# Anti-plane anisotropic crack tip fields

Equilibrium : $\dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} = 0$

Kinematics : $\gamma_{xz} = \dfrac{\partial w}{\partial x}$ , $\gamma_{yz} = \dfrac{\partial w}{\partial y}$

Material law : $\sigma_{xz} = \mu_{xx}\gamma_{xz} + \mu_{xy}\gamma_{yz}$

$$\sigma_{yz} = \mu_{xy}\gamma_{xz} + \mu_{yy}\gamma_{yz}$$

Note that $\begin{bmatrix} \mu_{xx} & \mu_{xy} \\ \mu_{xy} & \mu_{yy} \end{bmatrix} = \begin{bmatrix} C_{55} & C_{45} \\ C_{45} & C_{44} \end{bmatrix}$ ← Voight notation

$ML \rightarrow K \rightarrow Eq \rightarrow$ $\mu_{xx}\dfrac{\partial^2 w}{\partial x^2} + 2\mu_{xy}\dfrac{\partial^2 w}{\partial x \partial y} + \mu_{yy}\dfrac{\partial^2 w}{\partial y^2} = 0$

Change of variables : $z = x + py$
$\bar{z} = x + \bar{p}y$

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial y} &= p\frac{\partial}{\partial z} + \bar{p}\frac{\partial}{\partial \bar{z}} \end{aligned}\right\} \quad \begin{aligned} \frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2} \\ \frac{\partial^2}{\partial y^2} &= p^2\frac{\partial^2}{\partial z^2} + 2p\bar{p}\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{p}^2\frac{\partial^2}{\partial \bar{z}^2} \\ \frac{\partial^2}{\partial x \partial y} &= p\frac{\partial^2}{\partial z^2} + (p+\bar{p})\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{p}\frac{\partial^2}{\partial \bar{z}^2} \end{aligned}$$

$$\rightarrow \frac{\partial^2}{\partial z^2}\left[\mu_{xx} + 2\mu_{xy}p + \mu_{yy}p^2\right]w$$
$$+ \frac{\partial^2}{\partial z \partial \bar{z}}\left[2\mu_{xx} + 2\mu_{xy}(p+\bar{p}) + 2\mu_{yy}p\bar{p}\right]w$$
$$+ \frac{\partial^2}{\partial \bar{z}^2}\left[\mu_{xx} + 2\mu_{xy}\bar{p} + \mu_{yy}\bar{p}^2\right]w = 0$$

To get rid of $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial \bar{z}^2}$ terms, take

$$\mu_{xx} + 2\mu_{xy} p + \mu_{yy} p^2 = 0$$

$$p = \frac{-2\mu_{xy} \pm \sqrt{4\mu_{xy}^2 - 4\mu_{xx}\mu_{yy}}}{2\mu_{yy}}$$

strain energy for any $\gamma_{xz}, \gamma_{yz} \geq 0 \rightarrow \begin{bmatrix} \mu_{xx} & \mu_{xy} \\ \mu_{xy} & \mu_{yy} \end{bmatrix}$

is positive definite $\rightarrow \mu_{xx} > 0,\ \mu_{yy} > 0,\ \mu_{xx}\mu_{yy} - \mu_{xy}^2 > 0$

$$\rightarrow p = \underbrace{-\frac{\mu_{xy}}{\mu_{yy}}}_{p_r} \pm i \underbrace{\frac{\sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}}{\mu_{yy}}}_{p_i} \rightarrow \begin{array}{l} p = p_r + i p_i \\ \bar{p} = p_r - i p_i \end{array}$$

Then $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0 \rightarrow w = F(z) + G(\bar{z})$

$$w \text{ real} \rightarrow G(\bar{z}) = \overline{F(z)}$$

$$w = F(z) + \overline{F(z)} = 2\,\mathrm{Re}[F(z)]$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} = F'(z) + \overline{F'(z)} = 2\,\mathrm{Re}[F'(z)]$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} = pF'(z) + \overline{pF'(z)} = 2\,\mathrm{Re}[pF'(z)]$$

$$\sigma_{xz} = \mu_{xx}(F' + \overline{F'}) + \mu_{xy}(pF' + \overline{pF'})$$

$$\sigma_{yz} = \mu_{xy}(F' + \overline{F'}) + \mu_{yy}(pF' + \overline{pF'})$$

Note $\mu_{xy} + \mu_{yy} p = i p_i \mu_{yy}$

$$\mu_{xx} + \mu_{xy} p = \frac{\mu_{xx}\mu_{yy} - \mu_{xy}^2}{\mu_{yy}} + i p_i \mu_{xy}$$

$$\frac{\mu_{xy} + \mu_{yy} p}{\sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}} = i$$

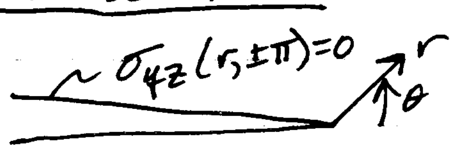
$$\frac{\mu_{xx} + \mu_{xy} p}{\sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}} = \frac{\sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}}{\mu_{yy}} + i\,\frac{\mu_{xy}}{\mu_{yy}} = -ip$$

Call $\sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2} = \mu$

$$\sigma_{yz} = \mu(iF' + \overline{iF'}) = -2\mu\,\mathrm{Im}[F'(z)]$$

$$\sigma_{xz} = \mu(-ipF' - \overline{ipF'}) = 2\mu\,\mathrm{Im}[pF'(z)]$$

<u>Crack solution</u>




$$z = \tilde{r} e^{i\tilde{\theta}}$$

on $\theta = \pm\pi \rightarrow \tilde{\theta} = \pm\pi,\ \tilde{r} = r$

Try $F'(z) = A z^s = (A_r + iA_i)\,\tilde{r}^s e^{is\tilde{\theta}}$

$$\sigma_{yz}(r,\pm\pi) = -2\mu r^s\left[\pm A_r \sin(s\pi) + A_i \cos(s\pi)\right] = 0$$

$\rightarrow \quad s = \frac{n}{2}$ , $n$ odd and $A_r = 0$

or $s = n \in I$ and $A_i = 0$

Finite energy $\rightarrow$ $s > -1$
most singular term $\rightarrow$ $s = -\frac{1}{2}$

$\therefore \quad F'(z) = iA_i/\sqrt{z}$

Irwin normalization $\rightarrow$ $\sigma_{yz}(r,0) = \dfrac{K_{III}}{\sqrt{2\pi r}}$
on $\theta=0$, $\hat{\theta}=0$, $\hat{r}=r$

$$\sigma_{yz}(r,0) = -2\mu r^{-1/2} A_i = \frac{K_{III}}{\sqrt{2\pi r}}$$

$$A_i = \frac{-K_{III}}{2\sqrt{2\pi}\,\mu}$$

$$\rightarrow F'(z) = -\frac{iK_{III}}{2\mu\sqrt{2\pi z}}$$

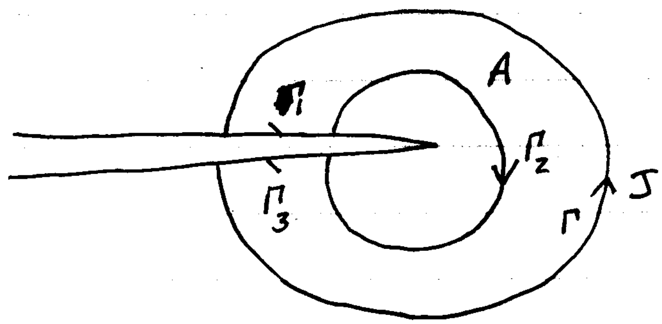
$$F(z) = -\frac{iK_{III}}{\mu}\sqrt{\frac{z}{2\pi}}$$

$$\delta = w(r,\pi) - w(r,-\pi) = \frac{4K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}$$

$$\mathcal{G}\,\delta a = \frac{1}{2}\int_0^{\delta a} \frac{4K_{III}^2}{2\pi\mu}\sqrt{\frac{\delta a - r}{r}}\,dr = \frac{K_{III}^2}{2\mu}\,\delta a$$

$$\boxed{\mathcal{G} = \frac{K_{III}^2}{2\mu}, \quad \mu = \sqrt{\mu_{xx}\mu_{yy} - \mu_{xy}^2}}$$

## Domain Integral Method for J calculations – Useful in FEM




$$J = \int_{\Gamma} W n_1 - \sigma_{ji} n_j u_{i,1} \, d\Gamma$$

Define sufficiently smooth $g(x_1, x_2)$ such that $g = 1$ on $\Gamma$ and $g = 0$ on $\Gamma_2$.

Then $$J = \int_{\Gamma+\Gamma_1+\Gamma_2+\Gamma_3} W g n_1 - g \sigma_{ji} n_j u_{i,1} \, d\Gamma$$

Note that the contributions from $\Gamma_1$ and $\Gamma_3$ are zero because $n_1 = 0$ and $\sigma_{ji} n_j = 0$ on these paths.

Apply divergence thm. $\rightarrow$ $$J = \int_A \left( \cancel{\frac{\partial W}{\partial x_1}} g + W g_{,1} - g_{,j} \sigma_{ji} u_{i,1} - g \underbrace{\sigma_{ji,j}}_{0} u_{i,1} - g \cancel{\sigma_{ji} u_{i,j1}} \right) dA$$

$$\rightarrow \boxed{J = \int_A W g_{,1} - \sigma_{ji} g_{,j} u_{i,1} \, dA}$$

## Phase-Field Fracture Modeling

Griffith's idea $\rightarrow$ $\phi = \int_V \tilde{\psi}\, dV + \int_{S_c} \mathcal{G}_c\, dS - W$

$\tilde{\psi}$ = elastic strain energy density
$S_c$ = crack surfaces which are part of the solution
$W$ = work done by loads

Computing the evolution of $S_c$ is difficult, especially in 3D.

Replace first two integrals with $\int_V \psi\, dV$

where $\psi = \mu^2 \underbrace{\tfrac{1}{2} c_{ijkl}\, \varepsilon_{ij} \varepsilon_{kl}}_{\tilde{\psi}} + \frac{\mathcal{G}_c}{4 l_0}(\mu - 1)^2 + \mathcal{G}_c l_0 \mu_{,i} \mu_{,i}$

$\Gamma$-convergence proofs show that the minimizers of $\phi$ are the same as $l_0 \rightarrow 0$.

However, this limit is unphysical as it allows for infinite stresses and so $l_0$ should be considered as a material parameter that characterizes the process zone size for the relevant fracture mechanism being modeled.

Consider $\delta \phi = 0$.

$$\delta\phi = \int_V \delta\psi \, dV - \int_V b_i \delta u_i \, dV - \int_S t_i \delta u_i \, dS$$

$$= \int_V \frac{\partial\psi}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial\psi}{\partial\mu}\delta\mu + \frac{\partial\psi}{\partial\mu_{,i}}\delta\mu_{,i} \, dV - \int_V b_i \delta u_i \, dV - \int_S t_i \delta u_i \, dS$$

$$= \int_V \underbrace{\frac{\partial\psi}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij}}_{\left(\frac{\partial\psi}{\partial\varepsilon_{ij}}\delta u_i\right)_{,j} - \left(\frac{\partial\psi}{\partial\varepsilon_{ij}}\right)_{,j}\delta u_i} + \frac{\partial\psi}{\partial\mu}\delta\mu + \left(\frac{\partial\psi}{\partial\mu_{,i}}\delta\mu\right)_{,i} - \left(\frac{\partial\psi}{\partial\mu_{,i}}\right)_{,i}\delta\mu \, dV - \int_V b_i \delta u_i \, dV - \int_S t_i \delta u_i \, dS = 0$$

$$\int_V \left[\left(\frac{\partial\psi}{\partial\varepsilon_{ij}}\right)_{,j} + b_i\right]\delta u_i + \left[\left(\frac{\partial\psi}{\partial\mu_{,i}}\right)_{,i} - \frac{\partial\psi}{\partial\mu}\right]\delta\mu \, dV$$

$$+ \int_S \left[t_i - \frac{\partial\psi}{\partial\varepsilon_{ij}}n_j\right]\delta u_i + -\frac{\partial\psi}{\partial\mu_{,i}}n_i \, \delta\mu \, dS = 0$$

for arbitrary $\delta u_i$ & $\delta\mu$ we obtain

$$\underbrace{\left(\frac{\partial\psi}{\partial\varepsilon_{ij}}\right)}_{\sigma_{ij}}{}_{,j} + b_i = 0 \quad \rightarrow \quad \sigma_{ij,j} + b_i = 0 \quad \text{in } V$$

Equilibrium

$$\underbrace{\left(\frac{\partial\psi}{\partial\mu_{,i}}\right)}_{\xi_i}{}_{,i} - \underbrace{\frac{\partial\psi}{\partial\mu}}_{\eta} = 0 \quad \rightarrow \quad \xi_{i,i} - \eta = 0 \quad \text{in } V$$

Micro-force balance

$$\frac{\partial\psi}{\partial\varepsilon_{ij}}n_j = t_i \quad \rightarrow \quad \sigma_{ij}n_j = t_i \quad \text{on } S$$

$$\frac{\partial\psi}{\partial\mu_{,i}}n_i = 0 \quad \rightarrow \quad \xi_i n_i = 0 \quad \text{on } S$$

Homogeneous 1D solutions

$$\sigma = \frac{\partial \psi}{\partial \varepsilon} = \mu^2 E \varepsilon = \text{constant}$$

$$\eta = \frac{\partial \psi}{\partial \mu} = \mu E \varepsilon^2 + \frac{g_c}{2l_0}(\mu - 1) = 0$$

$$\mu = \frac{1}{1 + \frac{2El_0}{g_c}\varepsilon^2}$$

$$\sigma = \frac{E\varepsilon}{\left(1 + \frac{2El_0}{g_c}\varepsilon^2\right)^2}$$

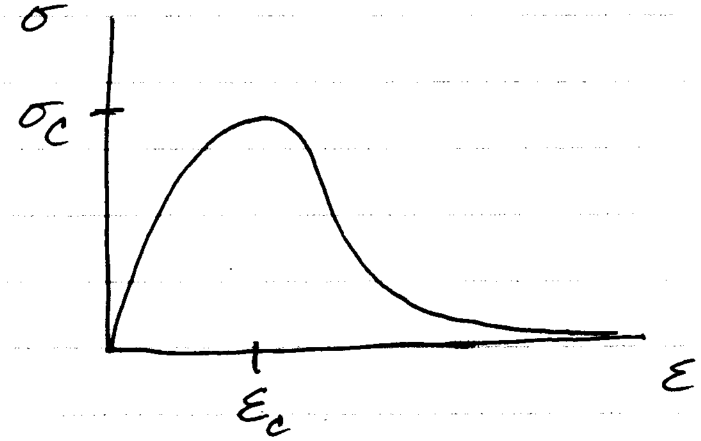
$$\frac{d\sigma}{d\varepsilon} = \frac{E}{\left(1 + \frac{2El_0}{g_c}\varepsilon^2\right)^2} - \frac{\frac{8El_0}{g_c}E\varepsilon^2}{\left(1 + \frac{2El_0}{g_c}\varepsilon^2\right)^3}$$

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} = E \checkmark$$

$$\frac{d\sigma}{d\varepsilon} = 0 \rightarrow 1 + \frac{2El_0}{g_c}\varepsilon^2 = \frac{8El_0}{g_c}\varepsilon^2$$

$$\rightarrow \boxed{\varepsilon_c = \frac{1}{\sqrt{6}}\sqrt{\frac{g_c}{El_0}}}$$

$$\rightarrow \sigma_c = \frac{\frac{1}{\sqrt{6}}\sqrt{\frac{g_c E}{l_0}}}{\left[1 + \frac{2El_0}{g_c}\frac{1}{6}\frac{g_c}{El_0}\right]^2} = \boxed{\frac{9}{16\sqrt{6}}\sqrt{\frac{g_c E}{l_0}} = \sigma_c}$$




Decending $\sigma-\varepsilon$ curves usually lead to localization. In this case the localization is a crack where $\sigma=0$ can occur by having $\varepsilon=0$ and $\mu$ = something (not neccesarily ~~[illegible]~~ one if $\nabla\mu\neq 0$) or $\varepsilon\to\infty$ and $\mu=0$.

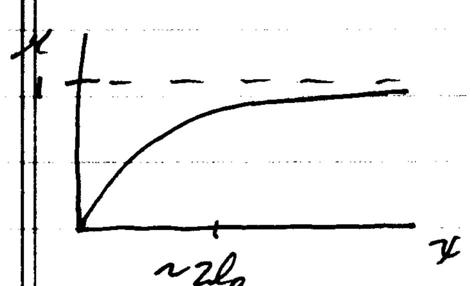
Let's look at a 1D solution with $\sigma=0$.

$$\xi_{i,i}-\eta=0 \qquad \left\{\begin{array}{l} \xi_x=\dfrac{\partial\psi}{\partial\mu_{,x}}=2\mathcal{G}_c l_0\dfrac{d\mu}{dx} \\ \eta=\cancelto{0 \text{ for } \sigma=0}{\mu E\varepsilon^2}+\dfrac{\mathcal{G}_c}{2l_0}(\mu-1)\end{array}\right.$$

$$2\mathcal{G}_c l_0\frac{d^2\mu}{dx^2}-\frac{\mathcal{G}_c}{2l_0}(\mu-1)=0$$

$$\mu=1+Ae^{-px}\rightarrow 2l_0p^2-\frac{1}{2l_0}=0\rightarrow p=\frac{1}{2l_0}$$

$$\mu(0)=0\rightarrow A=-1\rightarrow \boxed{\mu=1-e^{-x/2l_0}}$$




$$\text{Energy}=\underset{\uparrow \text{ 2 sides of surface}}{2}\int_0^\infty\psi\,dx=2\int_0^\infty\frac{\mathcal{G}_c}{4l_0}e^{-x/l_0}+\mathcal{G}_c l_0\frac{1}{4l_0^2}e^{-x/l_0}\,dx$$

$$\text{Energy}=\frac{\mathcal{G}_c}{l_0}\int_0^\infty e^{-x/l_0}\,dx=\mathcal{G}_c \checkmark$$

So the fracture surface energy is $\mathcal{G}_c$.

J-Integral for Phase-field

$$J = \int_\Gamma \psi n_1 - \sigma_{ji} n_j u_{i,1} - \xi_i n_i \mu_{,1} \, d\Gamma$$

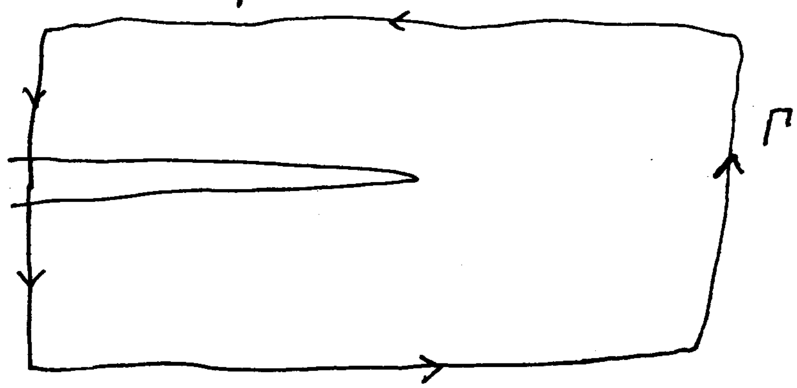
Show $J = 0$ on closed path.

$$J = \int_A \psi_{,1} - \cancel{\sigma_{ji,j}} u_{i,1} - \cancel{\sigma_{ji} u_{i,j1}} - \cancel{\xi_{i,i} \mu_{,1}} - \cancel{\xi_i \mu_{,i1}} \, dA$$

$$\cancel{\frac{\partial \psi}{\partial \varepsilon_{ij}} \varepsilon_{ij,1}} + \cancel{\frac{\partial \psi}{\partial \mu} \mu_{,1}} + \cancel{\frac{\partial \psi}{\partial \mu_{,i}} \mu_{,i1}}$$

$J = 0$ So what?

Consider a phase-field crack.



For all of $\Gamma$ except a small angular sector on the crack flanks $\mu \to 1$ and the solution looks like a linear elastic K-field. So we then only need to account for contributions from the $\mu$-field on the crack flanks.

$$J = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} - \int_{\infty}^{-\infty} \psi(-dy) = 0$$

$$\rightarrow \mathcal{G} = \frac{K_I^2 + K_{II}^2}{E'} = \mathcal{G}_c$$