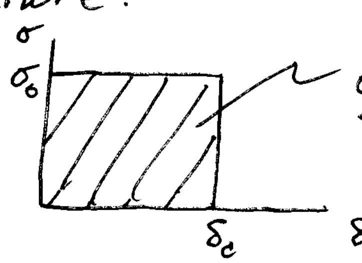
The Dugdale-Barenblatt Model

K is related to the coefficient on the singular $1/\sqrt{r}$ stress terms arising near a crack tip in a linear elastic material. However, we have noted that real materials cannot support infinite stresses. Why can K still be used as a fracture criterion if it doesn't represent reality?

Consider the following model for the non-linear behavior near the crack tip.

We will only analyze non-linear behavior on the plane ahead of the crack tip and we will assume that σ_{yy} is limited to a peak value of σ_0.

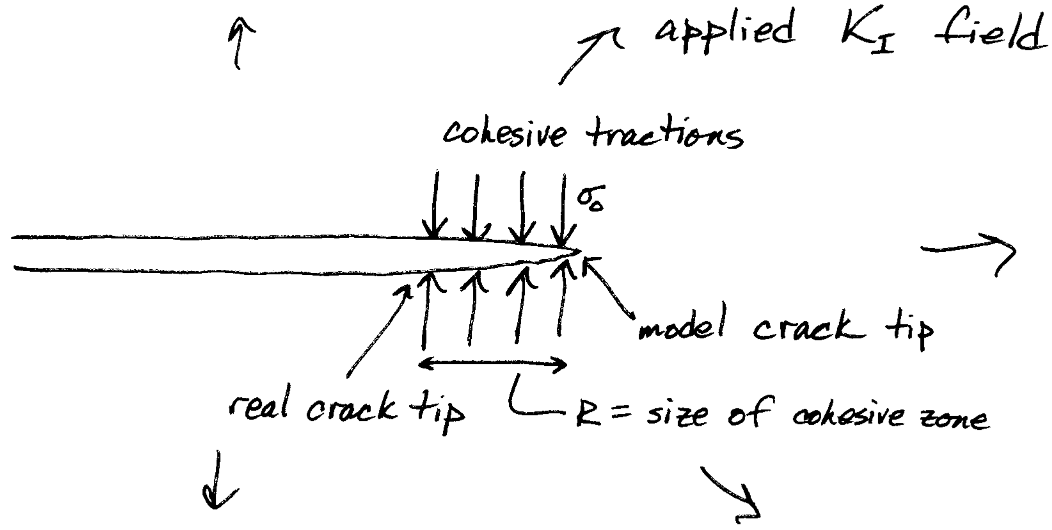
This is a so-called cohesive zone law or fracture process zone law. It can be used to represent a vast number of failure mechanisms including plastic tearing in plane stress, atomic decohesion, and fiber pull-out in composite fracture.



$\mathcal{G}_c = \sigma_0 \delta_c$

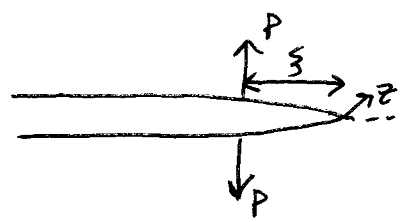
Fracture energy of the material.

The crack model look like



Note that we are beginning immediately with SSY conditions. We can set our origin at the real crack tip or the model crack tip. Since we will be focusing on the model crack tip we will set $z = x + iy = 0$ there.

Recall :
(You will show this in HW#2)



$$Z_I = \frac{P\sqrt{\xi}}{\pi\sqrt{z}\,(z+\xi)}$$

$$K_I = P\sqrt{\frac{2}{\pi\xi}}$$

At the model crack tip $K_I = K_{Applied} +$ K due to cohesive tractions.

$$\therefore\ K_{tip} = K_I - \int_0^R \frac{\sigma_0\sqrt{2}}{\sqrt{\pi}\ \sqrt{\xi}}\, d\xi .$$

b/c cohesive tractions are negative, i.e. they close the crack

Now, if $K_{tip} \neq 0$ then σ_{yy} ahead of the model tip will go to ∞, however this is prohibited by the cohesive zone model. $\therefore K_{tip} = 0$

$$K_I - \int_0^R \sigma_0 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\xi}} d\xi = 0$$

$$K_I - \sigma_0 \sqrt{\frac{2}{\pi}} \frac{4}{2} \xi^{1/2} \Big]_0^R = 0$$

$$K_I - \sigma_0 \sqrt{\frac{8}{\pi}} \sqrt{R} = 0$$

$$\therefore \boxed{R = \frac{\pi}{8}\left(\frac{K_I}{\sigma_0}\right)^2}$$

Dugdale plastic zone size

But when does the crack propagate?
The crack propagates when the COD at $z = -R = Re^{\pm i\pi}$ is equal to δ_c. Therefore we need to calculate the COD. To do this let's first compute Z_I.

$$Z_I = \frac{K_I}{\sqrt{2\pi z}} - \int_0^R \frac{\sigma_0 \sqrt{\xi}}{\pi \sqrt{z}\,(z+\xi)} d\xi$$

$$= \frac{K_I}{\sqrt{2\pi z}} - \frac{\sigma_0}{\pi\sqrt{z}}\left[2\sqrt{R} - 2\sqrt{z} \arctan\sqrt{\frac{R}{z}}\right]$$

$$= \cancel{\frac{K_I}{\sqrt{2\pi z}}} - \cancel{\frac{2\sigma_0}{\pi\sqrt{z}} \sqrt{\frac{\pi}{8}} \frac{K_I}{\sigma_0}} + \frac{2\sigma_0}{\pi} \arctan\sqrt{\frac{R}{z}}$$

$$\boxed{Z_I = \frac{2\sigma_0}{\pi} \arctan\sqrt{\frac{R}{z}}}$$

$$2\mu u_y = \frac{1}{2}(\chi+1)\,\mathrm{Im}\,\hat{Z}_I - y\,\mathrm{Re}\,Z_I$$

$$\hat{Z}_I = \frac{2\sigma_0}{\pi}\left[\sqrt{Rz} + z\arctan\sqrt{\frac{R}{z}} - R\arctan\sqrt{\frac{z}{R}}\right]$$

we need to be careful with arctan

Recall $\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$, $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$
these can be used to show

$$\arctan(x) = \frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right)$$

on the crack faces $z = re^{\pm i\pi}$, $y = 0^{\pm}$

$$\rightarrow \hat{Z}_I = \frac{2\sigma_0}{\pi}\left[\sqrt{Rr}\,e^{\pm i\frac{\pi}{2}} + re^{\pm i\pi}\frac{1}{2i}\ln\left(\frac{1+i\sqrt{\frac{R}{r}}e^{\mp i\frac{\pi}{2}}}{1-i\sqrt{\frac{R}{r}}e^{\mp i\frac{\pi}{2}}}\right)\right.$$

$$\left. - R\frac{1}{2i}\ln\left(\frac{1+i\sqrt{\frac{r}{R}}e^{\pm i\frac{\pi}{2}}}{1-i\sqrt{\frac{r}{R}}e^{\pm i\frac{\pi}{2}}}\right)\right]$$

Note:
$\mathrm{Re}\{\ln(\frac{1+x}{1-x})\}$
$= \ln|\frac{1+x}{1-x}|$
& $\mathrm{Re}\{\ln(\frac{1-x}{1+x})\}$
$= \ln|\frac{1-x}{1+x}| = -\ln|\frac{1+x}{1-x}|$
$\therefore \mathrm{Re}\{\ln(\frac{1\pm x}{1\mp x})\}$
$= \pm\ln|\frac{1+x}{1-x}|$

$$= \frac{2\sigma_0}{\pi}\left[\pm i\sqrt{Rr} + i\frac{r}{2}\ln\left(\frac{1\pm\sqrt{\frac{R}{r}}}{1\mp\sqrt{\frac{R}{r}}}\right)\right.$$

$$\left. + i\frac{R}{2}\ln\left(\frac{1\mp\sqrt{\frac{r}{R}}}{1\pm\sqrt{\frac{r}{R}}}\right)\right]$$

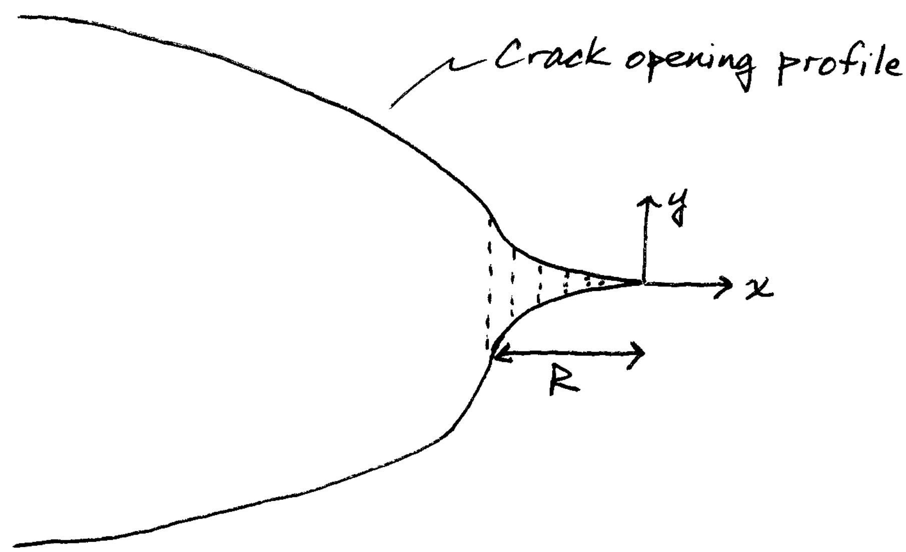
$$= \frac{2\sigma_0}{\pi}\left[\pm i\sqrt{Rr} + i\frac{r}{2}\ln\left(-\frac{1\pm\sqrt{\frac{r}{R}}}{1\mp\sqrt{\frac{r}{R}}}\right)\right.$$

$$\left. - i\frac{R}{2}\ln\left(\frac{1\pm\sqrt{\frac{r}{R}}}{1\mp\sqrt{\frac{r}{R}}}\right)\right]$$

$$\mathrm{Im}\,\hat{Z}_I = \frac{2\sigma_0}{\pi}\left[\pm\sqrt{Rr} \mp \frac{1}{2}(R-r)\ln\left|\frac{1+\sqrt{\frac{r}{R}}}{1-\sqrt{\frac{r}{R}}}\right|\right]$$

$$\therefore \; u_y(r, \theta = \pm\pi) = \frac{\varkappa+1}{4\mu}\frac{2\sigma_0}{\pi}\left[\pm\sqrt{Rr} \mp \frac{1}{2}(R-r)\ln\left|\frac{1+\sqrt{\frac{r}{R}}}{1-\sqrt{\frac{r}{R}}}\right|\right]$$

$$\frac{u_y}{R}(r, \theta=\pm\pi) = \pm\frac{4\sigma_0}{E'\pi}\left[\sqrt{\frac{r}{R}} - \frac{1}{2}\left(1-\frac{r}{R}\right)\ln\left|\frac{1+\sqrt{\frac{r}{R}}}{1-\sqrt{\frac{r}{R}}}\right|\right]$$



$$COD(r=R) = u_y(R,\pi) - u_y(R,-\pi) = \frac{8\sigma_0}{E'\pi}R$$

Propagation $\rightarrow$

$$\frac{8\sigma_0}{E'\pi}R = \delta_c$$

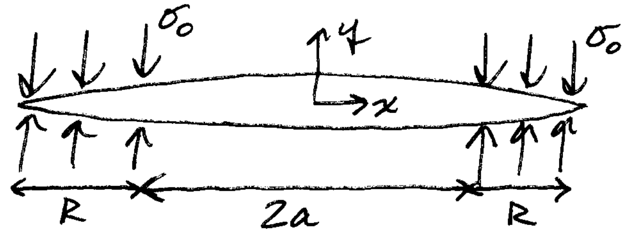
$$\frac{8\sigma_0}{E'\pi}\frac{\pi}{8}\frac{K_I^2}{\sigma_0^2} = \delta_c$$

$$\frac{K_I^2}{E'} = \sigma_0\delta_c$$

$$\mathcal{G} = \mathcal{G}_c$$

So, even though we have no singularity ahead of the crack tip we still get crack propagation when $\mathcal{G} = \mathcal{G}_c$ or equivalently when $K_I = K_c = \sqrt{\mathcal{G}_c E'} = \sqrt{\sigma_0 \delta_c E'}$.

A similar analysis can be made on the center crack geometry and we would find:



Remote $\sigma_{yy} = \sigma$

$$R = a\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right) - 1\right]$$

$$COD\,(x = \pm a) = \frac{8}{\pi}\frac{\sigma_0}{E'}\,a\,\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right]$$

In the limit as $R \ll a$ (i.e. $\sigma \ll \sigma_0$) and noting that $K_I = \sigma\sqrt{\pi(a+R)} \rightarrow \sigma\sqrt{\pi a}$ it can be shown that

$$COD(x = \pm a) \rightarrow \frac{\sigma^2 \pi a}{E'}\frac{1}{\sigma_0} = \delta_c$$

$$\rightarrow \frac{K_I^2}{E'} = \sigma_0 \delta_c$$

Under LSY propagation occurs when $\frac{8}{\pi}\frac{\sigma_0^2 a}{E'}\ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right] = \sigma_0\delta_c$ and under SSY propagation occurs when $\frac{\sigma^2 \pi a}{E'} = \sigma_0 \delta_c$.

We will introduce J soon, but these relationships imply

$$\frac{J}{J_{ssy}} = \frac{8}{\pi^2}\left(\frac{\sigma_0}{\sigma}\right)^2 \ln\left[\sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_0}\right)\right] \approx 1 + \frac{\pi^2}{24}\left(\frac{\sigma}{\sigma_0}\right)^2$$

LSY fracture criterion.

The J-integral

We will show that J is a path-independent integral and is equal to the energy release rate in non-linear elastic materials.

First we define the strain energy density in a non-linear elastic material as

$$W = \int_0^{\underset{\sim}{\varepsilon}} \sigma_{ij}\, d\varepsilon_{ij}$$

for linear elastic materials $W = \frac{1}{2} C_{ijkl}\, \varepsilon_{ij} \varepsilon_{kl}$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \qquad \text{i.e.} \quad W = \int_0^{\underset{\sim}{\varepsilon}} \frac{\partial W}{\partial \varepsilon_{ij}}\, d\varepsilon_{ij} = \int_0^{\underset{\sim}{\varepsilon}} dW$$

So our governing equations for a small deformation problem in this non-linear elastic material are

$\sigma_{ji,j} = 0$ Equilibrium, also $\sigma_{ji} n_j = t_i$ on S

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ Compatibility

$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$ Material Law

(* We could generalize to large deformations but we will not concern ourselves with this complication)

$J = \mathcal{G}$ Proof (Jin & Sun, IJF, 2004)

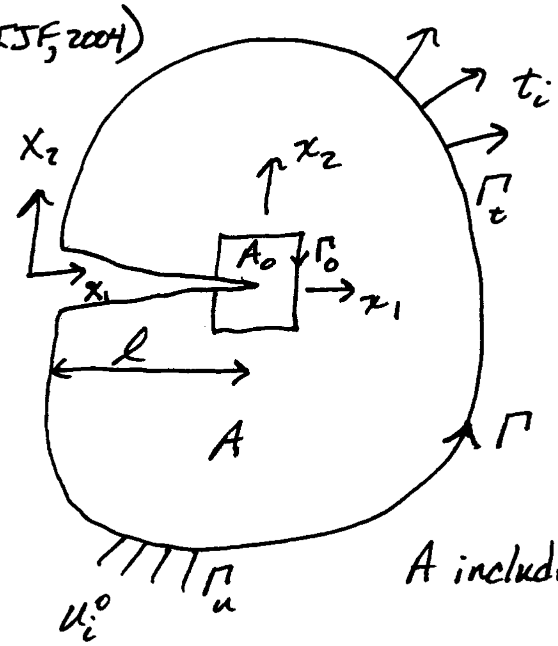
$\sigma_{ji,j} = 0$

$\sigma_{ji} n_j = t_i$ on Γ_t

$u_i = u_i^0$ on Γ_u

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$\sigma_{ji} = \partial W / \partial \varepsilon_{ij}$



A includes A_0

$$\Pi = \int_A W \, dA - \int_{\Gamma_t} t_i u_i \, d\Gamma$$

$$\mathcal{G} = -\frac{d\Pi}{dl} = -\frac{d}{dl}\int_A W \, dA + \underbrace{\int_{\Gamma} t_i \frac{du_i}{dl} \, d\Gamma}$$

$dt_i = 0$ on Γ_t

$du_i = 0$ on Γ_u

$t_i = 0$ on new crack surfaces associated with dl

$x_1 = X_1 - l$

$x_2 = X_2$

$$\frac{d}{dl} = \frac{\partial}{\partial l} + \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial l} = \frac{\partial}{\partial l} - \frac{\partial}{\partial x_1}$$

$$\mathcal{G} = -\int_{A-A_0} \frac{\partial W}{\partial l} \, dx_1 dx_2 + \int_{A-A_0} \frac{\partial W}{\partial x_1} \, dx_1 dx_2 - \frac{d}{dl}\int_{A_0} W \, dA$$

$$+ \int_{\Gamma} t_i \left(\frac{\partial u_i}{\partial l} - \frac{\partial u_i}{\partial x_1} \right) d\Gamma$$

Consider $\int_\Gamma t_i \frac{\partial u_i}{\partial l}\, d\Gamma = \int_{\Gamma+\Gamma_c} t_i \frac{\partial u_i}{\partial l}\, d\Gamma = \int_{\Gamma+\Gamma_c} \sigma_{ji} n_j \frac{\partial u_i}{\partial l}\, d\Gamma$

$\sigma_{ji}\frac{\partial u_i}{\partial l}$ non-singular $\longrightarrow$ $= \int_A \left(\sigma_{ji}\frac{\partial u_i}{\partial l}\right)_{,j} dx_1 dx_2$

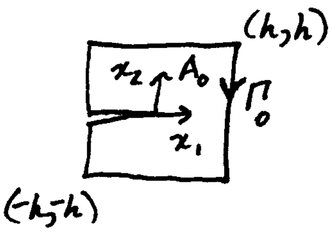
$$= \int_A \left(\overset{0}{\cancel{\sigma_{ji,j}}}\frac{\partial u_i}{\partial l} + \sigma_{ji}\frac{\partial \varepsilon_{ij}}{\partial l}\right) dx_1 dx_2$$

$$= \int_A \frac{\partial W}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial l}\, dx_1 dx_2 = \int_A \frac{\partial W}{\partial l}\, dx_1 dx_2$$

$$\rightarrow \mathcal{G} = \int_{A_0} \frac{\partial W}{\partial l}\, dx_1 dx_2 + \int_{A-A_0} \frac{\partial W}{\partial x_1}\, dx_1 dx_2 - \frac{d}{dl}\int_{A_0} W\, dA - \int_\Gamma t_i \frac{\partial u_i}{\partial x_1}\, d\Gamma$$

$$\int_{A-A_0} \frac{\partial W}{\partial x_1}\, dx_1 dx_2 = \int_{\Gamma+\Gamma_c+\Gamma_0} W n_1\, d\Gamma = \int_{\Gamma+\Gamma_0} W n_1\, d\Gamma$$

$$\rightarrow \mathcal{G} = \int_\Gamma W n_1 - t_i u_{i,1}\, d\Gamma + \int_{\Gamma_0} W n_1\, d\Gamma + \int_{A_0} \frac{\partial W}{\partial l}\, dx_1 dx_2 - \frac{d}{dl}\int_{A_0} W\, dA$$



$W = F(l)\,\tilde{W}(x_1, x_2)$ for small h (assumes some dominant behavior near crack tip)

$$\int_{\Gamma_0} W n_1\, d\Gamma = \int_{-h}^{h} F(l)\,\tilde{W}(-h, x_2)\, dx_2 + \int_{h}^{-h} -F(l)\,\tilde{W}(h, x_2)(-dx_2)$$

$$= \int_{-h}^{h} F(l)\left[\tilde{W}(-h, x_2) - \tilde{W}(h, x_2)\right] dx_2$$

$$\int_{A_0} \frac{\partial W}{\partial l}\, dx_1 dx_2 = \int_{A_0} \frac{dF}{dl} \tilde{W}(x_1, x_2)\, dx_1 dx_2$$

$$\frac{d}{dl}\int_{A_0} W\, dA = \frac{d}{dl}\int_{A_0} F(l)\, \tilde{W}(X_1 - l, X_2)\, dX_1 dX_2$$

$$= \lim_{\Delta l \to 0} \frac{1}{\Delta l}\Big\{ \int_{A_0} F(l+\Delta l)\, \tilde{W}(\overbrace{X_1 - l}^{x_1} - \Delta l, \overbrace{X_2}^{x_2})\, dX_1 dX_2$$

$$- \int_{A_0} F(l)\, \tilde{W}(\underbrace{X_1 - l}_{x_1}, X_2)\, dX_1 dX_2 \Big\}$$

I don't like these steps. They seem to assume Γ_0 moves with crack, or x_1, x_2 does not move during Δl.

$$= \lim_{\Delta l \to 0} \frac{1}{\Delta l}\Big[\int_{A_0} F(l+\Delta l)\, \tilde{W}(x_1 - \Delta l, x_2)\, dx_1 dx_2$$

$$- \int_{A_0} F(l)\, \tilde{W}(x_1, x_2)\, dx_1 dx_2 \Big]$$

$$\int_{A_0} F(l+\Delta l)\, \tilde{W}(x_1 - \Delta l, x_2)\, dx_1 dx_2 = F(l+\Delta l)\int_{A_0} \tilde{W}(x_1 - \Delta l, x_2)\, dx_1 dx_2$$

$$= F(l+\Delta l)\int_{-h}^{h}\Big[\int_{-h}^{h} \tilde{W}(x_1 - \Delta l, x_2)\, dx_1\Big] dx_2$$

This step → then fixes it.

$$x_1^* = x_1 - \Delta l \rightarrow \quad = F(l+\Delta l)\int_{-h}^{h}\Big[\int_{-h-\Delta l}^{h-\Delta l} \tilde{W}(x_1^*, x_2)\, dx_1^*\Big] dx_2$$

$$= F(l+\Delta l)\int_{-h}^{h}\Big[\int_{-h}^{h} \tilde{W}(x_1^*, x_2)\, dx_1^* + \int_{-h-\Delta l}^{-h} \tilde{W}(x_1^*, x_2)\, dx_1^*$$

$$+ \int_{h}^{h-\Delta l} \tilde{W}(x_1^*, x_2)\, dx_1^* \Big] dx_2$$

$$= F(l+\Delta l)\Big[\int_{A_0} \tilde{W}(x_1, x_2)\, dx_1 dx_2 + \int_{-h}^{h}\int_{-h-\Delta l}^{-h} \tilde{W}(x_1, x_2)\, dx_1 dx_2 + \int_{-h}^{h}\int_{h}^{h-\Delta l} \tilde{W}(x_1, x_2)\, dx_1 dx_2\Big]$$

as $\Delta l \to 0$

$$= \left[F(l) + \frac{dF}{dl}\Delta l\right]\left[\int_{A_0} \tilde{W}(x_1, x_2)\, dx_1\, dx_2 + \int_{-h}^{h} \tilde{W}(-h, x_2)\Delta l - \tilde{W}(h, x_2)\Delta l\, dx_2\right]$$

$$\to \frac{d}{dl}\int_{A_0} W\, dA = F(l)\left[\int_{-h}^{h} \tilde{W}(-h, x_2) - \tilde{W}(h, x_2)\, dx_2\right] + \frac{dF}{dl}\int_{A_0} \tilde{W}(x_1, x_2)\, dx_1\, dx_2$$

$$\therefore \quad \mathcal{G} = \int_\Gamma W n_1 - \sigma_{ji} n_j u_{i,1}\, d\Gamma + \cancel{\int_{-h}^{h} F(l)\left[\tilde{W}(-h, x_2) - \tilde{W}(h, x_2)\right] dx_2} + \cancel{\int_{A_0} \frac{dF}{dl} \tilde{W}(x_1, x_2)\, dx_1\, dx_2}$$
$$\cancel{- F(l)\int_{-h}^{h} \left[\tilde{W}(-h, x_2) - \tilde{W}(h, x_2)\right] dx_2} \cancel{- \frac{dF}{dl}\int_{A_0} \tilde{W}(x_1, x_2)\, dx_1\, dx_2}$$

$$\boxed{\begin{aligned}\mathcal{G} &= \int_\Gamma W n_1 - \sigma_{ji} n_j u_{i,1}\, d\Gamma \\ &= \int_\Gamma W\, dx_2 - \int_\Gamma t_i u_{i,1}\, d\Gamma\end{aligned}}$$

Path-independence then allows us to move the contour to any arbitrary Γ.

Let's take another look at the step I do not like.

$$\frac{d}{dl}\int_{A_0} W\,dA = \lim_{\Delta l \to 0} \frac{1}{\Delta l}\Big[\int_{A_0} F(l+\Delta l)\tilde{W}(X_1-l-\Delta l, X_2)\,dX_1 dX_2$$

$$- \int_{A_0} F(l)\tilde{W}(X_1-l, X_2)\,dX_1 dX_2\Big]$$

$$= \lim_{\Delta l \to 0} \frac{1}{\Delta l}\Big[\int_{-h}^{h}\int_{l-h}^{l+h} F(l+\Delta l)\tilde{W}(X_1-l-\Delta l, X_2)\,dX_1 dX_2$$

$$- \int_{-h}^{h}\int_{l-h}^{l+h} F(l)\tilde{W}(X_1-l, X_2)\,dX_1 dX_2\Big]$$

$x_1 = X_1 - l$ when crack length $= l$

$x_1 = X_1 - l - \Delta l$ when crack length $= l + \Delta l$

$$= \lim_{\Delta l \to 0} \frac{1}{\Delta l}\Big[\int_{-h}^{h}\int_{-h-\Delta l}^{h-\Delta l} F(l+\Delta l)\tilde{W}(x_1, x_2)\,dx_1 dx_2$$

$$- \int_{-h}^{h}\int_{-h}^{h} F(l)\tilde{W}(x_1, x_2)\,dx_1 dx_2\Big]$$

$$= \lim_{\Delta l \to 0} \frac{1}{\Delta l}\Big[\Big\{\cancel{F(l)} + \frac{dF}{dl}\Delta l\Big\}\int_{-h}^{h}\int_{-h}^{h} \tilde{W}(x_1, x_2)\,dx_1 dx_2$$

$$+ \Big\{F(l) + \frac{dF}{dl}\Delta l\Big\}\int_{-h}^{h} \tilde{W}(h, x_2)(-\Delta l) + \tilde{W}(-h, x_2)\Delta l\,dx_2$$

$$\cancel{- F(l)\int_{-h}^{h}\int_{-h}^{h} \tilde{W}(x_1, x_2)\,dx_1 dx_2}\Big]$$

$$= \frac{dF}{dl}\int_{A_0} \tilde{W}(x_1, x_2)\,dA + F(l)\int_{-h}^{h} \tilde{W}(-h, x_2) - \tilde{W}(h, x_2)\,dx_2$$

✓ Same as prior result [redacted]

Let's take a closer look at the need to separate $W(\ell, x_1, x_2)$ as $F(\ell)\hat{W}(x_1, x_2)$.

$$\int_{\Gamma_0} W n_1 \, d\Gamma = \int_{-h}^{h} W(\ell, -h, x_2)\, dx_2 + \int_{h}^{-h} - W(\ell, h, x_2)(dx_2)$$

$$= \int_{-h}^{h} \left[W(\ell, -h, x_2) - W(\ell, h, x_2) \right] dx_2$$

$$\frac{d}{d\ell} \int_{A_0} W \, dA = \lim_{\Delta\ell \to 0} \frac{1}{\Delta\ell} \left\{ \left[\int_{A_0} W dA \right] \Big|_{\ell+\Delta\ell} - \left[\int_{A_0} W dA \right] \Big|_{\ell} \right\}$$

$$= \lim_{\Delta\ell \to 0} \frac{1}{\Delta\ell} \left\{ \int_{-h}^{h} \int_{-h-\Delta\ell}^{h-\Delta\ell} W(\ell+\Delta\ell, x_1, x_2)\, dx_1\, dx_2 - \int_{-h}^{h} \int_{-h}^{h} W(\ell, x_1, x_2)\, dx_1 dx_2 \right\}$$

$$\int_{-h-\Delta\ell}^{h-\Delta\ell} W(\ell+\Delta\ell, x_1, x_2)\, dx_1 = \int_{-h}^{h} W(\ell, x_1, x_2) + \frac{\partial W}{\partial \ell}\Delta\ell + \dots \, dx_1$$

$$+ \int_{-h-\Delta\ell}^{-h} W(\ell, x_1, x_2) + \frac{\partial W}{\partial \ell}\Delta\ell + \dots \, dx_1$$

$$- \int_{h-\Delta\ell}^{h} W(\ell, x_1, x_2) + \frac{\partial W}{\partial \ell}\Delta\ell + \dots \, dx_1$$

Keeping terms up to $O(\Delta l)$:

$$\int_{-h-\Delta l}^{h-\Delta l} W(l+\Delta l, x_1, x_2)\,dx_1 = \int_{-h}^{h} W(l, x_1, x_2) + \frac{\partial W}{\partial l}\Delta l\,dx_1$$

$$+ W(l, -h, x_2)\Delta l$$

$$- W(l, h, x_2)\Delta l$$

$$\rightarrow \lim_{\Delta l \to 0} \frac{1}{\Delta l}\left\{ \quad \right\} = \int_{-h}^{h}\int_{-h}^{h} \frac{\partial W}{\partial l}\,dx_1\,dx_2$$

$$+ \int_{-h}^{h} \left[W(l, -h, x_2) - W(l, h, x_2) \right] dx_2$$

$\therefore$ The ansatz that $W(l, x_1, x_2) = F(l)\tilde{W}(x_1, x_2)$ is not necessary.

Define $$J \equiv \int_\Gamma W n_1 - \sigma_{ji} n_j \frac{\partial u_i}{\partial x_1} \, d\Gamma$$

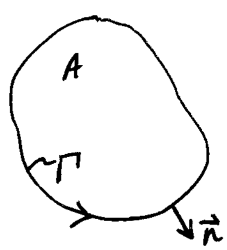
$$= \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

$$= \int_\Gamma W n_1 - t_i u_{i,1} \, d\Gamma$$

$$= \int_\Gamma W \, dx_2 - t_i u_{i,1} \, d\Gamma$$

where Γ is any contour

Consider J around any closed contour <u>not</u> surrounding a singularity.



$$J = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

$$= \int_A W_{,1} \, dA - \int_\Gamma \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

$$= \int_A \frac{\partial W}{\partial \varepsilon_{ij}} \varepsilon_{ij,1} \, dA - \int_\Gamma t_i u_{i,1} \, d\Gamma$$

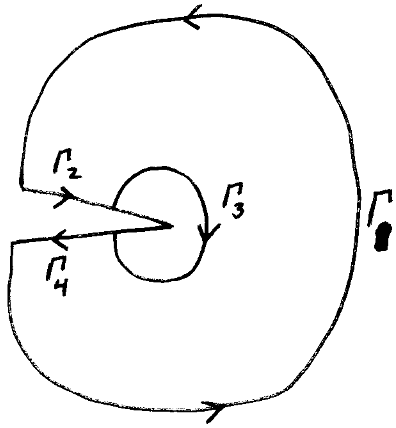
$$= \int_A \sigma_{ij} \varepsilon_{ij,1} \, dA - \int_\Gamma t_i u_{i,1} \, d\Gamma$$

$$= \int_A (\sigma_{ij} u_{i,1})_{,j} - \cancelto{0}{\sigma_{ij,j}} u_{i,1} \, dA - \int_\Gamma t_i u_{i,1} \, d\Gamma$$

$$= \int_\Gamma \underbrace{\sigma_{ij} n_j}_{t_i} u_{i,1} \, d\Gamma - \int_\Gamma t_i u_{i,1} \, d\Gamma$$

$$= 0$$

Back to our cracked body



$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

Γ is a closed contour not enclosing a singularity,

$\therefore \; J_\Gamma = 0 = J_{\Gamma_1} + J_{\Gamma_2} + J_{\Gamma_3} + J_{\Gamma_4}$

Recall $\Gamma_0 = \Gamma_1$ + crack faces, but $n_1 = 0$ and $\sigma_{ji} n_j = 0$ on crack faces.

$$\therefore \; \mathcal{G} = \int_{\Gamma_0} W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma = \int_{\Gamma_1} W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma = J_{\Gamma_1}$$

$J_{\Gamma_2} = J_{\Gamma_4}$ b/c $n_1 = 0$ and $\sigma_{ij} n_j = 0$ on Γ_2 and Γ_4

$$\therefore \; J_\Gamma = J_{\Gamma_1} + J_{\Gamma_3} = 0$$

Now take the direction around Γ_3 to be counterclockwise,

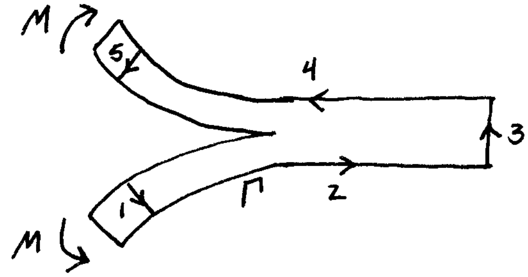
$$\therefore \; J_{\Gamma_3} = -J_{-\Gamma_3} \rightarrow J_{-\Gamma_3} = J_{\Gamma_1} = \mathcal{G}$$

Note that $J_{-\Gamma_3}$ is <u>any</u> arbitrary countour surrounding the crack tip in a counterclockwise sense.

$$\therefore \; J = \mathcal{G} \quad \text{and}$$

J is path-independent for any "counterclockwise" contour surrounding the crack tip.
(for cracks growing to the right)

Examples from HW#1 : $J = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$



$\Gamma_{2,4}$: $n_1 = 0$ and $t_i = 0 \rightarrow J_{2,4} = 0$

Γ_3 : $t_i = 0$ and if the crack is not to close it will be completely stress free $\rightarrow W = 0$
$\rightarrow J_3 = 0$

Γ_1 : Take y from center of the arm. Then,

$$\sigma_{xx} = \frac{12My}{bh^3}, \quad \varepsilon_{xx} = \frac{12My}{Ebh^3}, \text{ all other } \sigma_{ij}, \varepsilon_{ij} = 0$$

$\hookrightarrow \varepsilon_{xx} = u_{x,x}$

$$W = \frac{144M^2y^2}{2Eb^2h^6}$$

$$n_1 = n_x = -1, \quad n_2 = n_y = 0, \quad d\Gamma = -dy$$

$$J_1 = \int_{h/2}^{-h/2} -\frac{72M^2y^2}{Eb^2h^6} + \frac{144M^2y^2}{Eb^2h^6} (-dy)$$

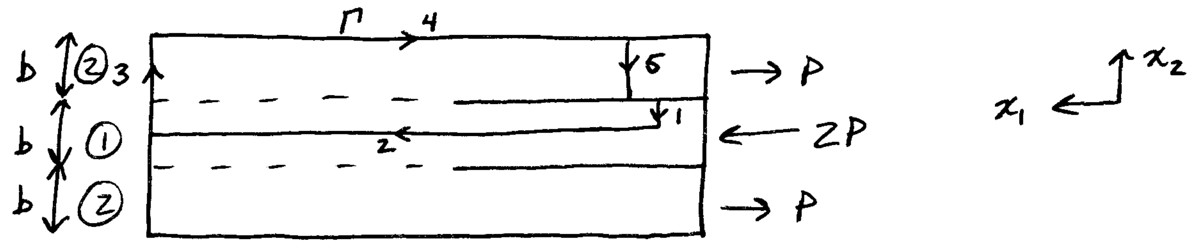
$$= \int_{-h/2}^{h/2} \frac{72M^2y^2}{Eb^2h^6} dy = \frac{72M^2}{Eb^2h^6} \frac{1}{3}\left(\frac{h^3}{4}\right) = \frac{6M^2}{Eb^2h^3}$$

Γ_5 : Again take y from center. $\sigma_{xx} = \frac{-12My}{bh^3}$, $\varepsilon_{xx} = \frac{-12My}{Ebh^3}$

$W = \frac{72M^2y^2}{Eb^2h^6}$, $n_1 = -1$, $n_2 = 0$, $d\Gamma = -dy$

and we will find $J_5 = J_1$

$$\longrightarrow \boxed{J = \frac{12M^2}{Eb^2h^3}}$$



The x_1 direction must be chosen in the direction of crack advance. "Counterclockwise" is then determined using the right-hand rule by sweeping the x_1 direction into x_2. In this case "counterclockwise" is actually clockwise.

Γ_4: $n_1 = 0$ and $t_i = 0$ → $J_4 = 0$

Γ_2: Γ_2 is along the mid-line such that due to symmetry $\sigma_{12} = 0$, however there can be a σ_{22} component. Also $n_1 = 0$, $d\Gamma = dx_1$.
$n_2 = -1$

$$\therefore \quad J_2 = \int_0^L -\sigma_{22}\, u_{2,1}\, dx_1$$

However, due to symmetry $u_2 = 0$ along the entire centerline and $\therefore$ $u_{2,1} = 0$ as well.
→ $J_2 = 0$.

Γ_3: If far from cracks then Γ_3 is stress free → $J_3 = 0$

Γ_1: $\sigma_{11} = \dfrac{-2P}{bt}$, $\varepsilon_{11} = \dfrac{-2P}{btE_1'}$, $W = \dfrac{4P^2}{2b^2t^2E_1'}$, $n_1 = -1$, $n_2 = 0$, $d\Gamma = -dx_2$

$$J_1 = \int_0^{-b/2} -\frac{4P^2}{2b^2t^2E_1'} + \frac{4P^2}{b^2t^2E_1'}\left(-dx_2\right) = \frac{2P^2}{b^2t^2E_1'}\,\frac{b}{2}$$

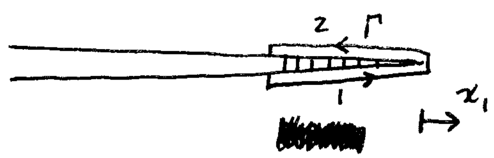
$\Gamma_5: \quad \sigma_{11} = \dfrac{P}{bt}, \quad \varepsilon_{11} = \dfrac{P}{btE_2'}, \quad W = \dfrac{P^2}{2b^2t^2E_2'}, \quad n_1 = -1, n_2 = 0, d\Gamma = -dx_2$

$$J_5 = \int_0^{-b} -\frac{P^2}{2b^2t^2E_2'} + \frac{P^2}{b^2t^2E_2'} \left(-dx_2\right) = \frac{P^2}{2b^2t^2E_2'}\, b$$

$$\therefore \; J = J_1 + J_5 = \frac{P^2}{bt^2E_1'} + \frac{P^2}{2bt^2E_2'}$$

$$\boxed{J = \frac{P^2}{bt^2}\left[\frac{1-\nu_1^2}{E_1} + \frac{1}{2}\,\frac{1-\nu_2^2}{E_2'}\right]}$$

Consider our cohesive zone model:



Take Γ to run exactly along the crack faces where the cohesive tractions act.

$$J = \int_{-R}^{0} \underbrace{\sigma_{22}}_{\sigma_0} u_{2,1}^- \, dx_1 + \int_0^{-R} -\underbrace{\sigma_{22}}_{\sigma_0} u_{2,1}^+ \left(-dx_1\right)$$

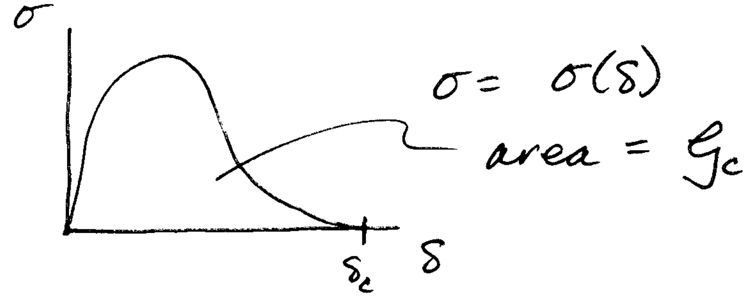
$$= +\sigma_0 \int_0^{-R} \underbrace{\left(u_2^+ - u_2^-\right)}_{\delta}{}_{,1} \, dx_1$$

$$= +\sigma_0 \int_0^{-R} \frac{d\delta}{dx_1}\, dx_1 = +\sigma_0 \, \delta \Big]_0^{-R}$$

$$= +\sigma_0 \left(\delta_c - 0\right)$$

$$\boxed{J = \sigma_0 \delta_c}$$

For a more general cohesive zone model we have,



Again, $J = \int_{-R}^{0} \sigma_{22} u^{-}_{2,1} \, dx_1 + \int_{0}^{-R} -\sigma_{22} u^{+}_{2,1} (-dx_1)$

$$= \int_0^{-R} \sigma(\delta) \left(u_2^+ - u_2^- \right)_{,1} dx_1$$

$$= \int_0^{-R} \sigma(\delta) \frac{d\delta}{dx_1} dx_1$$

$$= \int_{\delta(x=0)}^{\delta(x=-R)} \sigma(\delta) \, d\delta$$

$$= \int_0^{\delta_c} \sigma(\delta) \, d\delta \;=\; \text{area under } \sigma\text{-}\delta \text{ curve.}$$

For HW you will show $J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$
for the asymptotic fields.

$\therefore$ under SSY conditions and Mode I

$$J = \frac{K_I^2}{E'} = \mathcal{G}_c$$

So it actually doesn't matter what the cohesive zone law looks like.