Note that the equations for stress are singular when $\mathcal{D}=0$.

$$\mathcal{D} = (1+\alpha_s^2)^2 - 4\alpha_s\alpha_D = 0$$

When $v=0$, $\alpha_s=\alpha_D=1$ and $\therefore \mathcal{D}=0$. This singularity does not cause problems. In fact, the limiting procedure of $\lim_{v\to 0}\sigma_{ij}$ yields the asymptotic fields for the static crack tip solution.

For $v\neq 0$ we need to solve,

$$\left\{1+\left[1-\left(\tfrac{v}{c_s}\right)^2\right]\right\}^2 - 4\sqrt{\left[1-\left(\tfrac{v}{c_s}\right)^2\right]\left[1-\left(\tfrac{v}{c_D}\right)^2\right]} = 0$$

Ultimately this equation must be solved numerically and the solution can be written as

$$v = \underbrace{f(\nu)\,c_s}_{c_R \,\equiv\, \text{Rayleigh wave speed}} \quad \text{with } f(\nu) \approx \frac{0.862+1.14\nu}{1+\nu} \text{ for } \nu>0$$

Hence, this solution <u>suggests</u> that cracks must grow at speeds slower than the Rayleigh wave speed.

Accurate resuts for c_R :

ν	c_R/c_s
0.5	0.955313
0.3	0.927413
0.2	0.910996
0	0.874032
-0.5	0.774239
-1	0.688892

A Path-Independent Integral for Steady Crack Growth

$$U = \int_0^{\varepsilon} \sigma_{ij}\, d\varepsilon_{ij} \quad = \text{ history dependent stress work density}$$

$$T = \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i \quad = \text{ kinetic energy density } (\rho \text{ uniform})$$

Consider $\int_{\Gamma} (U+T)n_1 - \sigma_{ij} n_j u_{i,1}\, dS$ under SS conditions

$$\int_{\Gamma} (U+T) n_1\, d\Gamma = \int_A U_{,1} + T_{,1}\, dA$$

$$= \int_A \underbrace{\sigma_{ij}\, \varepsilon_{ij,1}}_{\text{See page 123}} + \left(\tfrac{1}{2}\rho \dot{u}_i \dot{u}_i\right)_{,1} dA$$

$$= \int_A \sigma_{ij} u_{i,1j} + \rho \dot{u}_i \dot{u}_{i,1}\, dA$$

but $\dot{u}_i = -v u_{i,1}$ $\rightarrow$ $= \int_A (\sigma_{ij} u_{i,1})_{,j} - \sigma_{ij,j} u_{i,1} - v\rho u_{i,1} \dot{u}_{i,1}\, dA$

but $\ddot{u}_i = -v \dot{u}_{i,1}$ $\rightarrow$ $= \int_A (\sigma_{ij} u_{i,1})_{,j} - \underbrace{(\sigma_{ij,j} - \rho \ddot{u}_i)}_{=0 \text{ by Newton's 2nd}} u_{i,1}\, dA$

$$= \int_{\Gamma} \sigma_{ij} n_j u_{i,1}\, d\Gamma$$ no siqularities in A.

$\therefore \int_{\Gamma} (U+T) n_1 - \sigma_{ij} n_j u_{i,1}\, d\Gamma = 0$ on a closed contour containing no singularities.

<u>Interpretation</u> of $\int_\Gamma (U+T)n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$

Let Γ move with velocity v in the x_1-direction.

$v \int_\Gamma (U+T) n_1 \, d\Gamma$ is the rate of flow of mechanical energy (work density plus kinetic) into Γ

$-v \int_\Gamma \sigma_{ij} n_j u_{i,1} \, d\Gamma$ is the rate of work done by the tractions acting on Γ.

For SS, the mechanical energy inside Γ must be constant and $\therefore$

$$\int_\Gamma (U+T)n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma = 0$$

in SS.

Now consider Γ as a contour around a crack tip. Standard arguments apply to prove path independance of $I = \int_\Gamma (U+T)n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$ around a crack tip with traction free crack faces.

In general $I \neq 0$ and I represents the energy flowing out of the crack tip.

i.e. $I = \mathcal{G}^D$ under SS conditions.

This integral can be used to determine $\mathcal{G}^D - K^D$ relationships for dynamic crack growth.

The Singularity in an Elastic-viscous Material

Constitutive Law: $\dot{\varepsilon}_{ij} = \frac{1+\nu}{E}\dot{\sigma}_{ij} - \frac{\nu}{E}\dot{\sigma}_{kk}\delta_{ij} + \underbrace{b\bar{\sigma}^{n-1}s_{ij}}_{\text{Power-law viscous (i.e. creep) term}}$

$$\bar{\sigma} = \sqrt{\tfrac{3}{2}s_{ij}s_{ij}}$$

For mathematical simplicity we will consider Mode III steady crack growth, but the arguments apply to in-plane modes as well.

Equilibrium: $\dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} = \rho \dfrac{\partial^2 w}{\partial t^2}$

Steady State $\rightarrow$ $\dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} = \rho v^2 \dfrac{\partial^2 w}{\partial x^2} = 2\rho v^2 \dfrac{\partial \varepsilon_{xz}}{\partial x}$

Constitutive Law + SS $\rightarrow$ $-v\dfrac{\partial \varepsilon_{xz}}{\partial x} = -\dfrac{v}{2\mu}\dfrac{\partial \sigma_{xz}}{\partial x} + b\bar{\sigma}^{n-1}\sigma_{xz}$

$$-v\frac{\partial \varepsilon_{yz}}{\partial x} = -\frac{v}{2\mu}\frac{\partial \sigma_{yz}}{\partial x} + b\bar{\sigma}^{n-1}\sigma_{yz}$$

$$\bar{\sigma} = \sqrt{3(\sigma_{xz}^2 + \sigma_{yz}^2)}$$

Assume stresses of the form (asymptotic fields)

$$\sigma_{iz} = \frac{1}{r^p}\tilde{\sigma}_{iz}(\theta) \qquad i = x, y \,.\ \text{as } r \to 0$$

We will now proceed to develop arguments to determine p.

First, assume that near the crack tip the elastic strain rates, $\dot{\varepsilon}^e_{ij}$, dominate the viscous strain rates, $\dot{\varepsilon}^v_{ij}$, i.e.

$$\lim_{r \to 0} \sqrt{\dot{\varepsilon}^v_{ij}\dot{\varepsilon}^v_{ij}} \Big/ \sqrt{\dot{\varepsilon}^e_{ij}\dot{\varepsilon}^e_{ij}} = 0$$

This assumption then implies that the material behaves elastically near the crack tip, and the asymptotic fields are identical to the dynamic elastic fields, i.e.

$$\sigma_{iz} \sim \frac{1}{\sqrt{r}}\,\tilde{\sigma}_{iz}(\theta)$$

Next, we must check the consistency of this result with the original assumption.

Recall, $\dot{(\;)} = -v\frac{\partial}{\partial x} = -v\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$

$$\dot{\varepsilon}^e_{ij} \sim \dot{\sigma}_{ij} \sim \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{r}}\hat{\sigma}_{iz}(\theta)\right) \sim \frac{1}{r^{3/2}}$$

$$\dot{\varepsilon}^v_{ij} \sim \bar{\sigma}^n \sim \frac{1}{r^{n/2}}$$

$$\therefore \sqrt{\dot{\varepsilon}^e_{ij}\dot{\varepsilon}^e_{ij}} \gg \sqrt{\dot{\varepsilon}^v_{ij}\dot{\varepsilon}^v_{ij}} \quad \text{if} \quad 3 > n$$

So, it is possible to have $p = 1/2$ if $n < 3$. Now we will look at the case of viscous strain rates dominating elastic rates.

Next, assume $\lim_{r \to 0} \sqrt{\dot{\varepsilon}^e_{ij} \dot{\varepsilon}^e_{ij}} \Big/ \sqrt{\dot{\varepsilon}^v_{ij} \dot{\varepsilon}^v_{ij}} = 0$

$\therefore$ the constitutive law $\rightarrow$ $v \left| \frac{\partial \varepsilon_{iz}}{\partial x} \right| \sim b \bar{\sigma}^{n-1} |\sigma_{iz}| \gg \frac{v}{2\mu} \left| \frac{\partial \sigma_{iz}}{\partial x} \right|$

but if this is true them equilibrium implies

$$\frac{\partial \varepsilon_{xz}}{\partial x} \sim 0 \quad \text{which is not}$$

consistent with the constitutive law argument.

Therefore, the only other possibility is that the viscous and elastic rates are of the same order, i.e.

$$\sqrt{\dot{\varepsilon}^e_{ij} \dot{\varepsilon}^e_{ij}} \sim \sqrt{\dot{\varepsilon}^v_{ij} \dot{\varepsilon}^v_{ij}}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r^p}\right) \sim \frac{1}{r^{np}}$$

$$\therefore \; p + 1 = np$$

$$p = \frac{1}{n-1}$$

Can $p = \frac{1}{n-1}$ be valid for $n < 3$?

Check: $\sigma \sim r^{\frac{-1}{n-1}} \rightarrow \varepsilon \sim r^{\frac{-1}{n-1}}$
$\rightarrow \sigma \varepsilon \sim r^{\frac{-2}{n-1}} \gg r^{-1}$ if $n < 3$

$$\therefore \; p = \frac{1}{n-1} \rightarrow \mathcal{G} \rightarrow \infty \quad \text{if } n < 3$$

$$\therefore \quad \boxed{\begin{aligned} p &= \tfrac{1}{2} \quad \text{if} \quad n < 3 \\ p &= \frac{1}{n-1} \quad \text{if} \quad n \geq 3 \end{aligned}}$$

The angular dependance of the fields can then be determined in a fashion similar to how we determined the HRR fields. In other words, we can find the non-linear equations governing the Angular dependence for these p values and then solve these equations with a shooting method (i.e. numerically).

For $n<3$ and $\therefore p=\frac{1}{2}$ the asymptotic solution [redacted] is like the dynamic elastic asymptotic field and there is an undetermined parameter analogous to K.

However, for $n>3$ and $p=\frac{1}{n-1}$ there is actually no undetermined parameter analogous to K. See Freund's book for a more detailed discussion on pages 206-214.