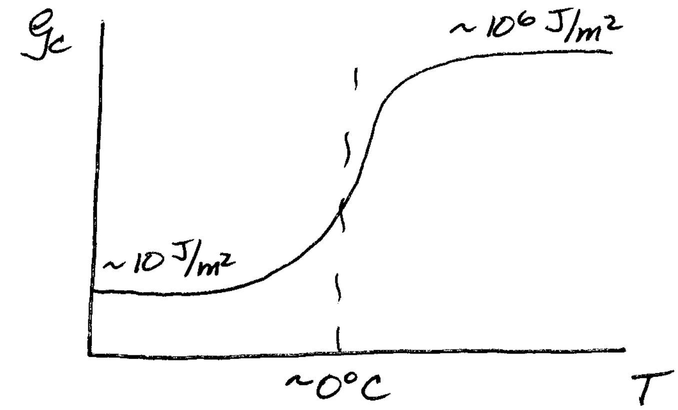
$$\mathcal{G} = \frac{K_I^2}{E'}$$

Under SSY situations fracture occurs when

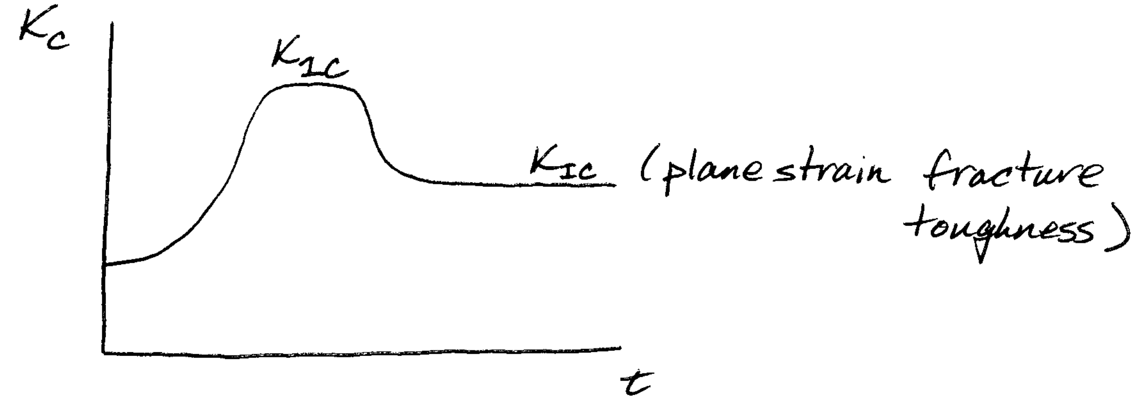
$$\mathcal{G} = \mathcal{G}_c \quad \text{or equivalently} \quad K_I = K_c$$

For many materials, especially metals, K_c is temperature dependent.

Steel:



For metals there is also a thickness effect.



For small thickness structures there is a "loss of constraint" on the plastic zone near the crack tip.

ASTM condition for a valid K_{Ic} measurement

$$t \geq 2.5 \left(\frac{K_{Ic}}{\sigma_y}\right)^2 \quad \text{ensures plane strain}$$

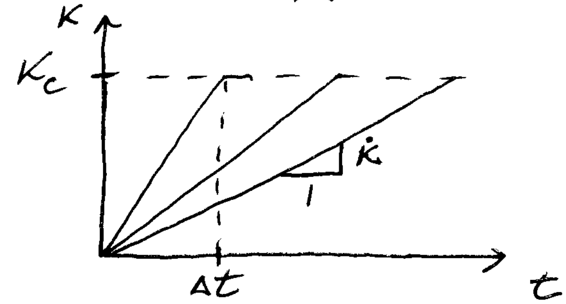
$$\text{Min}(a, c, \ldots) \geq 2.5 \left(\frac{K_{Ic}}{\sigma_y}\right)^2 \quad \text{ensures SSY}$$

Al: $K_{Ic} \sim 40\ MPa\sqrt{m}$, $\sigma_y \sim 300\ MPa$

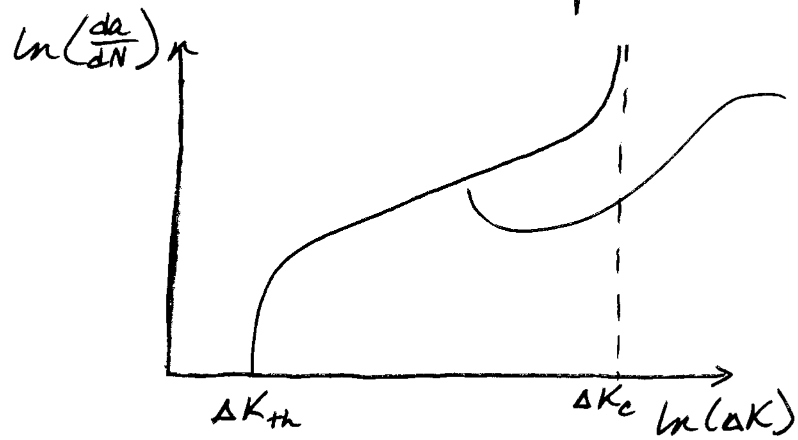
$\rightarrow t \geq 44\ mm$

Slow loading → time to load to failure is much larger than the time for a stress wave to propagate through the material specimen.

i.e. $\Delta t = \dfrac{K_c}{\dot{K}} \gg \dfrac{\text{Max}(a, b, t, \ldots)}{c}$, $c = \sqrt{\dfrac{E}{\rho}}$



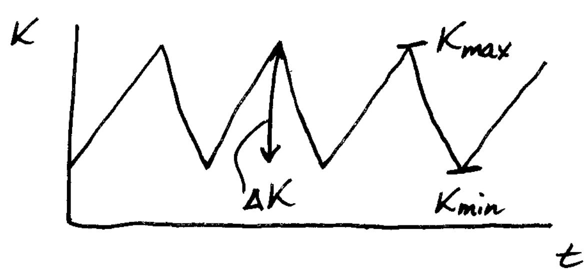
Mode I Cyclic Loading (Fatigue)



Paris law → $\dfrac{da}{dN} = \beta(\Delta K)^n$

$n \sim 2\text{–}4$

$\dfrac{da}{dN} = f(\Delta K, R)$

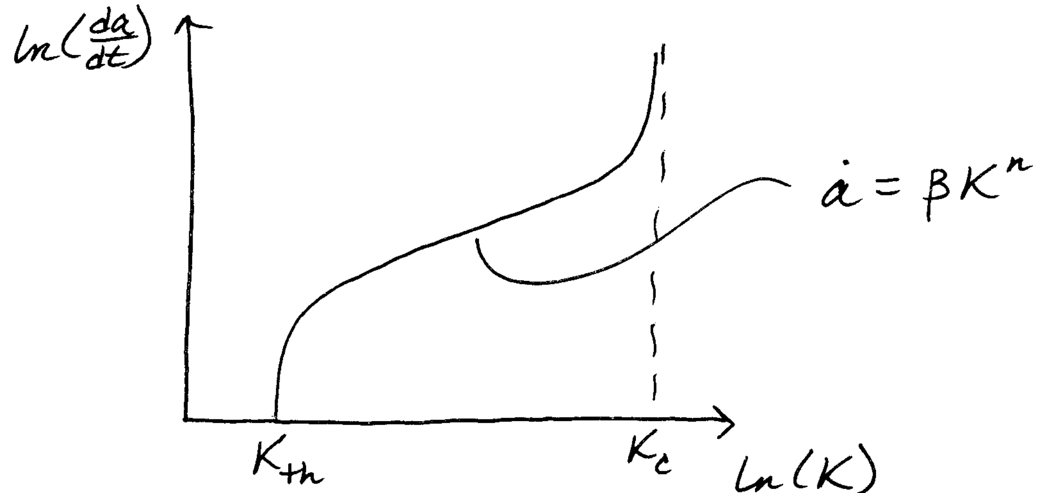


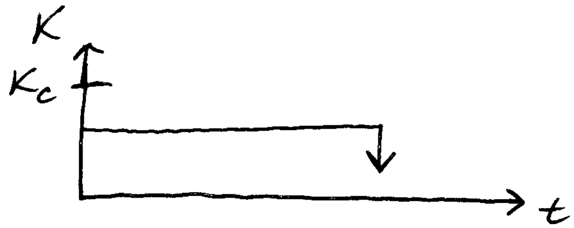
$R = \dfrac{K_{min}}{K_{max}}$

N = number of cycles

$\Delta K_{th} = \Delta K$ threshold, i.e. no crack growth below ΔK_{th}

Mode I Static Loading (static fatigue)



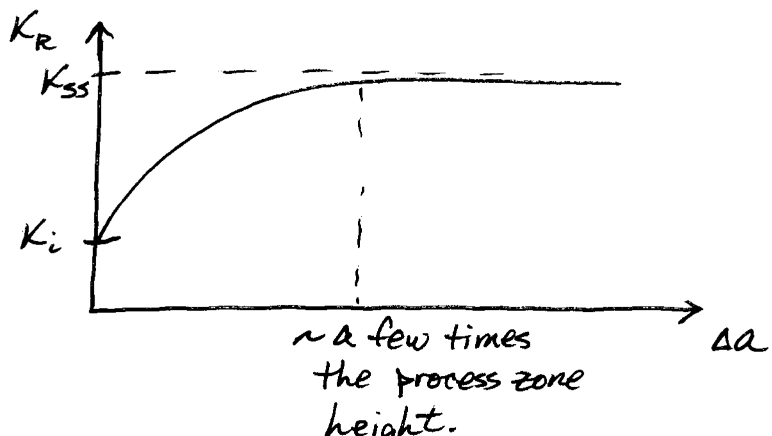


← Loading below K_c

Also called "subcritical crack growth"

This phenomenon is especially prominent in glass.

R-Curve Behavior



A process zone evolves as crack growth proceeds. More energy, i.e. K, must be fed into the crack to supply the dissipation in the process zone.

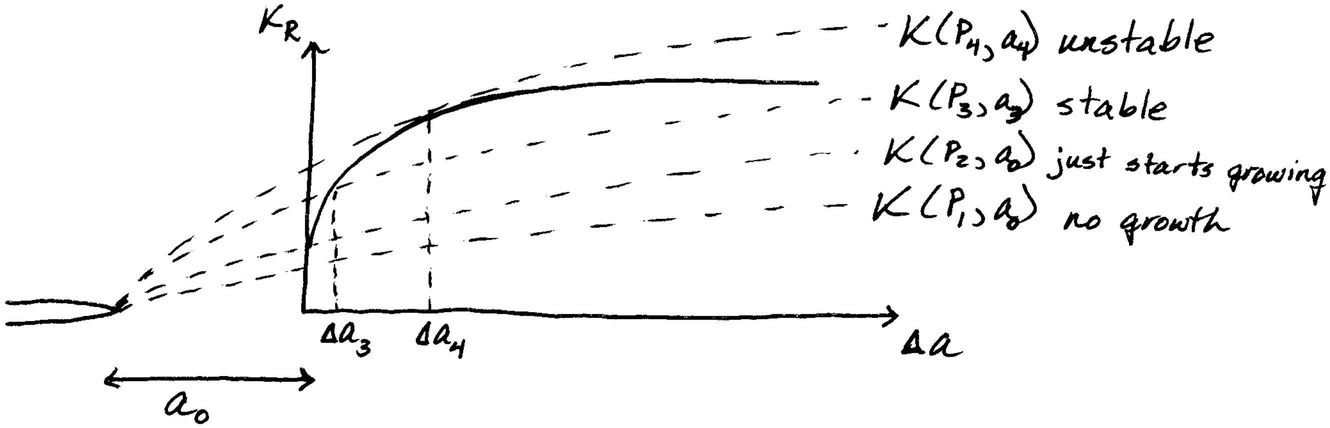
$\Delta a = a - a_0$

a = current crack length

a_0 = initial crack length

Crack growth occurs in an "R-curve" material when $K_{APP} = K_R$. Growth is unstable when

$$\left.\frac{dK_{APP}}{da}\right|_a > \left.\frac{dK_R}{d(\Delta a)}\right|_{\Delta a = a - a_0}$$

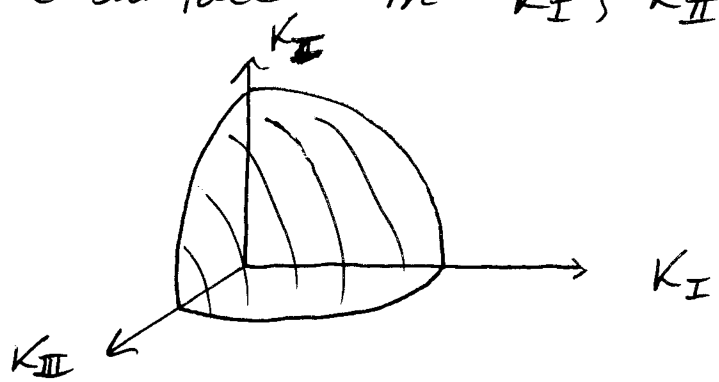


In these materials $\frac{\partial \mathcal{G}}{\partial \ell} > 0$ is not sufficient for unstable crack growth.

Mixed Mode Loading

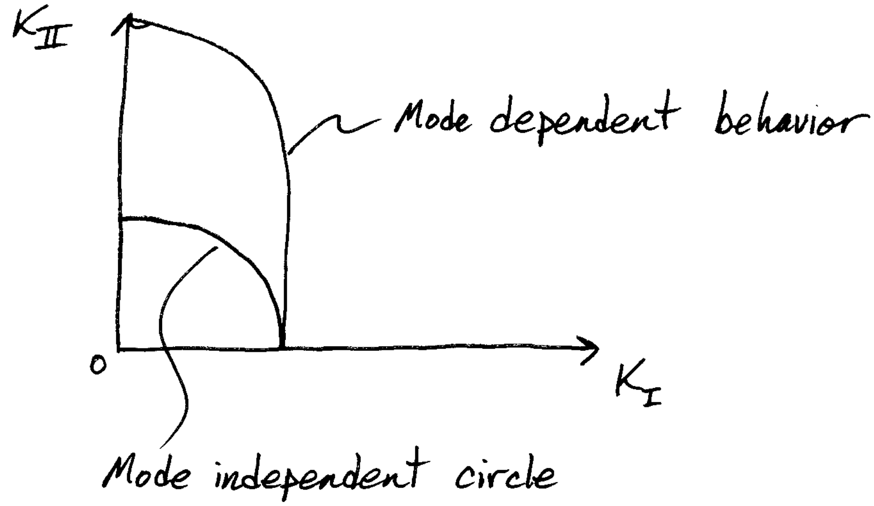
Recall: $\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$

If the criterion $\mathcal{G} = \mathcal{G}_c$ is valid for fracture then "$\mathcal{G} = \mathcal{G}_c$" represents an ellipsoidal "fracture surface" in K_I, K_{II}, K_{III} space.



Some materials do obey this type of criterion, however most exhibit "mode dependent" critical fracture energy release rates.

i.e.



Mode dependent critical fracture energy is especially important in the failure of bimaterial interfaces.

Furthermore, cracks loaded predominantly in Mode II tend to kink out of the crack plane.

For metals mode dependence is due to differences in the development of the yield/process zone in different modes. For brittle interfaces frictional effects have been used to explain this behavior.