Fracture Mechanics

Fracture mechanics concerns the study of failure of materials and structures, and the conditions (mechanical, thermal, electrical, chemical, etc.) under which failure occurs.

In this course we will focus primarily on the mechanical conditions leading to failure.

The determination of when a body will fail by the growth of a dominant crack requires two things.

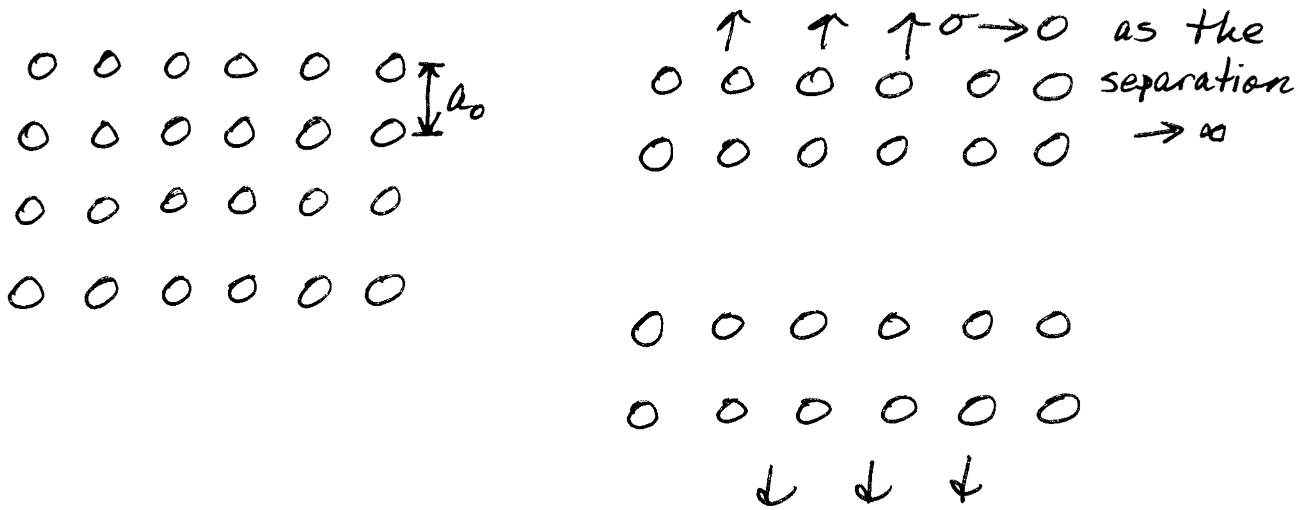
1) The determination of the stress and strain fields in the body. Or, at the very least, the determination of some mechanical quantity (usually the energy) that characterizes the intensity of loading.

2) A criterion for crack advance.

The first ~~usually~~ consideration usually requires the solution of a boundary value problem. The second requires the measurement of or a model for some basic material property.

Theoretical Tensile Strength

Consider the separation of an infinite atomic lattice into 2 semi-infinite lattices.



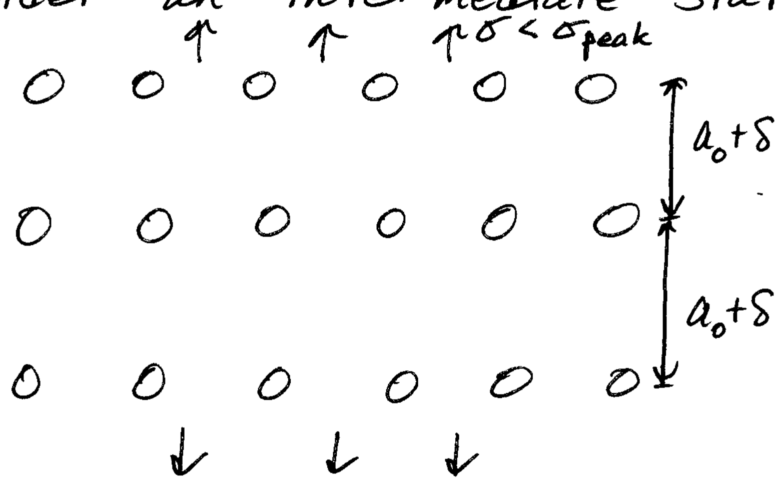
a_0 is the equilibrium lattice spacing

What is the difference (in energy) between these two states?

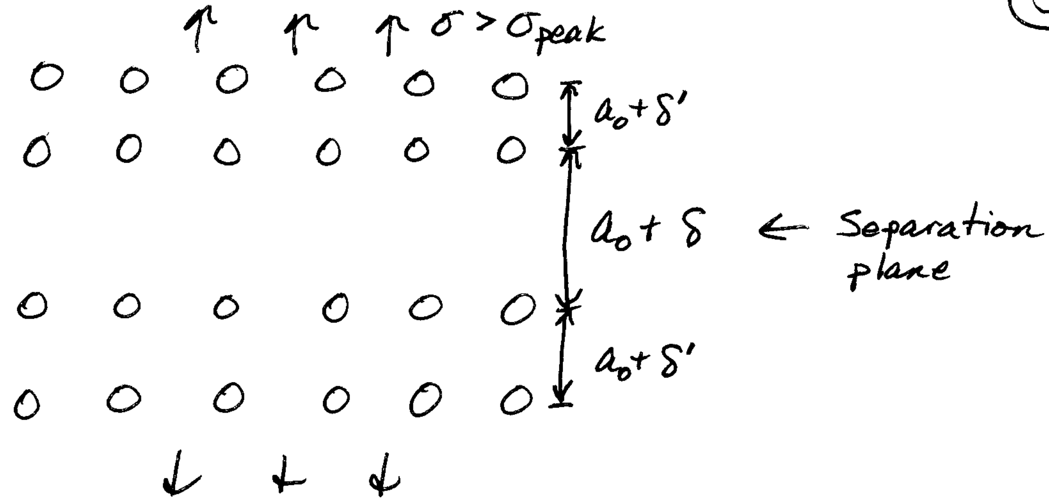
Answer: 2γ = 2 times the surface energy per unit area of surface

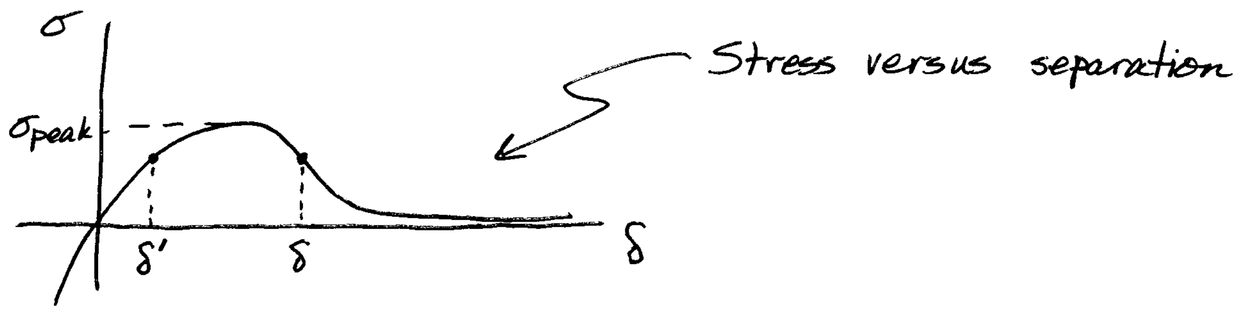
Now consider an intermediate state.

Prior to localization

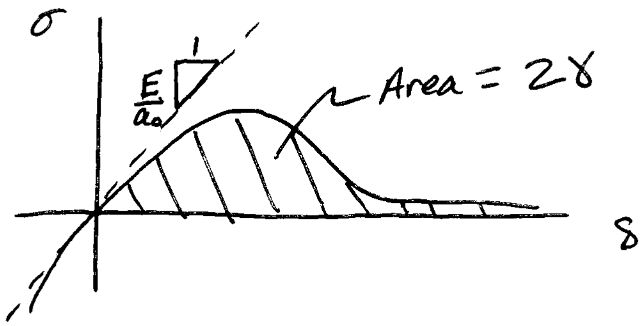


After localization





Let's estimate σ_{peak} using a "back of the envelope" type of approximation and our knowledge of some material properties.



Assume the following functional form:

$$\sigma = A \delta e^{-\delta/\beta}$$

$$\left.\frac{d\sigma}{d\delta}\right|_{\delta=0} = E/a_0$$

$$\left. Ae^{-\delta/\beta} - A\frac{\delta}{\beta}e^{-\delta/\beta} \right|_{\delta=0} = E/a_0$$

$$\boxed{A = E/a_0}$$

$$\int_0^\infty \sigma \, d\delta = 2\gamma$$

$$\int_0^\infty \frac{E}{a_0}\delta e^{-\delta/\beta} \, d\delta = 2\gamma$$

$$\frac{E}{a_0}\left(-\beta\delta e^{-\delta/\beta} - \beta^2 e^{-\delta/\beta}\right)\Big]_0^\infty = 2\gamma$$

$$\frac{E}{a_0}\left(0 - (-\beta^2)\right) = 2\gamma$$

$$\therefore \boxed{\beta = \sqrt{\frac{2\gamma a_0}{E}}}$$

Determine $\sigma_{peak} \rightarrow \quad \frac{d\sigma}{d\delta} = Ae^{-\delta/\beta} - A\frac{\delta}{\beta}e^{-\delta/\beta} = 0$

$$\therefore 1 - \frac{\delta}{\beta} = 0$$

$$\delta = \beta = \sqrt{\frac{2\gamma a_0}{E}}$$

$$\therefore \sigma_{peak} = \frac{E}{a_0}\sqrt{\frac{2\gamma a_0}{E}}\, e^{-1} = \frac{\sqrt{2}}{e}\sqrt{\frac{\gamma E}{a_0}} \approx \frac{1}{2}\sqrt{\frac{\gamma E}{a_0}}$$

$$\boxed{\sigma_{peak} \approx \frac{1}{2}\sqrt{\frac{\gamma E}{a_0}}}$$

Some characteristic values:

$$E \sim 100\ \text{GPa} \rightarrow 100 \times 10^9\ N/m^2$$
$$\gamma \sim 1\ J/m^2$$
$$a_0 \sim 1\ \text{Å} \rightarrow 1 \times 10^{-10}\ m$$

$$\therefore\ \sigma_{peak} \sim \frac{1}{2}\sqrt{\frac{10^{11}}{10^{-10}}} = \frac{\sqrt{10}}{2} \cdot 10^{10} \approx 15 \times 10^9\ N/m^2$$

$$\sigma_{peak} \sim 15\ \text{GPa} \sim E/10$$

This is a very high strength, but for nearly perfect (defect free) crystals, it is the correct order of magnitude.

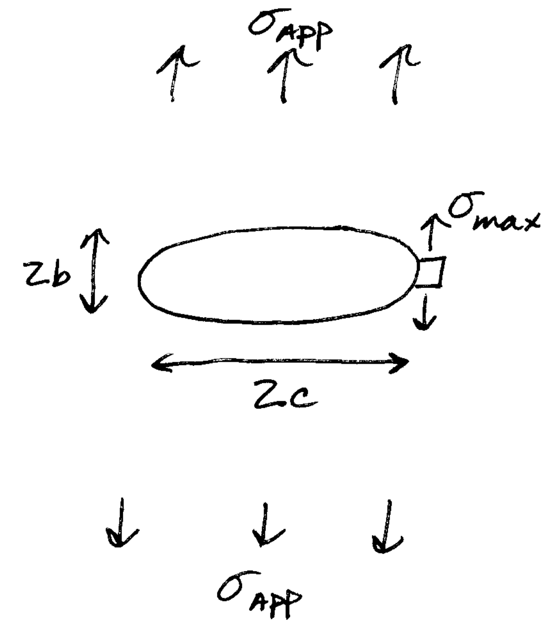
Why are materials so weak?

Answer: <u>Flaws</u> (stress concentrators)

<u>A Strength of Materials Approach accounting for Flaws</u>

Recall from elasticity theory that the stress concentration near an elliptical flaw is given as:

$$\frac{\sigma_{max}}{\sigma_{APP}} = 1 + 2\frac{c}{b}$$



$$\frac{\sigma_{max}}{\sigma_{APP}} = 1 + 2\frac{c}{b}$$

The radius of curvature of the ellipse is given as $\rho = \frac{b^2}{c}$.

Then we have $\sigma_{max} = \sigma_{APP}\left(1 + 2\sqrt{\frac{c}{\rho}}\right)$

and if $c >> b$ then $\sigma_{max} \approx \sigma_{APP}\, 2\sqrt{\frac{c}{\rho}}$

Now set $\sigma_{max} = \sigma_{peak}$ to determine $\sigma_{failure}$

$$\sigma_{failure}\, 2\sqrt{\frac{c}{\rho}} = \frac{1}{2}\sqrt{\frac{\gamma E}{a_o}}$$

$$\sigma_{failure} = \frac{1}{4}\left(\frac{\gamma E \rho}{c\, a_o}\right)^{1/2}$$

Even if the crack is atomistically sharp we must approximate ρ as $\rho \sim a_o$. Then if we take c to be some microstructural length like grain size, $c \sim 1\mu m$, we can approximate $\sigma_{failure}$ as

$$\sigma_{failure} \sim \frac{1}{4}\left(\frac{10^{11}}{10^{6}}\right)^{1/2} = \frac{\sqrt{10}}{4}10^{8} \sim 75\ MPa \sim \frac{E}{1000}$$

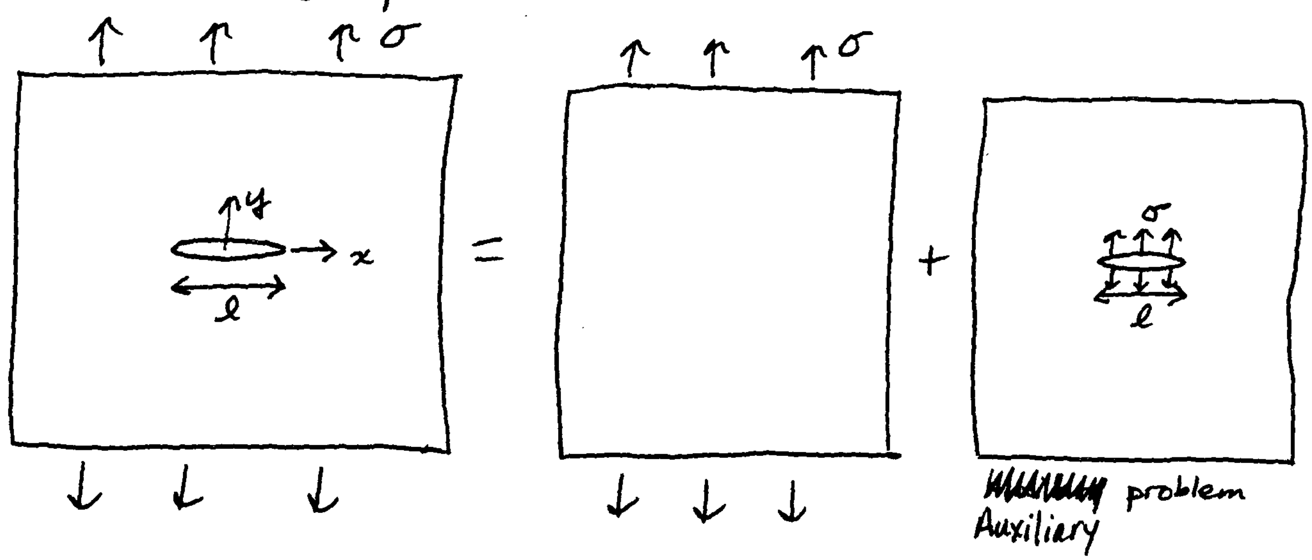
This is closer to the range for materials we are familiar with. Note that we are always assuming that the material does not deform plastically in these models. Hence the models are most applicable to glasses and ceramics at relatively low temperature.

Finally, note the $\underline{\underline{1/\sqrt{c}}}$ dependence of the strength

For brittle materials failure strength is <u>not</u> a material property.

<u>An energetic approach - Griffith Theory</u>

Linear elastic, isotropic, homogeneous, perfectly brittle small cracks, plane strain



We want to find the potential energy of the system wrt crack length.

Recall from elasticity theory (the solution of a boundary value problem) the crack opening displacement is given as

$$\delta = \frac{4\sigma(1-\nu^2)}{E}\sqrt{\frac{\ell^2}{4}-x^2}$$

The work done by σ in the auxiliary problem is given as

$$W_\sigma = \int_{-\ell/2}^{\ell/2} \sigma\,\delta\,dx$$

$$= \frac{4\sigma^2(1-\nu^2)}{E}\int_{-\ell/2}^{\ell/2}\sqrt{\frac{\ell^2}{4}-x^2}\,dx$$

$$\sin^2 u = \frac{4}{\ell^2}x^2$$

$$2\sin u\cos u\,du = \frac{8}{\ell^2}x\,dx$$

$$dx = \frac{2\ell^2\sin u\cos u}{8x}du = \frac{2\ell^2\sin u\cos u}{8\frac{\ell}{2}\sin u}du = \frac{\ell}{2}\cos u\,du$$

$$= \frac{4\sigma^2(1-\nu^2)}{E}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\frac{\ell^2}{4}-\frac{\ell^2}{4}\sin^2 u}\;\frac{\ell}{2}\cos u\,du$$

$$= \frac{\sigma^2(1-\nu^2)\ell^2}{E}\underbrace{\int_{-\frac{\pi}{2}}^{\pi/2}\cos^2 u\,du}_{\frac{\pi}{2}}$$

$$\therefore\; W_\sigma = \frac{\pi}{2}\frac{(1-\nu^2)}{E}\sigma^2\ell^2$$

Note that in linear elasticity problems the strain energy stored in the body is always 1/2 the work done by all applied loads, ∴

$$W_{SE} = \frac{1}{2} W_\sigma = \frac{\pi}{4} \frac{1-\nu^2}{E} \sigma^2 \ell^2$$

Then the total potential energy of the system is given as

$$W = W_{SE} - W_\sigma + W_0 = W_0 - \frac{\pi}{4} \frac{1-\nu^2}{E} \sigma^2 \ell^2$$

reference energy of uniform stress problem, i.e. energy when $\ell = 0$.

When the crack grows, 2 new surfaces are created. The surface energy of the crack is $2\gamma\ell$ per unit thickness.

Now the total energy of the system is

$$\phi = W + 2\gamma\ell$$

$$= W_0 - \frac{\pi}{4} \frac{1-\nu^2}{E} \sigma^2 \ell^2 + 2\gamma\ell$$

The system will be in thermodynamic equilibrium when $\dfrac{\partial\phi}{\partial\ell} = 0$

$$\frac{\partial \phi}{\partial l} = -\frac{\pi}{2}\frac{1-\nu^2}{E}\sigma^2 l + 2\gamma = 0$$

So the critical stress when the crack should propagate is

$$\sigma_c = \sqrt{\frac{4}{\pi}\frac{E}{1-\nu^2}\frac{\gamma}{l}}$$

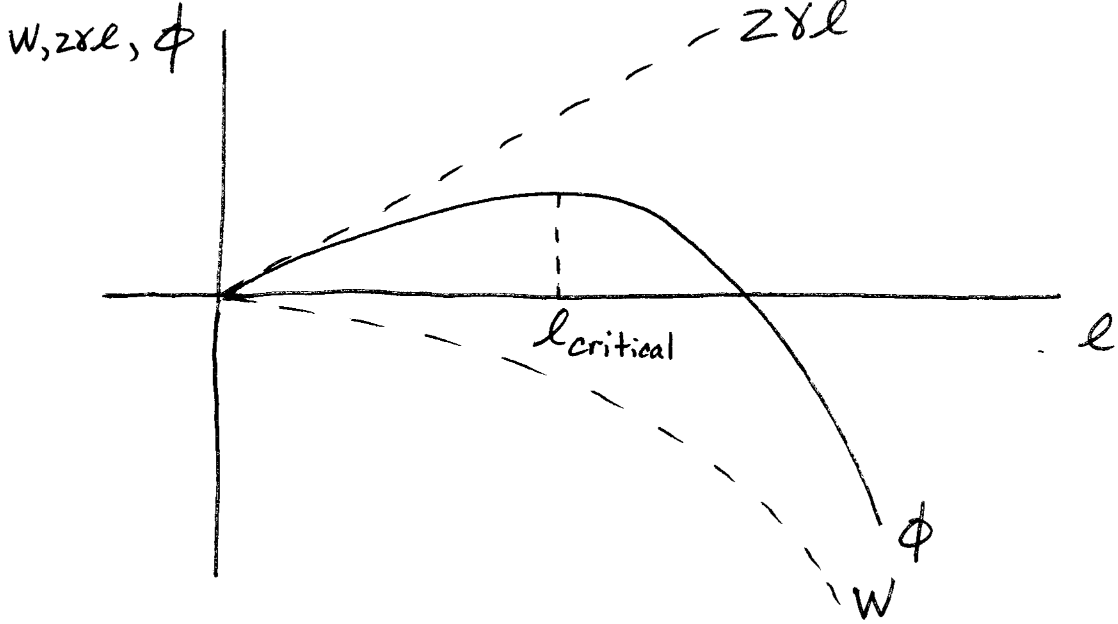
Definition of energy release rate $\mathcal{G}$

$$\mathcal{G} = -\frac{\partial W}{\partial l},$$ think of as the driving force for crack propagation.

Crack propagation occurs when $\mathcal{G} = 2\gamma$ for a "perfectly brittle" material.

More generally propagation occurs when $\mathcal{G} = \mathcal{G}_c$.

Is growth stable?



Note that if $l > l_{critical}$ then ϕ decreases. Hence l will continue to increase in order to drive ϕ down.

if $\dfrac{\partial^2 \phi}{\partial l^2} < 0$ (i.e. $\dfrac{\partial \mathcal{G}}{\partial l} > 0$) [redacted] then crack growth is unstable

if $\dfrac{\partial^2 \phi}{\partial l^2} > 0$ (i.e. $\dfrac{\partial \mathcal{G}}{\partial l} < 0$) crack growth is stable

For this problem $\dfrac{\partial^2 \phi}{\partial l^2} = -\dfrac{\pi}{2}\dfrac{1-\nu^2}{E}\sigma^2 < 0 \rightarrow$ unstable

Note again the $1/\sqrt{l}$ dependence of strength on crack length.

* A note of caution: In many case we call the crack length $2a$.

Be careful not to confuse a with <u>total</u> crack length.

$$\text{i.e. } \mathcal{G} = -\frac{\partial W}{\partial l} = -\frac{\partial W}{\partial (2a)}$$

What if the crack length is shorter than $l_{critical}$?

Well, the theory indicates that the crack length will decrease, i.e. the crack will heal.

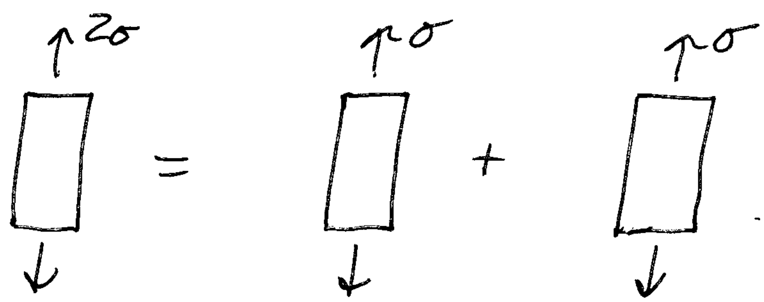
In air this does not actually happen because once a crack forms, a barrier to healing is created. For most materials this is an oxide or other passivation layer.

Crack healing can and does occur in a vacuum. In space, designers have to deal with the problem of cold welding.

Let's return to our calculation of the strain energy in our problem.

In general you <u>cannot</u> add energies from two superposed problems.

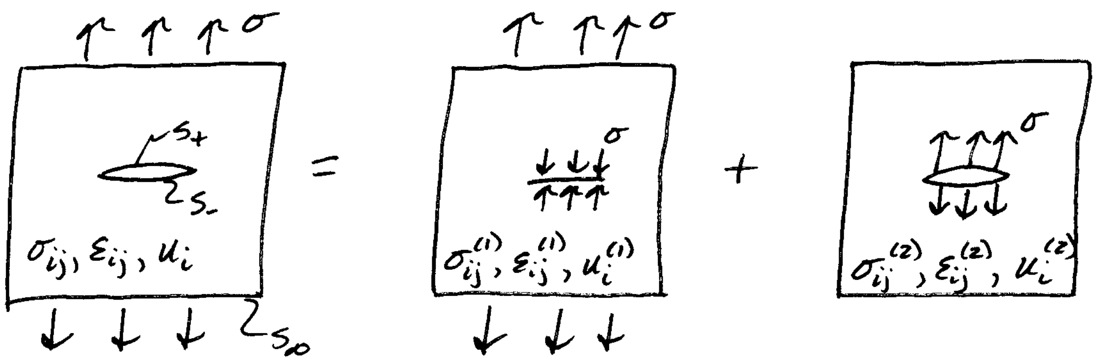
Ex)



$$W_{se} = \frac{1}{2E}(2\sigma)^2 \qquad \frac{1}{2E}\sigma^2 \qquad \frac{1}{2E}\sigma^2$$

$$\frac{1}{2E}4\sigma^2 \neq \frac{1}{2E}\sigma^2 + \frac{1}{2E}\sigma^2$$

Why did it work for our problem?



$$W_{SE} = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\,dV$$

$$= \int_S \frac{1}{2} T_i u_i \, dS \quad \text{by principle of virtual work}$$

$$= \int_{S_+} \frac{1}{2} T_i u_i \, dS + \int_{S_-} \frac{1}{2} T_i u_i \, dS + \int_{S_\infty} \frac{1}{2} T_i u_i \, dS$$

$$= \int_{S_+} \frac{1}{2}\left(T_i^{(1)} + T_i^{(2)}\right)\left(u_i^{(1)} + u_i^{(2)}\right) dS$$

$$+ \int_{S_-} \frac{1}{2}\left(T_i^{(1)} + T_i^{(2)}\right)\left(u_i^{(1)} + u_i^{(2)}\right) dS$$

$$+ \int_{S_\infty} \frac{1}{2}\left(T_i^{(1)} + T_i^{(2)}\right)\left(u_i^{(1)} + u_i^{(2)}\right) dS$$

but note that $u_i^{(1)}$ on S_+ $= u_i^{(1)}$ on S_-

and $T_i^{(1)}$ on S_+ $= -T_i^{(1)}$ on S_-

also $T_i^{(2)}$ on S_+ $= -T_i^{(2)}$ on S_-

$\therefore$ $T_i^{(2)}$ on $S_\infty = 0$.

$$\rightarrow W_{SE} = \int_{S_\infty} \frac{1}{2} T_i^{(1)} u_i^{(1)} \, dS + \int_{S_+ + S_-} \frac{1}{2} T_i^{(2)} u_i^{(2)} \, dS + \int_S \frac{1}{2} T_i^{(1)} u_i^{(2)} \, dS$$

$$W_{SE} = W_{SE}^{(1)} + W_{SE}^{(2)} + \int_S \tfrac{1}{2} T_i^{(1)} u^{(2)} \, dS$$

but $\displaystyle\int_S \tfrac{1}{2} T_i^{(1)} u^{(2)} \, dS = \int_S \tfrac{1}{2} T_i^{(2)} u_i^{(1)} \, dS$

by the reciprocal theorem

with $T_i^{(2)} = 0$ on S_∞

$$\rightarrow \int_S \tfrac{1}{2} T_i^{(2)} u_i^{(1)} \, dS = \int_{S_+ + S_-} \tfrac{1}{2} T_i^{(2)} u_i^{(1)} \, dS = 0$$

b/c $u_i^{(1)}$ on $S_+ = u_i^{(1)}$ on S_-

and $T_i^{(2)}$ on $S_+ = -T_i^{(2)}$ on S_-

$$\therefore \int_S \tfrac{1}{2} T_i^{(1)} u^{(2)} \, dS = 0$$

and $W_{SE} = W_{SE}^{(1)} + W_{SE}^{(2)}$ for our problem.

Again, this is not true in general.

A more general look at the strain energy release rate $\mathcal{G}$.

Recall: $\mathcal{G} = \dfrac{-\partial W}{\partial \ell}$

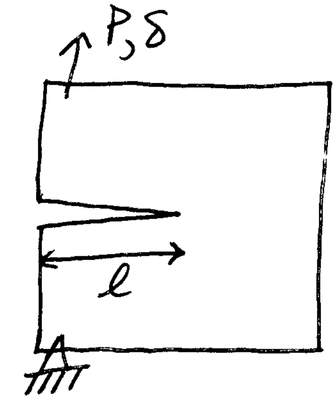
where W = stored strain energy $\overset{\text{minus}}{-}$ work done by loads
(per unit thickness)

ℓ = <u>total</u> crack length

W = total potential energy per unit thickness

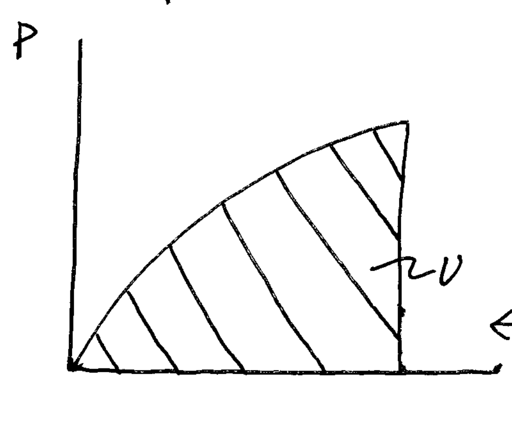
Consider a <u>non-linear</u> elastic body with a crack.

Applied point loading.



P is a force per unit thickness

The load (P) - deflection (δ) behavior of the body is given as:



The stored strain energy is

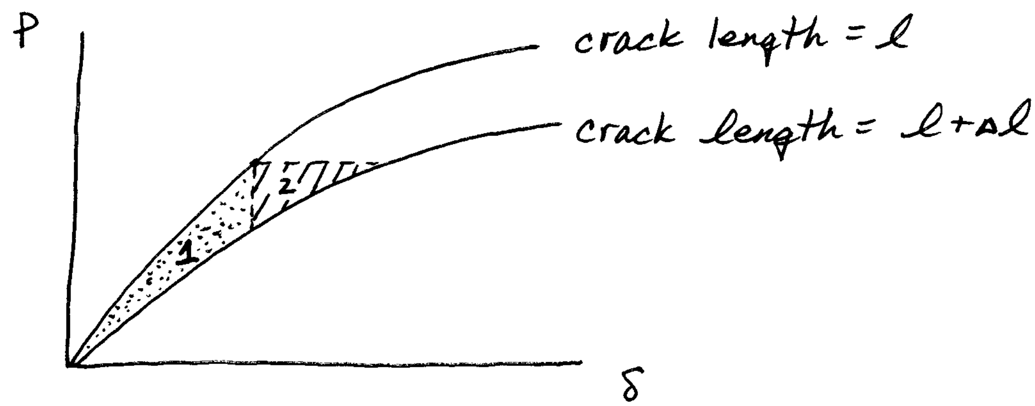
$$U = \int_0^\delta P \, d\delta$$

Structural behavior not material behavior

Note that $U = \int_0^\delta P\,d\delta \rightarrow P = \frac{\partial U}{\partial \delta}$.

Furthermore, U is a function of both δ and l, i.e. $U = U(\delta, l)$.

Now consider a specimen with a crack length of $l + \Delta l$. The structural behavior will now be softer.



Let's compute $\mathcal{G}$ for 2 loading situations: load control where P is fixed ~~and~~ displacement control where δ is fixed.

Schematically, ΔU = area 1 under displacement control and area 1 + area 2 under load control. We will now show that area 2 is of order Δl^2 and in the limit as $\Delta l \rightarrow 0$ $\mathcal{G}$ is identical for either loading condition.

A) Displacement control

$$W = U$$

$$W(l+\Delta l) = U(\delta, l+\Delta l)$$

$$= U(\delta, l) + \frac{\partial U}{\partial l}\Delta l + O(\Delta l^2)$$

$$\Delta W = W(l+\Delta l) - W(l) = \frac{\partial U}{\partial l}\Delta l + O(\Delta l^2)$$

$$\therefore \quad \mathcal{G} = \lim_{\Delta l \to 0} -\frac{\Delta W}{\Delta l} = -\frac{\partial U}{\partial l}\bigg|_{\delta \text{ fixed}}$$

B) Load control

$$W = U - P\delta$$

$$W(l+\Delta l) = U(\delta+\Delta\delta, l+\Delta l) - P[\delta+\Delta\delta]$$

$$= U(\delta, l) + \frac{\partial U}{\partial \delta}\Delta\delta + \frac{\partial U}{\partial l}\Delta l - P\delta - P\Delta\delta + O(\underbrace{\Delta\delta^2, \Delta l^2, \Delta\delta\Delta l}_{\Delta^2})$$

$$= U(\delta, l) + P\Delta\delta + \frac{\partial U}{\partial l}\Delta l + O(\Delta^2) - P\delta - P\Delta\delta$$

$$= \underbrace{U(\delta, l) - P\delta}_{W(l)} + \frac{\partial U}{\partial l}\Delta l + O(\Delta^2)$$

$$\Delta W = W(l+\Delta l) - W(l) = \frac{\partial U}{\partial l}\Delta l + O(\Delta^2)$$

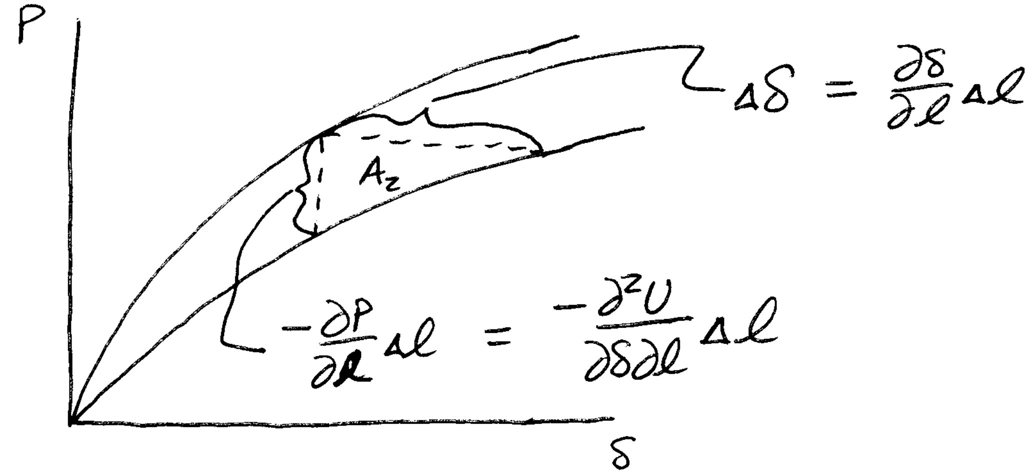
Note: this is $\frac{\partial U}{\partial l}\Big|_{\delta \text{ fixed}}$

Also $\Delta\delta \approx \frac{\partial \delta}{\partial l}\Delta l \to \Delta^2 \to \Delta l^2$

$$\therefore \mathcal{G} = \lim_{\Delta l \to 0} -\frac{\Delta W}{\Delta l} = -\left.\frac{\partial U}{\partial l}\right|_{\delta \text{ fixed}}$$

→ $\mathcal{G}$ is the same under load or displacement control.

Let's consider our area Z again.



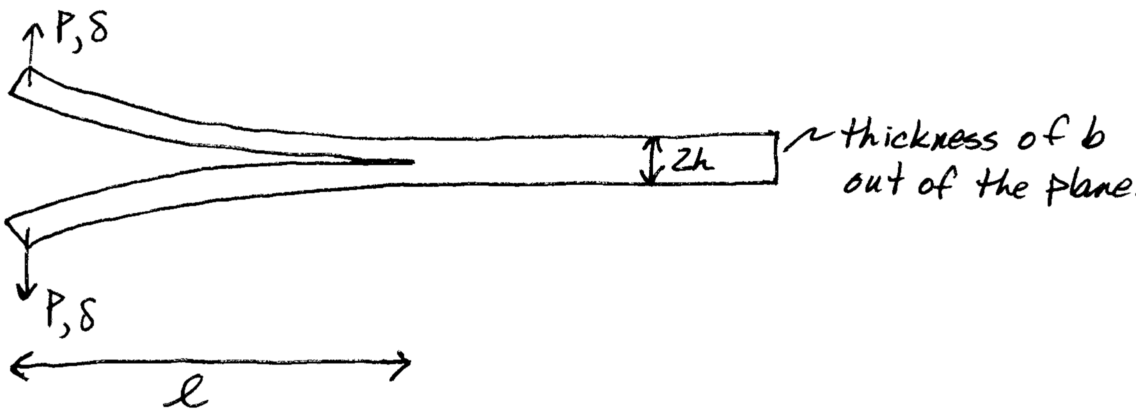
$$\therefore A_z \approx -\frac{1}{2}\frac{\partial^2 U}{\partial \delta \partial l}\frac{\partial \delta}{\partial l}\Delta l^2 = O(\Delta l^2)$$

For point loading we have shown that $\mathcal{G}$ does not depend on the loading conditions and that $\mathcal{G} = -\left.\frac{\partial U}{\partial l}\right|_{\delta}$ where U is the strain energy and not the total potential energy.

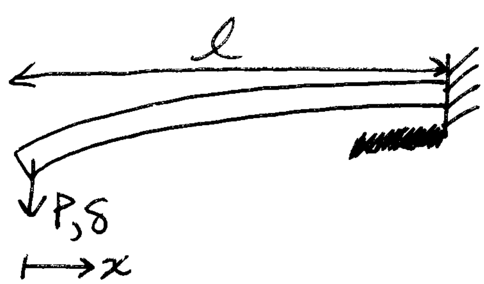
For more general loading it can also be shown that $\mathcal{G}$ does not depend on the loading conditions.

While $\mathcal{G}$ does not depend on fixed load versus fixed displacement conditions, crack growth stability <u>does</u> depend on these conditions.

Ex) The double cantilever beam specimen.



If $l \gg h$ and the crack tip is not too close to the end of the specimen, then we can treat the energy in this problem as that of 2 cantilever beams of length l.



Analysis: Recall that $\sigma = \frac{My}{I}$

$M = Px$, $I = \frac{1}{12} bh^3$, y is measured from the nuetral axis.

$$\therefore \quad \sigma = 12 \frac{Pxy}{bh^3}$$

strain energy $= U = \int_V \frac{1}{2} \sigma \varepsilon \, dV$

$$= \int_V \frac{1}{2} \frac{\sigma^2}{E} dV \quad , \; dV = b\,dx\,dy$$

$$= \frac{144}{2E}\left(\frac{P}{bh^3}\right)^2 b \int_{-\frac{h}{2}}^{h/2} \int_0^{\ell} x^2 y^2 \, dx \, dy$$

$$= \frac{72}{E} \frac{P^2}{bh^6} \left[\frac{1}{3}\ell^3 \cdot \frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) \right]$$

$$U = \frac{2}{E} \frac{P^2 \ell^3}{bh^3}$$

recall $U = \frac{1}{2} P\delta$ for a linear elastic system

$$\rightarrow \quad \frac{1}{2} P\delta = \frac{2}{E} \frac{P^2 \ell^3}{bh^3}$$

$$\therefore \quad \delta = \frac{4}{E} \frac{P\ell^3}{bh^3}$$

<u>Fixed δ</u>

2 beams

$W = 2U -$ work ~~done~~ by loads $\rightarrow 0$ b/c δ fixed, i.e. we can set the reference energy at this level on δ.

$$W = 2\frac{2}{E} \frac{P^2 \ell^3}{bh^3}$$

but we need this in terms of δ

$$\therefore \quad W = 2\frac{2}{E} \frac{\ell^3}{bh^3} \left(\frac{E\delta bh^3}{4\ell^3} \right)^2 = \frac{2E\,bh^3 \delta^2}{8\ell^3}$$

Note that W = total potential energy

$$\therefore \quad W' = \frac{W}{b} = \frac{2Eh^3\delta^2}{8\ell^3} = \text{PE per unit thickness}$$

$$\mathcal{G} = -\frac{\partial W'}{\partial \ell} = -\frac{\partial W}{\partial(\ell b)} = -\frac{Eh^3\delta^2}{4}\left(\frac{-3}{\ell^4}\right)$$

$$\mathcal{G} = \frac{3}{4}\frac{Eh^3\delta^2}{\ell^4}$$

stability: $\frac{\partial \mathcal{G}}{\partial \ell} = -3\frac{Eh^3\delta^2}{\ell^5} < 0 \rightarrow$ <u>stable</u>

Note $\mathcal{G}$ can be thought of as minus the change in PE per thickness wrt crack length in 2D problems, or more generally as minus the change in <u>total</u> PE wrt crack <u>area</u>.

<u>Fixed P</u>

$$W = 2U - 2P\delta = -2U = -\frac{2\cdot 2}{E}\frac{P^2\ell^3}{bh^3} = \frac{-2\cdot 2}{E}\frac{P^2(\ell b)^3}{b^4h^3}$$

$$\mathcal{G} = -\frac{\partial W}{\partial(\ell b)} = -\left[\frac{-2\cdot 2P^2}{E\,b^4h^3}3(\ell b)^2\right]$$

$$\mathcal{G} = \frac{12P^2\ell^2}{Eb^2h^3} = \frac{12\ell^2}{Eb^2h^3}\left(\frac{E\delta bh^3}{4\ell^3}\right)^2 \stackrel{\checkmark}{=} \frac{3}{4}\frac{Eh^3\delta^2}{\ell^4}$$

stability: $\frac{\partial \mathcal{G}}{\partial \ell} = \frac{24P^2\ell}{Eb^2h^3} > 0 \rightarrow$ <u>unstable</u>