What about LSY conditions?

We can still take a J contour to closely surround the fracture process zone, in which case

$$J = \int_0^{\delta_t} \sigma(\delta)\, d\delta$$

where δ_t is the opening at the back edge of the process zone and this expression for J is valid even prior to propagation.

Propagation occurs when $\delta_t = \delta_c$ which corresponds to

$$J = \int_0^{\delta_c} \sigma(\delta)\, d\delta = \mathcal{G}_c$$

Hence, the model predicts propagation when J reaches a critical value of $\mathcal{G}_c$. Note that this is equivalent to a fracture criterion based on a critical crack opening displacement.

Also note that under LSY conditions $J \neq \frac{K_I^2}{E'}$

For the Dugdale model in the center-cracked panel

$$J \Big/ \left(\frac{K_I^2}{E'}\right) \approx 1 + \frac{\pi^2}{24}\left(\frac{\sigma}{\sigma_0}\right)^2 + \ldots$$

Note $J > \frac{K_I^2}{E'}$ in LSY

The HRR fields

Hutchinson JMPS v. 16 pp. 13-31, 1968.
Rice and Rosengren JMPS v. 16 pp. 1-12, 1968.

Asymptotic crack tip fields in a power-law hardening non-linear elastic material (a good model for elastic-plastic materials for proportional loading).

- Deformation theory plasticity
  1) valid for proportional loading, i.e. at any material point the stress history can be written as $\sigma_{ij}(t) = \alpha(t)\,\bar{\sigma}_{ij}$ where α is a monotonically increasing function of time and $\bar{\sigma}_{ij}$ is a constant tensor.

  2) hydrostatic stress does not cause plastic deformation

  $$\rightarrow \quad \sigma_{kk} = \frac{E}{1-2\nu}\,\varepsilon_{kk} \qquad \text{elastic hydrostatic response}$$

  3) Near the crack tip plastic strains will overwhelm elastic strains and so we will neglect <u>deviatoric</u> elastic strains.

  $$e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\,, \quad s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}$$

  $$\bar{\sigma} = \sqrt{\tfrac{3}{2}s_{ij}s_{ij}}\,, \quad \bar{\varepsilon} = \sqrt{\tfrac{2}{3}e_{ij}e_{ij}}$$

  $$\bar{\sigma} = \sigma_0\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^N \quad \text{or} \quad \bar{\varepsilon} = \varepsilon_0\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{1/N}$$

  $$e_{ij} = \frac{3}{2}\frac{\bar{\varepsilon}}{\bar{\sigma}}\,s_{ij} \qquad (J_2 \text{ deformation theory})$$

$\bar{\sigma}$ = uniaxial effective stress or von Mises stress
$\bar{\varepsilon}$ = uniaxial effective plastic strain

Non-linear constitutive law:

$$\varepsilon_{ij} = \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij} + \frac{3}{2}\varepsilon_0\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{\frac{1-N}{N}} s_{ij}/\sigma_0$$

or $$\sigma_{ij} = \frac{E}{3(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \frac{2}{3}\sigma_0\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{N-1}\frac{e_{ij}}{\varepsilon_0}$$

Strain energy density $W = \int_0^{\varepsilon} \sigma_{ij}\, d\varepsilon_{ij}$

$$W = \int_0^{\varepsilon_{ij}} \frac{E}{3(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \frac{2}{3}\sigma_0\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{N-1}\frac{e_{ij}}{\varepsilon_0}\, d\varepsilon_{ij}$$

$$= \int_0^{\varepsilon_{kk}} \frac{E}{3(1-2\nu)}\varepsilon_{kk}\, d\varepsilon_{jj} + \int_0^{e_{ij}} \frac{2}{3}\sigma_0\left(\frac{\sqrt{\frac{2}{3}e_{kl}e_{kl}}}{\varepsilon_0}\right)^{N-1}\frac{e_{ij}}{\varepsilon_0}\, de_{ij}$$

$$= \frac{E}{6(1-2\nu)}(\varepsilon_{kk})^2 + \int_0^{e_{ij}} \frac{2}{3}\sigma_0\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{N-1}\frac{e_{ij}}{\varepsilon_0}\, de_{ij}$$

Consider $d(\bar{\varepsilon}^N) = N\bar{\varepsilon}^{N-1}\frac{\partial\bar{\varepsilon}}{\partial e_{ij}}\, de_{ij}$

$$\frac{\partial\bar{\varepsilon}}{\partial e_{ij}} = \frac{\partial}{\partial e_{ij}}\left(\frac{2}{3}e_{kl}e_{kl}\right)^{1/2} = \frac{1}{2}\left(\frac{2}{3}e_{kl}e_{kl}\right)^{-1/2}\frac{2}{3}(\delta_{ik}\delta_{jl}e_{kl} + e_{kl}\delta_{ik}\delta_{jl})$$

$$= \frac{2}{3}\frac{e_{ij}}{\bar{\varepsilon}}$$

$$\therefore\ d(\bar{\varepsilon}^N) = N\bar{\varepsilon}^{N-1}\frac{2}{3}\frac{e_{ij}}{\bar{\varepsilon}}de_{ij} = \frac{2}{3}N\bar{\varepsilon}^{N-2}e_{ij}\,de_{ij}$$

$$\rightarrow d(\bar{\varepsilon}^{N+1}) = \frac{2}{3}(N+1)\bar{\varepsilon}^{N-1}e_{ij}\,de_{ij}$$

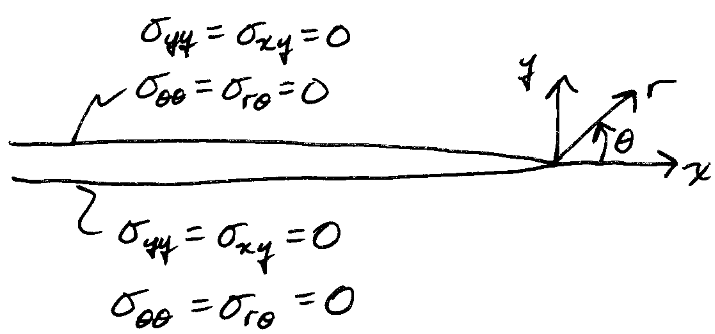
$$\therefore \int_0^{e_{ij}} \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^N}\bar{\varepsilon}^{N-1} e_{ij}\, de_{ij}$$

$$= \frac{2}{3}\frac{\sigma_0}{\varepsilon_0^N}\frac{3}{2(N+1)}\int_0^{\bar{\varepsilon}} d\left(\bar{\varepsilon}^{N+1}\right)$$

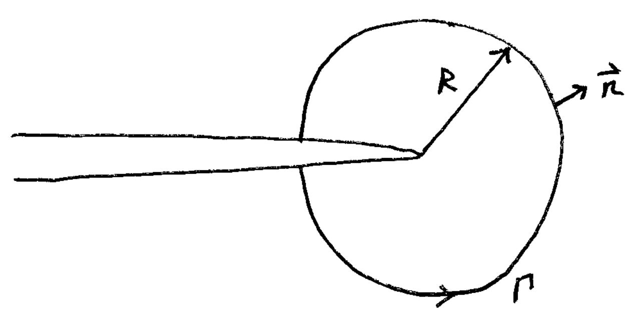
$$= \frac{\sigma_0 \varepsilon_0}{N+1}\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{N+1}$$

$$\therefore \boxed{W = \frac{E}{6(1-2\nu)}\left(\varepsilon_{kk}\right)^2 + \frac{\sigma_0 \varepsilon_0}{N+1}\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{N+1}}$$

Now consider a crack in such a material.



We know that J in a non-linear elastic material must be path-independent. Consider a circular J contour centered on the crack tip.



$n_x = \cos\theta$

$n_y = \sin\theta$

$$J = \int_\Gamma W n_x - \sigma_{ij} n_j u_{i,x}\, d\Gamma$$

$$\therefore J = \int_{-\pi}^{\pi}\left[W\cos\theta - \left(\sigma_{xx}u_{x,x}\cos\theta + \sigma_{xy}u_{x,x}\sin\theta + \sigma_{yx}u_{y,x}\cos\theta + \sigma_{yy}u_{y,x}\sin\theta\right)\right]R\,d\theta$$

In order for this integral to be path-independent, i.e. independent of the path radius R, the quantity in paretheses must have a $1/R$ dependence.

Consider $W = \frac{E}{6(1-2\nu)}(\varepsilon_{kk})^2 + \frac{\sigma_0\varepsilon_0}{N+1}\left(\frac{\bar{\varepsilon}}{\varepsilon_0}\right)^{N+1}$

$W \to \frac{1}{r}$ implies $\varepsilon_{kk} \to \frac{1}{\sqrt{r}}$ or $\bar{\varepsilon} \to \left(\frac{1}{r}\right)^{1/N+1}$ or both.

$\sigma_{kk} \to \frac{1}{\sqrt{r}}$ $\qquad \bar{\sigma} \to \left(\frac{1}{r}\right)^{N/N+1}$

Consider $u_{i,j}$:

$u_{x,x} = Ar^{-1/2} + Br^{-1/N+1}$, $u_{y,y} = Cr^{-1/2} + Dr^{-1/N+1}$, $u_{x,y} = Er^{-1/2} + Fr^{-1/N+1}$, $u_{y,x} = Gr^{-1/2} + Hr^{-1/N+1}$

Not allowed b/c $r^{-2/N+1}$ is more singular than r^{-1} for $0 \le N < 1$

$\varepsilon_{kk} \to \frac{1}{\sqrt{r}} \to (A+C)r^{-1/2} + (B+D)r^{-1/N+1} \to \frac{1}{\sqrt{r}} \quad \therefore B = -D$

$e_{ij} \to \left(\frac{1}{r}\right)^{1/N+1}$: $\quad e_{xy} = \varepsilon_{xy} = \frac{1}{2}\left[(E+G)r^{-1/2} + (F+H)r^{-1/N+1}\right] \quad \therefore E = -G$

$e_{xx} = \varepsilon_{xx} - \frac{\varepsilon_{kk}}{3} = Ar^{-1/2} + Br^{-1/N+1} - \frac{1}{3}(A+C)r^{-1/2} \quad \therefore 2A = C$

$e_{yy} = \varepsilon_{yy} - \frac{\varepsilon_{kk}}{3} = Cr^{-1/2} - Br^{-1/N+1} - \frac{1}{3}(A+C)r^{-1/2} \quad \therefore 2C = A$

otherwise hydrostatic components dominate W $\quad A = C = 0$

$\therefore \varepsilon_{kk} = 0$ for both the $\frac{1}{\sqrt{r}}$ and $\left(\frac{1}{r}\right)^{1/N+1}$ terms

Hence, the dominant strain term is like $\left(\frac{1}{r}\right)^{1/N+1}$

$$\boxed{\begin{aligned} \therefore\ \varepsilon_{ij} &= \left(\frac{1}{r}\right)^{1/N+1}\tilde{\varepsilon}_{ij}(\theta) \text{ with } \tilde{\varepsilon}_{kk}(\theta) = 0 \text{ as noted above} \\ \therefore\ \sigma_{ij} &= \left(\frac{1}{r}\right)^{N/N+1}\hat{\sigma}_{ij}(\theta) \end{aligned}}$$

Note that $\hat{\sigma}_{kk}(\theta) \neq 0$ because it leads to a volume strain term like $\varepsilon_{kk} = \frac{1-2\nu}{E}\left(\frac{1}{r}\right)^{N/N+1}\hat{\sigma}_{kk}(\theta)$ which is <u>less</u> singular than $\left(\frac{1}{r}\right)^{1/N+1}$ and is therefore allowed.

$\therefore$ the stress and strain singularities go like

$$\sigma_{ij} = \left(\frac{1}{r}\right)^{N/N+1} \hat{\sigma}_{ij}(\theta) \quad \text{and} \quad \varepsilon_{ij} = \left(\frac{1}{r}\right)^{1/N+1} \tilde{\varepsilon}_{ij}(\theta)$$

Check limits: $N=1 \rightarrow$ linear

$$\sigma_{ij} \rightarrow \left(\frac{1}{r}\right)^{1/2} \quad \text{and} \quad \varepsilon_{ij} \rightarrow \left(\frac{1}{r}\right)^{1/2}$$

$N=0 \rightarrow$ perfectly plastic

$$\sigma_{ij} \rightarrow \left(\frac{1}{r}\right)^{0}, \quad \varepsilon_{ij} \rightarrow \frac{1}{r}$$

These fields must satisfy equilibrium and compatibility and of course the material law.

Equilibrium is satisfied with Airy's stress function ϕ such that

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

$$\therefore \quad \phi = K r^{\frac{N+2}{N+1}} \tilde{\phi}(\theta)$$

scaling factor such that $\tilde{\sigma}(\theta)$ (to be introduced later) has a maximum value of unity

$$\therefore \quad \sigma_{rr} = K\left(\frac{1}{r}\right)^{N/N+1}\left[\frac{N+2}{N+1}\tilde{\phi} + \tilde{\phi}''\right]$$

$$\sigma_{\theta\theta} = K\left(\frac{1}{r}\right)^{N/N+1}\left[\frac{N+2}{(N+1)^2}\tilde{\phi}\right]$$

$$\sigma_{r\theta} = K\left(\frac{1}{r}\right)^{N/N+1}\left[\tilde{\phi}' - \frac{N+2}{N+1}\tilde{\phi}'\right] = K\left(\frac{1}{r}\right)^{N/N+1}\left[\frac{-1}{N+1}\tilde{\phi}'\right]$$

$$\bar{\sigma} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sqrt{\frac{3}{2}\left(s_{rr}^2 + s_{\theta\theta}^2 + s_{zz}^2 + 2s_{r\theta}^2\right)}$$

Plane strain $\rightarrow \quad \sigma_{zz} = \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})$

$$\begin{aligned}
\therefore \quad s_{rr} &= \sigma_{rr} - \tfrac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = \tfrac{1}{2}\sigma_{rr} - \tfrac{1}{2}\sigma_{\theta\theta} \\
s_{\theta\theta} &= \sigma_{\theta\theta} - \tfrac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = \tfrac{1}{2}\sigma_{\theta\theta} - \tfrac{1}{2}\sigma_{rr} \\
s_{zz} &= \sigma_{zz} - \tfrac{1}{3}(\sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}) = 0 \\
s_{r\theta} &= \sigma_{r\theta}
\end{aligned}$$

$$\rightarrow \quad \bar{\sigma} = \sqrt{\frac{3}{4}\underbrace{\left(\sigma_{rr}^2 - 2\sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}^2\right)}_{(\sigma_{rr} - \sigma_{\theta\theta})^2} + 3\sigma_{r\theta}^2}$$

$$\bar{\sigma} = K\left(\frac{1}{r}\right)^{N/N+1}\underbrace{\sqrt{\frac{3}{4}\left[\frac{N(N+2)}{(N+1)^2}\tilde{\phi} + \tilde{\phi}''\right]^2 + \frac{3}{(N+1)^2}\left(\tilde{\phi}'\right)^2}}_{\tilde{\sigma}(\theta) \text{ shorthand}}$$

Recall: $\varepsilon_{ij} = \frac{3}{2}\varepsilon_0\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{\frac{1-N}{N}}\frac{s_{ij}}{\sigma_0}$ when $\varepsilon_{kk} = 0$ ← This is true for our most singular term.

$$\varepsilon_{rr} = \frac{3}{2}\varepsilon_0 K^{1/N}\left(\frac{1}{r}\right)^{1/N+1}\frac{1}{\sigma_0^{1/N}}\underbrace{\tilde{\sigma}^{\frac{1-N}{N}}\frac{1}{2}\left[\frac{N(N+2)}{(N+1)^2}\tilde{\phi}+\tilde{\phi}''\right]}_{\hat{\varepsilon}_{rr}(\theta)}$$

$$\varepsilon_{\theta\theta} = \frac{3}{2}\varepsilon_0 K^{1/N}\left(\frac{1}{r}\right)^{1/N+1}\frac{1}{\sigma_0^{1/N}}\underbrace{\tilde{\sigma}^{\frac{1-N}{N}}\frac{1}{2}\left[-\frac{N(N+2)}{(N+1)^2}\tilde{\phi}-\tilde{\phi}''\right]}_{\hat{\varepsilon}_{\theta\theta}(\theta)=-\hat{\varepsilon}_{rr}(\theta)}$$

$$\varepsilon_{r\theta} = \frac{3}{2}\varepsilon_0 K^{1/N}\left(\frac{1}{r}\right)^{1/N+1}\frac{1}{\sigma_0^{1/N}}\underbrace{\tilde{\sigma}^{\frac{1-N}{N}}\left[\frac{-1}{N+1}\tilde{\phi}'\right]}_{\hat{\varepsilon}_{r\theta}(\theta)}$$

$$\varepsilon_{ij} = \underbrace{\frac{3}{2}\varepsilon_0\left(\frac{K}{\sigma_0}\right)^{1/N}\hat{\varepsilon}_{ij}(\theta)}_{\tilde{\varepsilon}_{ij}(\theta)\text{ from page 92}}\left(\frac{1}{r}\right)^{1/N+1}$$

Compatibility in polar coordinates:

$$\begin{cases}\varepsilon_{rr} = \dfrac{\partial u_r}{\partial r}\\[4pt] \varepsilon_{\theta\theta} = \dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\\[4pt] \varepsilon_{r\theta} = \dfrac{1}{2}\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}+\dfrac{1}{2}\dfrac{\partial u_\theta}{\partial r}-\dfrac{1}{2}\dfrac{u_\theta}{r}\end{cases}$$

$$\rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varepsilon_{\theta\theta})+\frac{1}{r^2}\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2}-\frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r}-\frac{2}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial\varepsilon_{r\theta}}{\partial\theta}\right)=0$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r^{\frac{N}{N+1}}\tilde{\varepsilon}_{\theta\theta}\right)+r^{\frac{-2N-3}{N+1}}\tilde{\varepsilon}''_{rr}+\frac{1}{N+1}r^{\frac{-2N-3}{N+1}}\tilde{\varepsilon}_{rr}-\frac{2}{r^2}\frac{\partial}{\partial r}\left(r^{\frac{N}{N+1}}\tilde{\varepsilon}'_{r\theta}\right)=0$$

$$-\frac{N}{(N+1)^2}r^{\frac{-2N-3}{N+1}}\tilde{\varepsilon}_{\theta\theta}+r^{\frac{-2N-3}{N+1}}\tilde{\varepsilon}''_{rr}+\frac{1}{N+1}r^{\frac{-2N-3}{N+1}}\tilde{\varepsilon}_{rr}-\frac{2N}{N+1}r^{\frac{-2N-3}{N+1}}\tilde{\varepsilon}'_{r\theta}=0$$

$$\therefore -\frac{N}{(N+1)^2}\tilde{\varepsilon}_{\theta\theta}+\tilde{\varepsilon}''_{rr}+\frac{1}{N+1}\tilde{\varepsilon}_{rr}-\frac{2N}{N+1}\tilde{\varepsilon}'_{r\theta}=0$$

$\rightarrow$ This is a non-linear ODE for $\tilde{\phi}(\theta)$ (implicitly)

Using $\tilde{\varepsilon}_{\theta\theta} = -\tilde{\varepsilon}_{rr}$

$$\longrightarrow \boxed{\tilde{\varepsilon}_{rr}'' + \frac{2N+1}{(N+1)^2}\tilde{\varepsilon}_{rr} - \frac{2N}{N+1}\tilde{\varepsilon}_{r\theta}' = 0}$$

define: $\hat{\sigma}(\theta) = \tilde{\sigma}(\theta)^2 = \frac{3}{4}\left[\tilde{\phi}'' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}\right]^2 + \frac{3}{(N+1)^2}\tilde{\phi}'^2$

$$\therefore\ \hat{\sigma}' = \frac{3}{2}\left[\tilde{\phi}'' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}\right]\left[\tilde{\phi}''' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}'\right] + \frac{6}{(N+1)^2}\tilde{\phi}'\tilde{\phi}''$$

$$\hat{\sigma}'' = \frac{3}{2}\left[\tilde{\phi}''' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}'\right]^2 + \frac{3}{2}\left[\tilde{\phi}'' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}\right]\left[\tilde{\phi}'''' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}''\right] + \frac{6}{(N+1)^2}\left[\tilde{\phi}''^2 + \tilde{\phi}'\tilde{\phi}'''\right]$$

Then, after some calculus & algebra, the governing ODE is

$$\boxed{\begin{aligned}&\tilde{\phi}'''' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}'' + \frac{1-N}{N}\frac{\hat{\sigma}'}{\hat{\sigma}}\left[\tilde{\phi}''' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}'\right]\\ &+ \left[\frac{1-N}{2N}\frac{\hat{\sigma}''}{\hat{\sigma}} + \frac{1-N}{2N}\frac{1-3N}{2N}\left(\frac{\hat{\sigma}'}{\hat{\sigma}}\right)^2\right]\left[\tilde{\phi}'' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}\right]\\ &+ \frac{2N+1}{(N+1)^2}\left[\tilde{\phi}'' + \frac{N(N+2)}{(N+1)^2}\tilde{\phi}\right] + \frac{2N}{(N+1)^2}\left[2\tilde{\phi}'' + \frac{1-N}{N}\frac{\hat{\sigma}'}{\hat{\sigma}}\tilde{\phi}'\right] = 0\end{aligned}}$$

BCs: $\tilde{\phi}(\theta=0) = 1$ and can be re-normalized later

$\tilde{\phi}'(\theta=0) = 0$ b/c $\tilde{\sigma}_{r\theta}(\theta=0) = 0$ due to Mode I symmetry

$\tilde{\phi}'''(\theta=0) = 0$ b/c $\tilde{\sigma}_{rr}'(\theta=0) = 0$ due to Mode I symmetry

$$\left\{\begin{array}{ll}\tilde{\phi}(\theta=\pi) = 0 & \text{b/c } \tilde{\sigma}_{\theta\theta}(\theta=\pi) = 0 \text{ on crack face}\\ \tilde{\phi}'(\theta=\pi) = 0 & \text{b/c } \tilde{\sigma}_{r\theta}(\theta=\pi) = 0 \text{ on crack face}\end{array}\right.$$

↳ These two conditions imply one another due to the path-independence of J. See Rice & Rosengren.