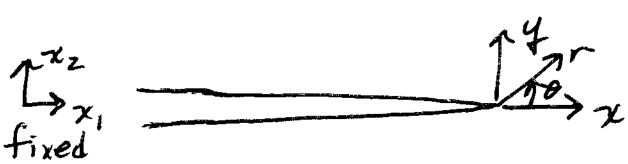
Dynamic Fracture Mechanics

The subject of dynamic fracture mechanics is as vast as the study of static cracks (and studies involving the quasi-static assumption) and a bit more complex both mathematically and physically. For more detail on this subject you should go to the book Dynamic Fracture Mechanics by Ben Freund. We will look at some of the simpler problems.

Mode III Steady-State Propagation

BCs: $\sigma_{yz}(r, \theta = \pm\pi, t) = 0$



$\rightarrow v$ — crack speed (constant)

x, y (r, θ) moves with the crack tip

i.e. $\quad y = x_2 \qquad\qquad x = x_1 - vt$

Consider the displacement field $w(x_1, x_2, t)$. Steady-state conditions imply that $w = w(x, y)$. In other words, an observer moving along with the crack tip does not see any changes in the fields over time.

Then: $$dw = \frac{\partial w}{\partial x_1} dx_1 + \frac{\partial w}{\partial x_2} dx_2 + \frac{\partial w}{\partial t} dt$$
$$= \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$$

but $\quad dy = dx_2 \quad$ and $\quad dx = dx_1 - v\,dt$

$$\therefore \quad \frac{\partial w}{\partial x_1} dx_1 + \frac{\partial w}{\partial x_2} dx_2 + \frac{\partial w}{\partial t} dt = \frac{\partial w}{\partial x}(dx_1 - v\,dt) + \frac{\partial w}{\partial y} dx_2$$

This equality must hold for arbitrary combinations of dx_1, dx_2 and dt. Therefore

$$\frac{\partial w}{\partial x_1} = \frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial x_2} = \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial t} = -v \frac{\partial w}{\partial x}$$

or more generally $\dfrac{\partial}{\partial x_1} = \dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial x_2} = \dfrac{\partial}{\partial y}$, $\dfrac{\partial}{\partial t} = -v \dfrac{\partial}{\partial x}$

<u>Governing equations</u>

Newton's 2nd Law : $\quad \dfrac{\partial \sigma_{13}}{\partial x_1} + \dfrac{\partial \sigma_{23}}{\partial x_2} = \rho \dfrac{\partial^2 w}{\partial t^2}$

Strain-Displacement : $\quad \varepsilon_{13} = \dfrac{1}{2} \dfrac{\partial w}{\partial x_1} \qquad \varepsilon_{23} = \dfrac{1}{2} \dfrac{\partial w}{\partial x_2}$

Hooke's Law : $\quad \sigma_{13} = 2\mu\,\varepsilon_{13} \qquad \sigma_{23} = 2\mu\,\varepsilon_{23}$

S-D into HL into Newton's 2nd

$$\mu \frac{\partial^2 w}{\partial x_1^2} + \mu \frac{\partial^2 w}{\partial x_2^2} = \rho \frac{\partial^2 w}{\partial t^2} \qquad \text{Note: } c_s = \sqrt{\frac{\mu}{\rho}}$$

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} = \frac{1}{c_s^2} \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \left(\frac{v}{c_s}\right)^2 \frac{\partial^2 w}{\partial x^2}$$

$$\therefore \quad \left[1-\left(\frac{v}{c_s}\right)^2\right]\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

This is almost Laplace's Equation, but not quite. However, any time you see an equation like this it can be transformed into Laplace's Equation by rescaling one of the coordinate directions. It is almost always the case that the y direction is rescaled.

Take $\quad \tilde{y} = \underbrace{\sqrt{1-\left(\frac{v}{c_s}\right)^2}}_{\alpha_s}\, y \quad \rightarrow \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{y}}\frac{d\tilde{y}}{dy} = \alpha_s \frac{\partial}{\partial \tilde{y}}$

$$\therefore \quad \alpha_s^2 \frac{\partial^2 w}{\partial x^2} + \alpha_s^2 \frac{\partial^2 w}{\partial \tilde{y}^2} = 0$$

$$\text{or} \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \tilde{y}^2} = 0 \quad \text{i.e.} \quad \tilde{\nabla}^2 w = 0$$

This is Laplace's Equation which can always be solved by the real or imaginary part of an analytic function of $z = x + i\tilde{y}$.

$$z = x + i\tilde{y} \qquad x = \tfrac{1}{2}(z+\bar{z})$$
$$\bar{z} = x - i\tilde{y} \qquad \tilde{y} = \tfrac{1}{2i}(z-\bar{z})$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial z}\frac{\partial z}{\partial \tilde{y}} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial \tilde{y}} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$\therefore \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial \tilde{y}^2} = -\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \tilde{y}^2} = \tilde{\nabla}^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

$$\therefore 4\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$$

$$\therefore w = F(z) + G(\bar{z})$$

but w is real, $\therefore G(\bar{z}) = \bar{F}(\bar{z})$

$$w = F(z) + \bar{F}(\bar{z}) = 2\,\mathrm{Re}[F(z)]$$

Equivalently, we could have chosen $w = 2\,\mathrm{Im}[F(z)]$

BCs: $\sigma_{yz}(r, \theta = \pm\pi) = 0$

$$2\mu \left.\frac{\partial w}{\partial y}\right|_{r, \theta = \pm\pi} = 0$$

$$2\mu \alpha_s \left.\frac{\partial w}{\partial \tilde{y}}\right|_{\tilde{r}, \tilde{\theta} = \pm\pi} = 0$$

note $r = \sqrt{x^2 + y^2}$ but $\tilde{r} = \sqrt{x^2 + \tilde{y}^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$ $\quad$ $\tilde{\theta} = \arctan\left(\frac{\tilde{y}}{x}\right)$

when $\theta = \pm\pi$ $\quad$ $\tilde{\theta} = \pm\pi$ as well.

$$\frac{\partial w}{\partial x} = \frac{dF}{dz}\frac{\partial z}{\partial x} + \frac{d\bar{F}}{d\bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{dF}{dz} + \overline{\frac{dF}{dz}} = 2\,\mathrm{Re}\left[\frac{dF}{dz}\right]$$

$$\frac{\partial w}{\partial \tilde{y}} = \frac{dF}{dz}\frac{\partial z}{\partial \tilde{y}} + \frac{d\bar{F}}{d\bar{z}}\frac{\partial \bar{z}}{\partial \tilde{y}} = i\frac{dF}{dz} - i\overline{\frac{dF}{dz}} = -2\,\mathrm{Im}\left[\frac{dF}{dz}\right]$$

$$\therefore \quad \frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} = 2\frac{dF}{dz}$$

$$2\mu\left(\frac{1}{2}\frac{\partial w}{\partial x}\right) - i\,2\mu\left(\frac{1}{2}\frac{\partial w}{\partial \tilde{y}}\right) = 2\mu\frac{dF}{dz}$$

$$\mu\frac{\partial w}{\partial x} - i\frac{\mu}{\alpha_s}\frac{\partial w}{\partial y} = 2\mu\frac{dF}{dz}$$

$$\boxed{\sigma_{xz} - i\frac{1}{\alpha_s}\sigma_{yz} = 2\mu\frac{dF}{dz}, \quad w = F(z) + \overline{F(z)}}$$

Solution: Try $2\mu\frac{dF}{dz} = A z^p$ (A complex)

$$\begin{aligned}\therefore \quad \sigma_{yz} &= -\alpha_s\,\mathrm{Im}\left[A z^p\right] \\ &= -\alpha_s\,\mathrm{Im}\left[A \tilde{r}^p e^{ip\tilde{\theta}}\right] \\ &= -\alpha_s \tilde{r}^p\left[\mathrm{Re}(A)\sin p\tilde{\theta} + \mathrm{Im}(A)\cos p\tilde{\theta}\right]\end{aligned}$$

BCs $\rightarrow \sigma_{yz}(\tilde{r}, \tilde{\theta}=\pm\pi) = 0 \rightarrow \mathrm{Re}(A)=0$ and $p = \ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$

or $\mathrm{Im}(A)=0$ and $p = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\therefore \text{ in general} \quad 2\mu\frac{dF}{dz} = \sum_{n=-\infty}^{\infty}\left(A_n z^n + iB_n z^{n+\frac{1}{2}}\right)$$

where A_n & B_n are real

We want the most singular term with finite energy (kinetic & potential) near the crack tip.

$$PE \propto \varepsilon^2 \sim \left(\frac{dw}{dz}\right)^2 \sim z^{2p}$$

$$KE \propto v^2 \sim \left(\frac{dw}{dt}\right)^2 \sim \left(\frac{\partial w}{\partial x}\right)^2 \sim z^{2p}$$

particle velocities not crack tip velocity

$\therefore$ KE & PE have the same radial dependence and by the same energy argument used for the stationary crack, the exclusion principle based on a finite energy in a finite region near the crack tip implies that

$$2p > -2 \rightarrow p > -1$$

$\therefore$ the most singular term is $\boxed{p = -\frac{1}{2}}$

$$\therefore 2\mu \frac{dF}{dz} = i B_{-1} z^{-1/2}$$

Ahead of the crack tip we have

$$\sigma_{xz}(\tilde{r}, \tilde{\theta}=0) = 0 \qquad \sigma_{yz}(\tilde{r}, \tilde{\theta}=0) = \frac{K_{III}^D}{\sqrt{2\pi x}}$$

$$2\mu \frac{dF}{dz} = \sigma_{xz} - i \frac{1}{\alpha_s} \sigma_{yz} = -i \frac{1}{\alpha_s} \frac{K_{III}^D}{\sqrt{2\pi x}} \quad \text{on } \theta = 0$$

$$= i B_{-1} \frac{1}{\sqrt{x}}$$

$$\therefore \boxed{B_{-1} = -\frac{1}{\alpha_s} \frac{K_{III}^D}{\sqrt{2\pi}}}$$

$$\therefore \quad 2\mu \frac{dF}{dz} = -\frac{i}{\alpha_s} \frac{K_{III}^D}{\sqrt{2\pi}} z^{-1/2} \qquad z = \tilde{r} e^{i\tilde{\theta}}$$

$$\therefore \quad \sigma_{yz} = \frac{K_{III}^D}{\sqrt{2\pi\tilde{r}}} \cos\frac{\tilde{\theta}}{2}$$

$$\sigma_{xz} = -\frac{K_{III}^D}{\alpha_s\sqrt{2\pi\tilde{r}}} \sin\frac{\tilde{\theta}}{2}$$

then $F(z) = \frac{1}{2\mu}\left(-\frac{i}{\alpha_s}\right)\frac{2K_{III}^D}{\sqrt{2\pi}} z^{1/2}$

$$w = 2\,\mathrm{Re}[F(z)] = \frac{2K_{III}^D}{\mu\alpha_s}\sqrt{\frac{\tilde{r}}{2\pi}} \sin\frac{\tilde{\theta}}{2}$$

Recall: $\tilde{r} = \sqrt{x^2+\tilde{y}^2} = \sqrt{x^2+\alpha_s^2 y^2}$

$$= \sqrt{r^2\cos^2\theta + \left(1-\frac{v^2}{c_s^2}\right)r^2\sin^2\theta}$$

$$= r\sqrt{1-\left(\frac{v\sin\theta}{c_s}\right)^2}$$

$$= r\,\gamma_s(\theta; v)$$

$$\tan\tilde{\theta} = \frac{\tilde{y}}{x} = \alpha_s\frac{y}{x} = \alpha_s\tan\theta$$

$$\therefore \quad \tilde{\theta} = \arctan\left[\alpha_s\tan\theta\right]$$

$$\therefore \quad \boxed{\begin{aligned} \sigma_{yz} &= \frac{K_{III}^D}{\sqrt{2\pi r}}\frac{1}{\sqrt{\gamma_s(\theta;v)}}\cos\frac{\tilde{\theta}}{2} \\ \sigma_{xz} &= -\frac{K_{III}^D}{\alpha_s\sqrt{2\pi r}}\frac{1}{\sqrt{\gamma_s(\theta;v)}}\sin\frac{\tilde{\theta}}{2} \\ w &= \frac{2K_{III}^D}{\mu\alpha_s}\sqrt{\frac{r}{2\pi}}\sqrt{\gamma_s(\theta;v)}\sin\frac{\tilde{\theta}}{2} \end{aligned}}$$

Modes I & II Steady-State Dynamic Crack Growth (plane strain) (isotropic)

[redacted] $$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = \rho \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = \rho \frac{\partial^2 u_2}{\partial t^2}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\sigma_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{11} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{22}$$

$$\sigma_{22} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{11} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{22}$$

$$\sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12}$$

Use Helmholtz decomposition to introduce ϕ and ψ

$$\rightarrow \quad u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_2} \qquad u_2 = \frac{\partial \phi}{\partial x_2} + \frac{\partial \psi}{\partial x_1}$$

Strain-Displacement into Hooke's Law into Newton's 2nd

$$① \ \frac{E}{1+\nu}\left[\frac{1-\nu}{(1-2\nu)}(\phi_{,111} - \psi_{,112}) + \frac{\nu}{(1-2\nu)}(\phi_{,122} + \psi_{,112}) + \phi_{,122} - \tfrac{1}{2}\psi_{,222} + \tfrac{1}{2}\psi_{,112}\right] = \rho\frac{\partial^2}{\partial t^2}(\phi_{,1} - \psi_{,2})$$

and

$$② \ \frac{E}{1+\nu}\left[\frac{1-\nu}{(1-2\nu)}(\phi_{,222} + \psi_{,122}) + \frac{\nu}{(1-2\nu)}(\phi_{,112} - \psi_{,122}) + \phi_{,112} - \tfrac{1}{2}\psi_{,122} + \tfrac{1}{2}\psi_{,111}\right] = \rho\frac{\partial^2}{\partial t^2}(\phi_{,2} + \psi_{,1})$$

$$\frac{\partial}{\partial x_1}① + \frac{\partial}{\partial x_2}② \rightarrow \frac{E}{1+\nu}\left[\frac{1-\nu}{1-2\nu}(\phi_{,1111} - \psi_{,1112} + \phi_{,2222} + \psi_{,1222}) + \frac{\nu}{1-2\nu}(\phi_{,1122} + \psi_{,1112} + \phi_{,1122} - \psi_{,1222}) \right.$$

$$\left. + \phi_{,1122} - \tfrac{1}{2}\psi_{,1222} + \tfrac{1}{2}\psi_{,1112} + \phi_{,1122} - \tfrac{1}{2}\psi_{,1222} + \tfrac{1}{2}\psi_{,1112}\right] = \rho\frac{\partial^2}{\partial t^2}(\phi_{,11} + \phi_{,22})$$

$$\rightarrow \frac{E}{1+\nu}\left[\frac{1-\nu}{1-2\nu}\left(\phi_{,1111}+2\phi_{,1122}+\phi_{,2222}\right)\right] = \rho\frac{\partial^2}{\partial t^2}\left(\phi_{,11}+\phi_{,22}\right)$$

$$\rightarrow \quad \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\nabla^4\phi = \rho\frac{\partial^2}{\partial t^2}\nabla^2\phi$$

$$\rightarrow \quad \boxed{\nabla^2\phi = \frac{1}{C_D^2}\frac{\partial^2\phi}{\partial t^2}} \qquad C_D = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}$$

(Dilatational wave speed)

$$\frac{\partial}{\partial x_1}② - \frac{\partial}{\partial x_2}① \rightarrow \frac{E}{1+\nu}\Big[\frac{1-\nu}{1-2\nu}\left(\phi_{,1222}+\psi_{,1122}-\phi_{,1112}+\psi_{,1122}\right)$$

$$+\frac{\nu}{1-2\nu}\left(\phi_{,1112}-\psi_{,1122}-\phi_{,1222}-\psi_{,1122}\right)$$

$$+\phi_{,1112}-\frac{1}{2}\psi_{,1122}+\frac{1}{2}\psi_{,1111}-\phi_{,1222}+\frac{1}{2}\psi_{,2222}-\frac{1}{2}\psi_{,1122}\Big]$$

$$= \rho\frac{\partial^2}{\partial t^2}\left(\psi_{,11}+\psi_{,22}\right)$$

$$\rightarrow \frac{E}{1+\nu}\left[\frac{1}{2}\psi_{,1111}+\psi_{,1122}+\frac{1}{2}\psi_{,2222}\right] = \rho\frac{\partial^2}{\partial t^2}\left(\psi_{,11}+\psi_{,22}\right)$$

$$\frac{E}{2(1+\nu)}\nabla^4\psi = \rho\frac{\partial^2}{\partial t^2}\nabla^2\psi$$

$$\boxed{\nabla^2\psi = \frac{1}{C_S^2}\frac{\partial^2\psi}{\partial t^2}} \qquad C_S = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}}$$

(shear wave speed)

Again, SS $\rightarrow$ $\left(1-\frac{v^2}{C_D^2}\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$

and

$$\left(1-\frac{v^2}{C_S^2}\right)\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

Again these equations imply

$$\phi = F(z_D) + \overline{F(z_D)} \quad , \quad \psi = G(z_S) + \overline{G(z_S)}$$

$$z_D = x + i\alpha_D y \qquad\qquad z_S = x + i\alpha_S y$$

$$\alpha_D = \sqrt{1-(v/c_D)^2} \qquad\qquad \alpha_S = \sqrt{1-(v/c_S)^2}$$

BCs: (A) $\sigma_{yy}(x<0, y=0) = 0$ (B) $\sigma_{xy}(x<0, y=0) = 0$

$$\text{(A)} \rightarrow \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\phi_{,xx} - \psi_{,xy}\right) + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\phi_{,yy} + \psi_{,xy}\right) = 0 \text{ on } \begin{matrix} x<0 \\ y=0 \end{matrix}$$

$$\text{(B)} \rightarrow \frac{1}{2}\frac{E}{1+\nu}\left(\phi_{,xy} - \psi_{,yy} + \phi_{,xy} + \psi_{,xx}\right) = 0 \text{ on } \begin{matrix} x<0 \\ y=0 \end{matrix}$$

Recall: $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$ [redacted]

$$\frac{\partial}{\partial y} = \text{[redacted]}\, i\alpha\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2} \quad , \quad \frac{\partial^2}{\partial y^2} = -\alpha^2\left(\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right)$$

$$\frac{\partial^2}{\partial x \partial y} = i\alpha\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \bar{z}^2}\right)$$

$$\therefore \text{(A)} \rightarrow \nu\left(\frac{d^2F}{dz_D^2} + \overline{\frac{d^2F}{dz_D^2}} - i\alpha_S\frac{d^2G}{dz_S^2} + i\alpha_S\overline{\frac{d^2G}{dz_S^2}}\right)$$

$$+ (1-\nu)\left(-\alpha_D^2\frac{d^2F}{dz_D^2} - \alpha_D^2\overline{\frac{d^2F}{dz_D^2}} + i\alpha_S\frac{d^2G}{dz_S^2} - i\alpha_S\overline{\frac{d^2G}{dz_S^2}}\right) = 0 \text{ on } \begin{matrix} x<0 \\ y=0 \end{matrix}$$

$$\text{(B)} \rightarrow 2\left(i\alpha_D\frac{d^2F}{dz_D^2} - i\alpha_D\overline{\frac{d^2F}{dz_D^2}}\right) + (1+\alpha_S^2)\frac{d^2G}{dz_S^2} + (1+\alpha_S^2)\overline{\frac{d^2G}{dz_S^2}} = 0$$

$$A \quad \left[\nu - (1-\nu)\alpha_D^2\right]\left(F'' + \overline{F''}\right) + (1-2\nu)\alpha_s i\left(G'' - \overline{G''}\right) = 0 \text{ on } \theta = \pm\pi$$

$$B \quad 2\alpha_D i\left(F'' - \overline{F''}\right) + (1+\alpha_s^2)\left(G'' + \overline{G''}\right) = 0 \quad \text{on } \theta = \pm\pi$$

note
$$\begin{aligned}
\nu - (1-\nu)\alpha_D^2 &= \nu - (1-\nu)\left(1 - \frac{v^2}{C_D^2}\right) \\
&= \nu - (1-\nu)\left(1 - v^2 \frac{\rho(1+\nu)(1-2\nu)}{(1-\nu)E}\right) \\
&= \nu - (1-\nu) + v^2 \frac{\rho(1-2\nu)(1+\nu)}{E} \\
&= -1 + 2\nu + (1-2\nu)\, v^2 \frac{1}{2}\frac{\rho\, 2(1+\nu)}{E} \\
&= -(1-2\nu)\left[1 - \frac{1}{2}\frac{v^2}{C_s^2}\right] \\
&= -(1-2\nu)\left[\frac{1}{2} + \frac{1}{2}\alpha_s^2\right] \\
&= -\frac{1-2\nu}{2}(1+\alpha_s^2)
\end{aligned}$$

$$\therefore A \rightarrow -\frac{1-2\nu}{2}(1+\alpha_s^2)\left(F'' + \overline{F''}\right) + (1-2\nu)\alpha_s i\left(G'' - \overline{G''}\right) = 0$$

$$(1+\alpha_s^2)\left(F'' + \overline{F''}\right) - 2\alpha_s i\left(G'' - \overline{G''}\right) = 0 \text{ on } \theta = \pm\pi$$

Try: $F'' = (A+iB)z_D^p \qquad G'' = (C+iD)z_s^p$

A, B, C, D real

note on $\theta = \pm\pi \quad y = 0 \Rightarrow \theta_s = \theta_D = \theta = \pm\pi$
and $r_s = r_D = r = \sqrt{x^2}$

$$A \rightarrow 2r^p(1+\alpha_s^2)(A\cos p\pi \mp B\sin p\pi) + 4r^p\alpha_s(D\cos p\pi \pm C\sin p\pi) = 0$$

$$B \rightarrow -4r^p\alpha_D(B\cos p\pi \pm A\sin p\pi) + 2r^p(1+\alpha_s^2)(C\cos p\pi \mp D\sin p\pi) = 0$$

2 Cases: $\sin p\pi = 0$ with $(1+\alpha_s^2)A + 2\alpha_s D = 0$
i.e. $p = \ldots -1, 0, 1, \ldots$ and $-2\alpha_D B + (1+\alpha_s^2)C = 0$

or $\cos p\pi = 0$ with $-(1+\alpha_s^2)B + 2\alpha_s C = 0$
i.e. $p = \ldots -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$ and $-2\alpha_D A - (1+\alpha_s^2)D = 0$

Again, the most singular terms with integrable energy density correspond to $\sigma_{ij} \sim r^{-1/2}$

$$\sigma_{ij} \sim F'' \text{ and } G'' \rightarrow \boxed{p = -\frac{1}{2}}$$

$$\therefore\ p = -\frac{1}{2} \quad \text{and} \quad A = -\frac{1+\alpha_s^2}{2\alpha_D} D, \quad B = \frac{2\alpha_s}{1+\alpha_s^2} C$$

Normalizations: $\sigma_{yy} = \dfrac{K_I^D}{\sqrt{2\pi r}}$ on $\theta = 0$

$\sigma_{xy} = \dfrac{K_{II}^D}{\sqrt{2\pi r}}$ on $\theta = 0$

$$\text{Mode I}\ \therefore\ \frac{K_I^D}{\sqrt{2\pi \cancel{r}}} = \frac{E}{(1+\nu)\cancel{(1-2\nu)}}\cancel{r^{-1/2}}\left[\left(-\frac{\cancel{1-2\nu}}{\cancel{2}}\right)(1+\alpha_s^2)A\cancel{2} + \cancel{(1-2\nu)}\alpha_s i(2iD)\right]$$

$$\frac{K_I^D}{\sqrt{2\pi}} = \frac{E}{1+\nu}\left[\frac{(1+\alpha_s^2)^2}{2\alpha_D} - 2\alpha_s\right] D$$

$$\therefore\ \boxed{D = \frac{K_I^D}{\sqrt{2\pi}}\frac{1+\nu}{E}\frac{2\alpha_D}{\underbrace{(1+\alpha_s^2)^2 - 4\alpha_s\alpha_D}_{\mathcal{D}}} = \frac{K_I^D}{\sqrt{2\pi}}\frac{1+\nu}{E}\frac{2\alpha_D}{\mathcal{D}}}$$

$\mathcal{D} \leftarrow$ (opposite of that in Freund's Book)

$$A = -\frac{K_I^D}{\sqrt{2\pi}}\frac{1+\nu}{E}\frac{1+\alpha_s^2}{\mathcal{D}}$$

Mode II: $\dfrac{K_{II}^{D}}{\sqrt{2\pi r}} = \dfrac{E}{2(1+\nu)}\left[\dfrac{\partial^2\phi}{\partial x\partial y} - \dfrac{\partial^2\psi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial x\partial y} + \dfrac{\partial^2\psi}{\partial x^2}\right]$ on $\theta=0$

$$= \frac{E}{2(1+\nu)}\left[2i\alpha_D(F''-\bar{F}'') + (1+\alpha_s^2)(G''+\bar{G}'')\right]_{\text{on } \theta=0}$$

$$\frac{K_{II}^{D}}{\sqrt{2\pi \cancel{r}}} = \frac{E}{\cancel{2}(1+\nu)}\left[\cancel{2}i\alpha_D \cancel{r^{-\frac{1}{2}}} 2iB + (1+\alpha_s^2)\cancel{2}C\cancel{r^{-1/2}}\right]$$

$$= \frac{E}{1+\nu}\left[-2\alpha_D\frac{2\alpha_s}{1+\alpha_s^2} + (1+\alpha_s^2)\right]C$$

$$\therefore \boxed{\begin{aligned} C &= \frac{K_{II}^{D}}{\sqrt{2\pi}}\frac{1+\nu}{E}\frac{1+\alpha_s^2}{(1+\alpha_s^2)-4\alpha_s\alpha_D} = \frac{K_{II}^{D}}{\sqrt{2\pi}}\frac{1+\nu}{E}\frac{1+\alpha_s^2}{\mathcal{D}} \\ B &= \frac{K_{II}^{D}}{\sqrt{2\pi}}\frac{1+\nu}{E}\frac{2\alpha_s}{\mathcal{D}} \end{aligned}}$$

$\therefore$ <u>Mode I Stresses</u> (i.e. take $K_{II}^{D}=0$)

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[\underbrace{[\nu-(1-\nu)\alpha_D^2]}_{-\frac{1-2\nu}{2}(1+\alpha_s^2)} 2\,\mathrm{Re}F'' + (1-2\nu)\alpha_s i \cdot 2i\,\mathrm{Im}G''\right]$$

$$= \frac{\cancel{E}}{\cancel{(1+\nu)}\cancel{(1-2\nu)}}\left[+\frac{\cancel{1-2\nu}}{\cancel{2}}(1+\alpha_s^2)\cancel{2}\frac{K_I^{D}}{\sqrt{2\pi r_D}}\frac{\cancel{1+\nu}}{\cancel{E}}\frac{1+\alpha_s^2}{\mathcal{D}}\cos\frac{\theta_D}{2} - (\cancel{1-2\nu})\alpha_s 2\frac{K_I^{D}}{\sqrt{2\pi r_s}}\frac{\cancel{1+\nu}}{\cancel{E}}\frac{2\alpha_D}{\mathcal{D}}\cos\frac{\theta_s}{2}\right]$$

$$= +\frac{K_I^{D}}{\sqrt{2\pi r_D}}\frac{(1+\alpha_s^2)^2}{\mathcal{D}}\cos\frac{\theta_D}{2} - \frac{K_I^{D}}{\sqrt{2\pi r_s}}\frac{4\alpha_s\alpha_D}{\mathcal{D}}\cos\frac{\theta_s}{2}$$

Take $r_D = r\gamma_D$ where $\gamma_D = \sqrt{1-\left(\frac{v\sin\theta}{c_D}\right)^2}$

$r_s = r\gamma_s$ where $\gamma_s = \sqrt{1-\left(\frac{v\sin\theta}{c_s}\right)^2}$

Then Mode I →

$$\boxed{\sigma_{yy} = \frac{K_I^{D}}{\sqrt{2\pi r}}\left[\frac{(1+\alpha_s^2)^2}{\mathcal{D}}\frac{\cos\left(\frac{\theta_D}{2}\right)}{\sqrt{\gamma_D}} - \frac{4\alpha_s\alpha_D}{\mathcal{D}}\frac{\cos\left(\frac{\theta_s}{2}\right)}{\sqrt{\gamma_s}}\right]}$$

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\Big[\underbrace{[(1-\nu)-\nu\alpha_D^2]}\,2\,\mathrm{Re}\,F'' - (1-2\nu)\,i\alpha_s\,2i\,\mathrm{Im}\,G''\Big]$$

$$1-2\nu+\underbrace{\nu-(1-\nu)\alpha_D^2}+(1-2\nu)\alpha_D^2$$

$$-\frac{1-2\nu}{2}(1+\alpha_s^2)$$

$$\frac{1-2\nu}{2}\left(2+2\alpha_D^2-(1+\alpha_s^2)\right)$$

$$= \frac{\cancel{E}}{\cancel{(1+\nu)}\cancel{(1-2\nu)}}\Big[\frac{\cancel{1-2\nu}}{\cancel{2}}(1+2\alpha_D^2-\alpha_s^2)\,\cancel{2}\,\frac{-K_I^D}{\sqrt{2\pi r_D}}\,\frac{\cancel{1+\nu}}{\cancel{E}}\,\frac{1+\alpha_s^2}{\mathcal{D}}\cos\frac{\theta_D}{2}$$
$$+ \cancel{(1-2\nu)}\,\alpha_s\,2\,\frac{K_I^D}{\sqrt{2\pi r_s}}\,\frac{\cancel{1+\nu}}{\cancel{E}}\,\frac{2\alpha_D}{\mathcal{D}}\cos\frac{\theta_s}{2}\Big]$$

Mode I
$$\boxed{\sigma_{xx} = \frac{K_I^D}{\sqrt{2\pi r}}\left[\frac{-(1+\alpha_s^2)(1+2\alpha_D^2-\alpha_s^2)}{\mathcal{D}}\,\frac{\cos\left(\frac{\theta_D}{2}\right)}{\sqrt{\gamma_D}} + \frac{4\alpha_s\alpha_D}{\mathcal{D}}\,\frac{\cos\left(\frac{\theta_s}{2}\right)}{\sqrt{\gamma_s}}\right]}$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\Big[2i\alpha_D\,2i\,\mathrm{Im}\,F'' + (1+\alpha_s^2)\,2\,\mathrm{Re}\,G''\Big]$$

$$= \frac{\cancel{E}}{\cancel{2}\cancel{(1+\nu)}}\Big[-\overset{2}{\cancel{4}}\alpha_D\,\frac{-K_I^D}{\sqrt{2\pi r_D}}\,\frac{\cancel{1+\nu}}{\cancel{E}}\,\frac{(1+\alpha_s^2)}{\mathcal{D}}\left(-\sin\frac{\theta_D}{2}\right)$$
$$+ (1+\alpha_s^2)\,\cancel{2}\,\frac{K_I^D}{\sqrt{2\pi r_s}}\,\frac{\cancel{1+\nu}}{\cancel{E}}\,\frac{2\alpha_D}{\mathcal{D}}(i\cdot i)\left(-\sin\frac{\theta_s}{2}\right)\Big]$$

Mode I
$$\boxed{\sigma_{xy} = \frac{K_I^D}{\sqrt{2\pi r}}\left[\frac{-2\alpha_D(1+\alpha_s^2)}{\mathcal{D}}\,\frac{\sin\left(\frac{\theta_D}{2}\right)}{\sqrt{\gamma_D}} + \frac{2\alpha_D(1+\alpha_s^2)}{\mathcal{D}}\,\frac{\sin\left(\frac{\theta_s}{2}\right)}{\sqrt{\gamma_s}}\right]}$$

Recall $\mathcal{D} = (1+\alpha_s^2)^2 - 4\alpha_s\alpha_D$ (Opposite of Freund's def.)

<u>Mode II Stresses</u> (take $K_I^D = 0$)

$$\sigma_{yy} = \frac{E}{\cancel{(1+\nu)}\cancel{(1-2\nu)}}\Big[-\frac{\cancel{1-2\nu}}{\cancel{2}}(1+\alpha_s^2)\,\cancel{2}\,\frac{K_{II}^D}{\sqrt{2\pi r_D}}\,\frac{\cancel{1+\nu}}{E}\,\frac{2\alpha_s}{\mathcal{D}}(i\cdot i)\left(-\sin\frac{\theta_D}{2}\right)$$
$$+ \cancel{(1-2\nu)}\,\alpha_s\,2\,(i\cdot i)\,\frac{K_{II}^D}{\sqrt{2\pi r_s}}\,\frac{\cancel{1+\nu}}{E}\,\frac{1+\alpha_s^2}{\mathcal{D}}\left(-\sin\frac{\theta_s}{2}\right)\Big]$$

Mode II $$\boxed{\sigma_{yy} = \frac{K_{II}^{D}}{\sqrt{2\pi r}}\left[-\frac{2\alpha_s(1+\alpha_s^2)}{\mathcal{D}}\frac{\sin(\theta_D/2)}{\sqrt{\gamma_D}} + \frac{2\alpha_s(1+\alpha_s^2)}{\mathcal{D}}\frac{\sin(\theta_s/2)}{\sqrt{\gamma_s}}\right]}$$

$$\sigma_{xx} = \frac{\cancel{E}}{\cancel{(1+\nu)(1-2\nu)}}\left[\frac{\cancel{1-2\nu}}{\cancel{2}}(1+2\alpha_D^2-\alpha_s^2)\cancel{2}\,\frac{K_{II}^{D}}{\sqrt{2\pi r_D}}\frac{\cancel{1+\nu}}{\cancel{E}}\frac{2\alpha_s}{\mathcal{D}}(i\cdot i)\left(-\sin\frac{\theta_D}{2}\right)\right.$$

$$\left.-(\cancel{1-2\nu})\,\alpha_s\,2(i\cdot i)\frac{K_{II}^{D}}{\sqrt{2\pi r_s}}\frac{\cancel{1+\nu}}{\cancel{E}}\frac{1+\alpha_s^2}{\mathcal{D}}\left(-\sin\frac{\theta_s}{2}\right)\right]$$

Mode II $$\boxed{\sigma_{xx} = \frac{K_{II}^{D}}{\sqrt{2\pi r}}\left[\frac{2\alpha_s(1+2\alpha_D^2-\alpha_s^2)}{\mathcal{D}}\frac{\sin(\theta_D/2)}{\sqrt{\gamma_D}} - \frac{2\alpha_s(1+\alpha_s^2)}{\mathcal{D}}\frac{\sin(\theta_s/2)}{\sqrt{\gamma_s}}\right]}$$

$$\sigma_{xy} = \frac{\cancel{E}}{\cancel{2(1+\nu)}}\left[4\alpha_D(i\cdot i)\frac{K_{II}^{D}}{\sqrt{2\pi r_D}}\frac{\cancel{1+\nu}}{\cancel{E}}\frac{\cancel{2}\alpha_s}{\mathcal{D}}\cos\frac{\theta_D}{2}\right.$$

$$\left.+(1+\alpha_s^2)\cancel{2}\,\frac{K_{II}}{\sqrt{2\pi}}\frac{\cancel{1+\nu}}{\cancel{E}}\frac{1+\alpha_s^2}{\mathcal{D}}\cos\frac{\theta_s}{2}\right]$$

Mode II $$\boxed{\sigma_{xy} = \frac{K_{II}^{D}}{\sqrt{2\pi r}}\left[-\frac{4\alpha_s\alpha_D}{\mathcal{D}}\frac{\cos(\theta_D/2)}{\sqrt{\gamma_D}} + \frac{(1+\alpha_s^2)^2}{\mathcal{D}}\frac{\cos(\theta_s/2)}{\sqrt{\gamma_s}}\right]}$$

and once again note $\mathcal{D} = (1+\alpha_s^2)^2 - 4\alpha_s\alpha_D$

It is interesting to note that these asymptotic fields are also valid for <u>non-uniform</u> crack growth, i.e. not constant velocity in one direction. The analysis of this case can be found in Freund's book and is carried out with an asymptotic expansion in a small parameter ε.