So our elastic-plastic solution for stresses is

$$\left.\begin{aligned} \sigma_{\theta' z} &= \frac{K_{III}}{\sqrt{2\pi r'}} \cos\frac{\theta'}{2} \\ \sigma_{r' z} &= \frac{K_{III}}{\sqrt{2\pi r'}} \sin\frac{\theta'}{2} \end{aligned}\right\} \text{ for } r' > R_p^*$$

$$\left.\begin{aligned} \sigma_{\theta z} &= \tau_0 \\ \sigma_{rz} &= 0 \end{aligned}\right\} \text{ for } r' < R_p^* \qquad R_p^* = \frac{1}{2\pi}\left(\frac{K_{III}}{\tau_0}\right)^2$$

* Note the solution is easiest to write in this way using mixed coordinate systems.

We also need continuous displacement across the e-p boundary.

Elastic solution $\rightarrow$ $w = \frac{K_{III}}{\mu}\sqrt{\frac{2r'}{\pi}} \sin\frac{\theta'}{2}$

Plastic solution : For proportional loading the shear strains will be in the same ratio as the shear stresses.

λ can depend on position

i.e. $\varepsilon_{xz} = \lambda\sigma_{xz}$, $\varepsilon_{yz} = \lambda\sigma_{yz}$

or $\varepsilon_{rz} = \lambda\sigma_{rz}$, $\varepsilon_{\theta z} = \lambda\sigma_{\theta z}$

Compatibility $\rightarrow$ $\varepsilon_{rz} = \frac{1}{2}\frac{\partial w}{\partial r}$, $\varepsilon_{\theta z} = \frac{1}{2}\frac{1}{r}\frac{\partial w}{\partial \theta}$

$$\therefore \quad \frac{1}{2}\frac{\partial^2 w}{\partial r \partial \theta} = \frac{1}{2}\frac{\partial}{\partial r}\left(r\,\frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

$$\rightarrow \quad \frac{\partial \varepsilon_{rz}}{\partial \theta} - \frac{\partial}{\partial r}\left(r\,\varepsilon_{\theta z}\right) = 0$$

$$\frac{\partial \varepsilon_{\theta z}}{\partial r} + \frac{\varepsilon_{\theta z}}{r} - \frac{1}{r}\frac{\partial \varepsilon_{rz}}{\partial \theta} = 0$$

However, in the plastic region $\varepsilon_{rz} = \lambda \sigma_{rz} = 0$
$\varepsilon_{\theta z} = \lambda \sigma_{\theta z} = \lambda \tau_0$

$$\therefore \text{ since } \varepsilon_{rz} = 0 \rightarrow \underbrace{\frac{\partial \varepsilon_{\theta z}}{\partial r} + \frac{\varepsilon_{\theta z}}{r} = 0}_{\text{equidimensional}}$$

$$\therefore \quad \varepsilon_{\theta z} = A(\theta)\, r^{p}$$

$$\frac{\partial \varepsilon_{\theta z}}{\partial r} = A(\theta)\, p\, r^{p-1}$$

$$p + 1 = 0$$

$$p = -1$$

$$\therefore \quad \varepsilon_{\theta z} = A(\theta)\frac{1}{r}$$

Strain has a $\frac{1}{r}$ singularity. Recall J-integral argument implies $\sigma \cdot \varepsilon \rightarrow \frac{1}{r}$ and for perfect plasticity $\sigma \rightarrow r^{0}$ $\therefore$ $\varepsilon \rightarrow r^{-1}$.

$$\varepsilon_{\theta z} = \frac{1}{2}\frac{1}{r}\frac{\partial w}{\partial \theta} = A(\theta)\frac{1}{r}$$

$$\therefore \quad \frac{\partial w}{\partial \theta} = 2A(\theta)$$

$$\varepsilon_{rz} = \frac{1}{2}\frac{\partial w}{\partial r} = 0$$

$\therefore$ $w = f(\theta)$ in plastic zone.

Match w on e-p boundary.

$$\frac{K_{III}}{\mu}\sqrt{\frac{2R_p^*}{\pi}}\sin\frac{\theta'}{2} = f(\theta)$$

$$\frac{K_{III}}{\mu}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2\pi}}\frac{K_{III}}{\tau_0}\sin\theta = f(\theta)$$

$$\therefore f(\theta) = \frac{1}{\pi}\frac{K_{III}^2}{\mu\tau_0}\sin\theta$$

$$\therefore \boxed{\begin{array}{ll} w = \dfrac{1}{\pi}\dfrac{K_{III}^2}{\mu\tau_0}\sin\theta & \text{for } r' < R_p^* \\ & -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ w = \dfrac{K_{III}}{\mu}\sqrt{\dfrac{2r'}{\pi}}\sin\dfrac{\theta'}{2} & \text{for } r' > R_p^* \\ & -\pi \le \theta \le \pi \end{array}}$$

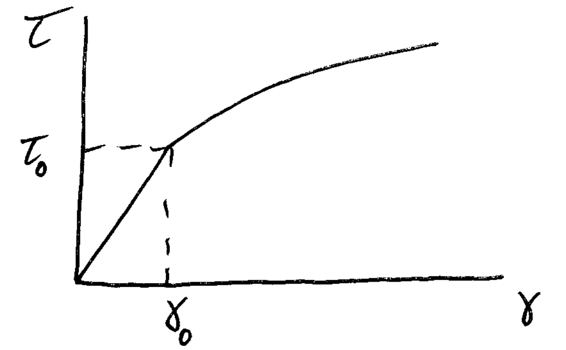
Crack tip sliding displacement $= w(\theta=\frac{\pi}{2}) - w(\theta=-\frac{\pi}{2})$

$$\delta_{tip} = CSD = \frac{1}{\pi}\frac{K_{III}^2}{\mu\tau_0}(1 - -1) = \frac{2}{\pi}\frac{K_{III}^2}{\mu\tau_0}$$

$$J = \frac{K_{III}^2}{2\mu} = \alpha\,\tau_0\,\delta_{tip} = \alpha\frac{2}{\pi}\frac{K_{III}^2}{\mu} \rightarrow \alpha = \frac{\pi}{4}$$

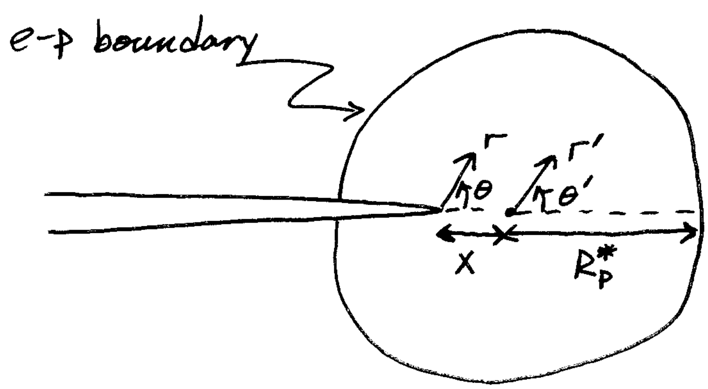
$$\therefore \boxed{J = \frac{\pi}{4}\tau_0\,\delta_{tip}}$$

Rice has also solved this problem for the elastic - power law hardening case.



$$\tau/\tau_0 = \gamma/\gamma_0 \quad \text{for } \tau \le \tau_0$$

$$\tau/\tau_0 = (\gamma/\gamma_0)^N \quad \text{for } \tau \ge \tau_0$$



$$R_p^* = \frac{1}{2\pi}\left(\frac{K_{III}}{\tau_0}\right)^2$$

$$X = \frac{1-N}{1+N} R_p^*$$

Note: $X = 0$ for $N = 1$
$X = R_p^*$ for $N = 0$

Again, outside the e-p boundary the elastic solution is:

$$\sigma_{r'z} = \frac{K_{III}}{\sqrt{2\pi r'}} \sin\frac{\theta'}{2}$$

$$\sigma_{\theta' z} = \frac{K_{III}}{\sqrt{2\pi r'}} \cos\frac{\theta'}{2}$$

Inside the plastic zone: $\sigma_{xz} = \tau_0 \left(\frac{R_p^*}{r}\right)^{\frac{N}{N+1}} \tilde{\sigma}_{xz}(\theta)$

$$\sigma_{yz} = \tau_0 \left(\frac{R_p^*}{r}\right)^{N/N+1} \tilde{\sigma}_{yz}(\theta)$$

$$\gamma_{xz} = \gamma_0 \left(\frac{R_p^*}{r}\right)^{1/N+1} \tilde{\gamma}_{xz}(\theta), \quad \gamma_{yz} = \gamma_0 \left(\frac{R_p^*}{r}\right)^{1/N+1} \tilde{\gamma}_{yz}(\theta)$$