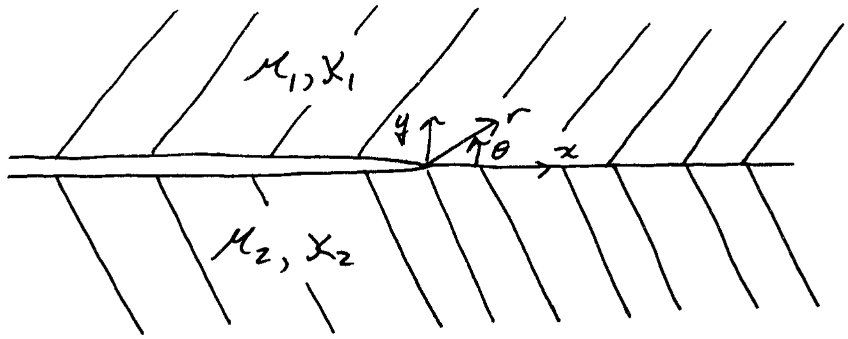
Interfacial Fracture Mechanics



The asymmetry induced by having different materials above and below the crack plane implies that we will not be able to use Westergaard functions for the solution. Instead, we will have to resort to using 2 complex potentials (in each material).

Recall from elasticity that any two analytic complex potentials $\phi(z)$ and $\psi(z)$ will satisfy the governing elasticity equations if,

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}\,\phi'(z) = 2\left[\phi'(z) + \bar{\phi}'(\bar{z})\right]$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\,\phi''(z) + \psi'(z)\right]$$

$$2\mu\,(u_x + i u_y) = \chi\,\phi(z) - z\,\bar{\phi}'(\bar{z}) - \bar{\psi}(\bar{z})$$

Now define a new potential $\Omega(z)$ to replace $\psi(z)$ such that

$$\Omega(z) = \psi(z) + z\,\phi'(z)$$

$$\rightarrow \psi(z) = \Omega(z) - z\,\phi'(z)$$

then, $$\sigma_{xx}+\sigma_{yy} = 2\left[\phi'(z)+\bar{\phi}'(\bar{z})\right]$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2\left[(\bar{z}-z)\phi''(z)-\phi'(z)+\Omega'(z)\right]$$

$$2\mu(u_x+iu_y) = \chi\phi(z)+(z-\bar{z})\bar{\phi}'(\bar{z})-\bar{\Omega}(\bar{z})$$

We will be using $\phi_1(z)$ and $\Omega_1(z)$ for the solution in material 1 and $\phi_2(z)$ and $\Omega_2(z)$ for the solution in material 2.

BCs : Ⓐ $\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)}$ and $\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)}$ for $y=0$

Ⓑ $\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)} = \sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} = 0$ for $y=0,\ x<0$

Ⓒ $u_x^{(1)} = u_x^{(2)}$ and $u_y^{(1)} = u_y^{(2)}$ for $y=0,\ x>0$

Ⓐ $\rightarrow$ $(\sigma_{yy}-i\sigma_{xy})_1 = (\sigma_{yy}-i\sigma_{xy})_2$ along interface

$$2\sigma_{yy}+2i\sigma_{xy} = 2\bar{\phi}'(\bar{z})+2(\bar{z}-z)\phi''(z)+2\Omega'(z)$$

$\frac{1}{2}$ complex conjugate

$$\longrightarrow \sigma_{yy}-i\sigma_{xy} = \phi'(z)+(z-\bar{z})\bar{\phi}''(\bar{z})+\bar{\Omega}'(\bar{z})$$

along interface $y=0 \rightarrow z=\bar{z}$

$$\therefore\ \phi_1'(z)+\bar{\Omega}_1'(\bar{z}) = \phi_2'(z)+\bar{\Omega}_2'(\bar{z}) \quad \text{for } y=0$$

$$\phi_1'(z)+\bar{\Omega}_1'(z) = \phi_2'(z)+\bar{\Omega}_2'(z) \quad \text{for } y=0$$

Note $\phi_1(z)$ and $\Omega_1(z)$ are analytic for $y \geq 0$
$\phi_2(z)$ and $\Omega_2(z)$ are analytic for $y \leq 0$

$\therefore$ $\bar{\phi}_1(z)$ and $\bar{\Omega}_1(z)$ are analytic for $y \leq 0$
$\bar{\phi}_2(z)$ and $\bar{\Omega}_2(z)$ are analytic for $y \geq 0$

Aside: Analytic continuation states that if $f_1(z)$ is analytic in R_1 and $f_2(z)$ is analytic in R_2 and $f_1(z) = f_2(z)$ in $R_1 \cap R_2$, then $f_1(z)$ and $f_2(z)$ are 2 different representations of the same analytic function $f_3(z)$ that is analytic in $R_1 \cup R_2$.

$\therefore$ By analytic continuation

Factor of 2 follows Rice's paper

$$\underbrace{\phi_1'(z) - \bar{\Omega}_2'(z)}_{\text{analytic in } y \geq 0} \underset{\substack{\text{equal on boundary} \\ \text{i.e. on interface}}}{=} \underbrace{\phi_2'(z) - \bar{\Omega}_1'(z)}_{\text{analytic on } y \leq 0} = \underbrace{2g(z)}_{\text{analytic everywhere}}$$

i.e. $$g(z) = \sum_{m=0}^{\infty} B_m z^m$$

$$\therefore \quad \bar{\Omega}_2'(z) = \phi_1'(z) - 2g(z)$$

$$\bar{\Omega}_1'(z) = \phi_2'(z) - 2g(z)$$

C $\rightarrow$ $(u_x + i u_y)_1 = (u_x + i u_y)_2$ on $y=0,\ x>0$
i.e. $z = \bar{z}$

$$\frac{1}{2\mu_1}\left[\chi_1 \phi_1(z) - \bar{\Omega}_1(z)\right] = \frac{1}{2\mu_2}\left[\chi_2 \phi_2(z) - \bar{\Omega}_2(z)\right] \text{ on } z=\bar{z}$$

$$\underbrace{\frac{\chi_1}{\mu_1}\phi_1(z) + \frac{1}{\mu_2}\bar{\Omega}_2(z)}_{\text{analytic on } y \geq 0} = \underbrace{\frac{\chi_2}{\mu_2}\phi_2(z) + \frac{1}{\mu_1}\bar{\Omega}_1(z)}_{\text{analytic on } y \leq 0}$$

equal on interface
$\therefore$ equal throughout z-plane

$$\therefore \boxed{\frac{\chi_1}{\mu_1}\phi_1'(z) + \frac{1}{\mu_2}\bar{\Omega}_2'(z) = \frac{\chi_2}{\mu_2}\phi_2'(z) + \frac{1}{\mu_1}\bar{\Omega}_1'(z) \quad \text{all } z}$$

$$\rightarrow \frac{\chi_1}{\mu_1}\phi_1' + \frac{1}{\mu_2}(\phi_1' - 2g) = \frac{\chi_2}{\mu_2}\phi_2' + \frac{1}{\mu_1}(\phi_2' - 2g)$$

$$\therefore \boxed{\begin{aligned} \phi_2'(z) &= \frac{\chi_1/\mu_1 + 1/\mu_2}{\chi_2/\mu_2 + 1/\mu_1}\phi_1'(z) + \frac{2(1/\mu_1 - 1/\mu_2)}{\chi_2/\mu_2 + 1/\mu_1} g(z) \\ \Omega_1'(z) &= \frac{\chi_1/\mu_1 + 1/\mu_2}{\chi_2/\mu_2 + 1/\mu_1}\bar{\phi}_1'(z) - \frac{2(\chi_2/\mu_2 + 1/\mu_2)}{\chi_2/\mu_2 + 1/\mu_1}\bar{g}(z) \\ \Omega_2'(z) &= \bar{\phi}_1'(z) - 2\bar{g}(z) \end{aligned}}$$

B $\rightarrow$ $(\sigma_{yy} - i\sigma_{xy})_1 = (\sigma_{yy} - i\sigma_{xy})_2 = 0$ for $y=0,\ x<0$

$$\phi_1'(z) + \bar{\Omega}_1'(\bar{z}) = 0 \quad \text{on } y=0 \quad x<0$$

$$\phi_1'(x)^+ + \bar{\Omega}_1'(x)^- = 0$$

$$\phi_1'(x)^+ + \frac{\chi_1/\mu_1 + 1/\mu_2}{\chi_2/\mu_2 + 1/\mu_1}\phi_1'(x)^- - \frac{2(\chi_2/\mu_2 + 1/\mu_2)}{\chi_2/\mu_2 + 1/\mu_1} g(x) = 0$$

i.e. $(x)^+ = (re^{i\pi})$ $\quad (x)^- = (re^{-i\pi})$ $\quad (x) = (re^{\pm i\pi})$

$$\therefore \left(\frac{\varkappa_2}{\mu_2}+\frac{1}{\mu_1}\right)\phi_1'(x)^+ + \left(\frac{\varkappa_1}{\mu_1}+\frac{1}{\mu_2}\right)\phi_1'(x)^- = \frac{2(\varkappa_2+1)}{\mu_2}\,g(x)$$

Eigenvalue problem: ~~Homogeneous~~ solution $\rightarrow$ $g(z)=0$

take $\phi_1'(z) = A z^p$

$$x^+ \rightarrow z = r e^{i\pi}$$
$$x^- \rightarrow z = r e^{-i\pi}$$

$$\left(\frac{\varkappa_2}{\mu_2}+\frac{1}{\mu_1}\right)\cancel{A r}^p e^{ip\pi} + \left(\frac{\varkappa_1}{\mu_1}+\frac{1}{\mu_2}\right)\cancel{A r}^p e^{-ip\pi} = 0$$

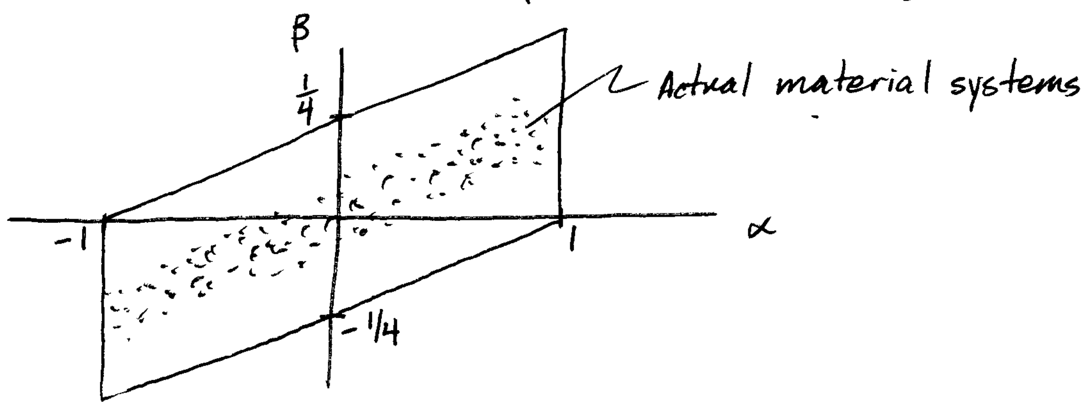
$$e^{2ip\pi} = -\frac{\varkappa_1/\mu_1 + 1/\mu_2}{\varkappa_2/\mu_2 + 1/\mu_1}$$

$$2ip\pi = \ln\left[\frac{\varkappa_1/\mu_1 + 1/\mu_2}{\varkappa_2/\mu_2 + 1/\mu_1}\, e^{in\pi}\right] \qquad n = \text{odd integers}$$

$$\therefore \boxed{\begin{aligned} p &= \frac{n}{2} - \frac{i}{2\pi}\ln\left[\frac{\varkappa_1/\mu_1 + 1/\mu_2}{\varkappa_2/\mu_2 + 1/\mu_1}\right] \\ p &= \frac{n}{2} - i\varepsilon \qquad n=\text{odd integers}\end{aligned}}$$

Dunder's Parameters: $\alpha = \dfrac{E_1' - E_2'}{E_1' + E_2'}$, $\beta = \dfrac{1}{2}\left[\dfrac{\mu_1(1-2\nu_2) - \mu_2(1-2\nu_1)}{\mu_1(1-\nu_2)+\mu_2(1-\nu_1)}\right]$ ← plane strain

$$\therefore \varepsilon = \frac{1}{2\pi}\ln\left[\frac{1-\beta}{1+\beta}\right] = \frac{1}{2\pi}\ln\left[\frac{\varkappa_1/\mu_1 + 1/\mu_2}{\varkappa_2/\mu_2 + 1/\mu_1}\right]$$



The particular solution is given by

$$\left(\frac{\chi_2}{\mu_2}+\frac{1}{\mu_1}\right)\phi_1'(z) + \left(\frac{\chi_1}{\mu_1}+\frac{1}{\mu_2}\right)\phi_1'(z) = \frac{2(\chi_2+1)}{\mu_2}\, g(z)$$

$$\phi_1'(z) = \frac{2(\chi_2+1)\mu_1}{\mu_1(\chi_2+1)+\mu_2(\chi_1+1)}\, g(z)$$

$\therefore$ The total solution for $\phi_1'(z)$ can be written as

$$\phi_1'(z) = \sum_{\substack{n \\ \text{odd}}} A_n\, z^{\frac{n}{2}-i\varepsilon} + \frac{2(\chi_2+1)\mu_1}{\mu_1(\chi_2+1)+\mu_2(\chi_1+1)} \sum_{m=0}^{\infty} B_m z^m$$

The most singular term with finite energy stored near the crack tip is associated with $n=-1$.

$$\therefore \quad \phi_1'(z) = A_{-1}\, z^{-\frac{1}{2}-i\varepsilon}$$

Take $\quad \sigma_{yy} + i\sigma_{xy} = \dfrac{K r^{i\varepsilon}}{\sqrt{2\pi r}} \quad$ on $\theta = 0$

(K is complex)

$$\therefore \quad \bar{\phi}_1'(\bar{z}) + \Omega_1'(z) = \frac{K r^{i\varepsilon}}{\sqrt{2\pi r}} \quad \text{on } \theta = 0$$

$$\bar{A}_{-1}\, r^{-1/2}\, r^{i\varepsilon} + \frac{\chi_1/\mu_1 + 1/\mu_2}{\chi_2/\mu_2+1/\mu_1}\, \bar{A}_{-1}\, r^{-1/2}\, r^{+i\varepsilon} = \frac{K r^{i\varepsilon}}{\sqrt{2\pi r}}$$

$$\bar{A}_{-1}\left[1 + \frac{\chi_1/\mu_1 + 1/\mu_2}{\chi_2/\mu_2+1/\mu_1}\right] = \frac{K}{\sqrt{2\pi}}$$

$$\bar{A}_{-1} = \frac{K}{\sqrt{2\pi}}\, \frac{\chi_2/\mu_2 + 1/\mu_1}{(\chi_2+1)/\mu_2 + (\chi_1+1)/\mu_1}$$

Due to the confusion over the use of analytic continuation, I thought it might be useful to use a more direct approach.

Let's assume $\phi_1' = A z^p \quad \Omega_1' = B z^p$

$$\phi_2' = C z^p \quad \Omega_2' = D z^p$$

where A, B, C, D and $\underline{\underline{p}}$ are complex

Traction continuity on $y=0$, $x>0$ or $z = re^{i\cdot 0}$

$$\rightarrow \phi_1'(z) + \bar{\Omega}_1'(\bar{z}) = \phi_2'(z) + \bar{\Omega}_2'(\bar{z}) \text{ on } z=\bar{z}=r>0$$

$$\rightarrow A r^p + \bar{B} r^{\bar{p}} = C r^p + \bar{D} r^{\bar{p}}$$

Here we have a problem because we cannot cancel out the r^p and $r^{\bar{p}}$. To fix this we can change our original assumption such that

$$\Omega_1' = B z^{\bar{p}} \text{ and } \Omega_2' = D z^{\bar{p}}$$

then traction continuity becomes:

$$A r^p + \bar{B} r^p = C r^p + \bar{D} r^p$$

$$A + \bar{B} = C + \bar{D} \quad (1)$$

Displacement continuity on $y=0$, $x>0$

$$\rightarrow \frac{1}{2\mu_1}\left[\kappa_1\phi_1(z) - \overline{\Omega}_1(\bar{z})\right] = \frac{1}{2\mu_2}\left[\kappa_2\phi_2(z) - \overline{\Omega}_2(\bar{z})\right]$$

on $z=\bar{z}=r>0$

Taking this equation and differentiating w.r.t. r

$$\frac{\kappa_1}{2\mu_1}A - \frac{1}{2\mu_1}\bar{B} = \frac{\kappa_2}{2\mu_2}C - \frac{1}{2\mu_2}\bar{D} \quad (2)$$

Solving (1) & (2) for B and D we find

$$B = \underbrace{\frac{\mu_1+\kappa_1\mu_2}{\mu_2-\mu_1}}_{a}\bar{A} - \underbrace{\frac{\mu_1+\kappa_2\mu_1}{\mu_2-\mu_1}}_{b}\bar{C}$$

$$D = \bar{A} - \bar{C} + a\bar{A} - b\bar{C}$$

Traction-free conditions on $z=re^{i\pi}$, $\bar{z}=re^{-i\pi}$

$$\phi_1'(z) + \overline{\Omega}_1'(\bar{z}) = 0 \quad \text{on } z=re^{i\pi}$$

$$Ae^{ip\pi} + aAe^{-ip\pi} - bCe^{-ip\pi} = 0$$

$$\rightarrow C = \frac{1}{b}\left(a+e^{i2\pi p}\right)A \quad (3)$$

Traction-free on $z=re^{-i\pi}$, $\bar{z}=re^{i\pi}$

$$\phi_2'(z) + \overline{\Omega}_2'(\bar{z}) = 0 \quad \text{on } z=re^{-i\pi}$$

$$(4)\quad Ce^{-i\pi p} + Ae^{i\pi p} - Ce^{i\pi p} + aAe^{i\pi p} - bCe^{i\pi p} = 0$$

$$(3)\to(4) \to \frac{a}{b}e^{-i\pi p} + \frac{1}{b}e^{i\pi p} + e^{i\pi p} - \frac{a}{b}e^{i\pi p} - \frac{1}{b}e^{i3\pi p} + \cancel{ae^{i\pi p}} - \cancel{ae^{i\pi p}} - e^{i3\pi p} = 0$$

Multiply by $e^{-i3\pi p}$ and define $x = {}^{-i2\pi p}$

$$-\left(\frac{1+b}{b}\right) + \left(\frac{1+b-a}{b}\right)x + \frac{a}{b}x^2 = 0$$

$$x = 1, \quad \underbrace{-\frac{1+b}{a} = -\frac{x_2/\mu_2 + 1/\mu_1}{x_1/\mu_1 + 1/\mu_2}}_{\text{These are the same solutions as on page (146)}}$$

$$x = 1 \to e^{-i2\pi p} = 1$$

$$\to \underbrace{p = \text{integers}}$$

These solutions with $p > 0$ correspond to the $g(z)$ solutions $p < 0$ are ruled out based on energy considerations

$$\boxed{\begin{aligned}
&\therefore \text{ For the asymptotic field}\\
&\phi_1'(z) = \frac{\bar{K}}{\sqrt{2\pi}} \frac{\varkappa_2/\mu_2 + 1/\mu_1}{(\varkappa_2+1)/\mu_2 + (\varkappa_1+1)/\mu_1}\, z^{-1/2 - i\varepsilon}\\
&\Omega_1'(z) = \frac{K}{\sqrt{2\pi}} \frac{\varkappa_1/\mu_1 + 1/\mu_2}{(\varkappa_2+1)/\mu_2 + (\varkappa_1+1)/\mu_1}\, z^{-\frac{1}{2} + i\varepsilon}\\
&\phi_2'(z) = \frac{\bar{K}}{\sqrt{2\pi}} \frac{\varkappa_1/\mu_1 + 1/\mu_2}{(\varkappa_2+1)/\mu_2 + (\varkappa_1+1)/\mu_1}\, z^{-1/2 - i\varepsilon}\\
&\Omega_2'(z) = \frac{K}{\sqrt{2\pi}} \frac{\varkappa_2/\mu_2 + 1/\mu_1}{(\varkappa_2+1)/\mu_2 + (\varkappa_1+1)/\mu_1}\, z^{-1/2 + i\varepsilon}
\end{aligned}}$$

Crack Opening/Sliding Displacement

$$(u_y + i u_x)_{\theta=\pi} - (u_y + i u_x)_{\theta=-\pi}$$
$$= K r^{i\varepsilon} \sqrt{\frac{r}{2\pi}} \, \frac{(\varkappa_1+1)\mu_2 + (\varkappa_2+1)\mu_1}{2\mu_1\mu_2} \, \frac{1}{(1+2i\varepsilon)\cosh(\pi\varepsilon)}$$

Energy Release Rate

$$\mathcal{G} = \frac{(\varkappa_1+1)\mu_2 + (\varkappa_2+1)\mu_1}{16\mu_1\mu_2} \, \frac{1}{\cosh^2(\pi\varepsilon)} \, K\bar{K}$$

$$\cosh(\pi\varepsilon) = \frac{e^{\pi\varepsilon} + e^{-\pi\varepsilon}}{2} = \frac{1}{2}\left[\frac{(\varkappa_1+1)/\mu_1 + (\varkappa_2+1)/\mu_2}{(\varkappa_2/\mu_2 + 1/\mu_1)\sqrt{\dfrac{\varkappa_1/\mu_1 + 1/\mu_2}{\varkappa_2/\mu_2 + 1/\mu_1}}}\right]$$

This type of stress field has features that are very different from the standard crack tip fields.

First, consider $\sigma_{yy} + i\sigma_{xy} = \dfrac{K r^{i\varepsilon}}{\sqrt{2\pi r}}$

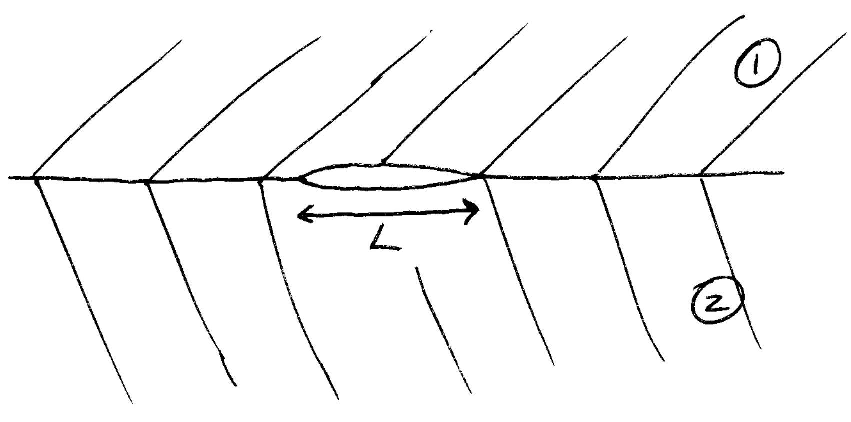
Recall: $r^{i\varepsilon} = e^{\ln r^{i\varepsilon}} = e^{i\varepsilon \ln r}$
$= \cos(\varepsilon \ln r) + i \sin(\varepsilon \ln r)$

$$\therefore \quad \sigma_{yy} = \frac{1}{\sqrt{2\pi r}}\left[\mathrm{Re}(K)\cos(\varepsilon \ln r) - \mathrm{Im}(K)\sin(\varepsilon \ln r)\right]$$

$$\sigma_{xy} = \frac{1}{\sqrt{2\pi r}}\left[\mathrm{Im}(K)\cos(\varepsilon \ln r) + \mathrm{Re}(K)\sin(\varepsilon \ln r)\right]$$

→ The stresses on $r, \theta = 0$ <u>oscillate</u>!

Next, consider the center crack problem.



$$K^{right} = \underbrace{(\sigma_{yy}^{\infty} + i\sigma_{xy}^{\infty})}_{Te^{i\psi}}(1+2i\varepsilon)L^{-i\varepsilon}\sqrt{\frac{\pi L}{2}}$$

$$T = \sqrt{\sigma_{yy}^{\infty\,2} + \sigma_{xy}^{\infty\,2}} \qquad \psi = \arctan\left(\frac{\sigma_{xy}^{\infty}}{\sigma_{yy}^{\infty}}\right)$$

Let's say we have two specimens with two different crack lengths, L_1 and L_2.
What are the relationships between T_1, T_2, ψ_1 and ψ_2 such that $K_1 = K_2$.

$$K_1 = K_2$$

$$T_1 e^{i\psi_1} \cancel{(1+2i\varepsilon)} L_1^{-i\varepsilon} \sqrt{\frac{\pi L_1}{\cancel{2}}} = T_2 e^{i\psi_2} \cancel{(1+2i\varepsilon)} L_2^{-i\varepsilon} \sqrt{\frac{\pi L_2}{\cancel{2}}}$$

$$T_1 \underbrace{e^{i\psi_1} L_1^{-i\varepsilon}}_{\text{sin's \& cos's}} \sqrt{L_1} = T_2 \underbrace{e^{i\psi_2} L_2^{-i\varepsilon}}_{\text{sin's \& cos's}} \sqrt{L_2}$$

$$\therefore \quad T_1 \sqrt{L_1} = T_2 \sqrt{L_2} \quad \leftarrow \text{Standard result}$$

$$\frac{T_1}{T_2} = \sqrt{\frac{L_2}{L_1}}$$

Ratio of traction magnitudes equals the inverse square root of the ratio of crack lengths.

But we also must have: $e^{i\psi_1} L_1^{-i\varepsilon} = e^{i\psi_2} L_2^{-i\varepsilon}$

$$e^{i\psi_1} e^{\ln L_1^{-i\varepsilon}} = e^{i\psi_2} e^{\ln L_2^{-i\varepsilon}}$$

$$e^{i\psi_1} e^{-i\varepsilon \ln L_1} = e^{i\psi_2} e^{-i\varepsilon \ln L_2}$$

$$e^{i(\psi_1 - \varepsilon \ln L_1)} = e^{i(\psi_2 - \varepsilon \ln L_2)}$$

$$\therefore \quad \psi_1 - \varepsilon \ln L_1 = \psi_2 - \varepsilon \ln L_2$$

$$\psi_2 = \psi_1 + \varepsilon \ln\left(\frac{L_2}{L_1}\right) \leftarrow$$

The angle of loading must change for $\varepsilon \neq 0$!