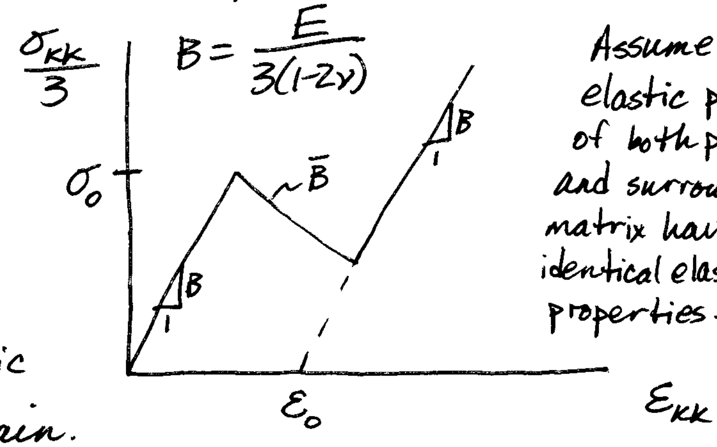
Dilatant Transformation Toughening

There exists a class of ceramics that are toughened by the addition of partially stablized zirconia (ZrO_2) particles. When the stress acting on these particles reaches a critical level, the particles undergo a martensitic type of phase transformation (i.e. from one crystal structure to another). The volume of the transformed phase is larger than that of the untransformed phase. As a crack passes by, the K field causes transformations and the transformed particles apply closure stresses to the crack thus reducing the level of K_{tip}.

Constitutive Law:

Non-linearity occurs only in the hydrostatic stress vs. volume strain.



Assume elastic properties of both phases and surrounding matrix have identical elastic properties.

If $\bar{B} \leq -\frac{4\mu}{3}$ then a particle cannot exist on points along the downward sloping branch of the curve. Hence, when a particle transforms, it transforms completely. We will not delve into the mechanics explanation of this phenomenon.

Fracture Model: Assume that the transforming particles do not perturb the isotropic elastic K-field stresses. Hence, this model is only valid to order ε_0, i.e. the solution for K_{tip} can be expanded as

$$K_{tip} = K_I + \frac{dK_{tip}}{d\varepsilon_0}\varepsilon_0 + \underbrace{\frac{1}{2}\frac{d^2K_{tip}}{d\varepsilon_0^2}\varepsilon_0^2 + \ldots}_{\text{We are neglecting these terms.}}$$

$$K_{tip} = K_I + \underbrace{K_I^{\text{Transformation}}}_{K_I \text{ due to particles transforming}}$$

In order to determine $K_I^{\text{Trans.}}$ we will perform the following procedure.

1) Determine the volume (area) of material that will transform.
2) Remove this material and allow it to expand freely within the plane (plane strain in z).
3) Apply appropriate surface tractions and body forces such that the strain at every point is returned to zero.
4) Put the material back in its hole and release the applied forces by essentially applying the opposite of these forces to the body.

5) Use weight function methods to determine K_I^{Trans} from the forces applied in step 4.

1) → $\frac{\sigma_{kk}}{3} = \sigma_0$ $\qquad \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy}) = \nu(\sigma_{rr}+\sigma_{\theta\theta})$ for plane strain

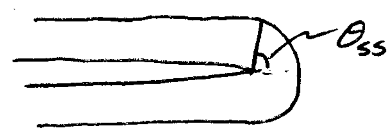
$$\frac{K_I}{\sqrt{2\pi R_T}} \frac{2(1+\nu)}{3} \cos\frac{\theta}{2} = \sigma_0$$

$$\therefore \quad R_T = \frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2 \cos^2\frac{\theta}{2}$$

We will discuss 2 cases, the stationary crack and the steady-state condition.

For the stationary crack the equation for R_T is valid for all θ, i.e. $-\pi \le \theta \le \pi$. However, for the growing crack this equation is only valid for the frontal boundary.

SS →



θ_{ss} occurs where $y = y_{max}$

$$y = R_T \sin\theta = \frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2 \sin\theta\cos^2\frac{\theta}{2}$$

$$\frac{dy}{d\theta} = \frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2\left(\cos\theta\cos^2\frac{\theta}{2} - 2\sin\theta\cos\frac{\theta}{2}\cdot\frac{1}{2}\sin\frac{\theta}{2}\right) = 0$$

$$\therefore \quad \cos\frac{\theta}{2} = 0 \quad \text{or} \quad \cos\theta\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2} = 0$$

$$\theta = \pm\pi \qquad\qquad \cos\frac{3\theta}{2} = 0$$

$$\theta = \pm\pi/3$$

For steady state there is a frontal zone with $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ along with two semi-infinite stips of height y_{max}.

2) $\rightarrow$

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}(\sigma_{yy}+\sigma_{zz}) + \frac{\varepsilon_0}{3}$$

$$\varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{zz}) + \frac{\varepsilon_0}{3}$$

plane strain

$$\varepsilon_{zz} = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy}) + \frac{\varepsilon_0}{3} = 0$$

$$\therefore \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy}) - \frac{E\varepsilon_0}{3}$$

$$\rightarrow \varepsilon_{xx} = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} + \frac{(1+\nu)\varepsilon_0}{3}$$

$$\varepsilon_{yy} = \frac{1-\nu^2}{E}\sigma_{yy} - \frac{\nu(1+\nu)}{E}\sigma_{xx} + \frac{(1+\nu)\varepsilon_0}{3}$$

For free in-plane expansion $\sigma_{xx} = \sigma_{yy} = 0$

$$\rightarrow \varepsilon_{xx} = \varepsilon_{yy} = \frac{1+\nu}{3}\varepsilon_0$$

3) $\rightarrow$

$$\varepsilon_{xx} = \varepsilon_{yy} = 0 \rightarrow \sigma_{xx} = \sigma_{yy} = \sigma$$

$$\frac{1-\nu^2}{E}\sigma - \frac{\nu+\nu^2}{E}\sigma + \frac{1+\nu}{3}\varepsilon_0 = 0$$

$$\frac{1-\nu-2\nu^2}{E}\sigma = -\frac{1+\nu}{3}\varepsilon_0$$

$$\frac{(1+\nu)(1-2\nu)}{E}\sigma = -\frac{1+\nu}{3}\varepsilon_0$$

$$\therefore \sigma_{xx} = \sigma_{yy} = \sigma = -\frac{E}{3(1-2\nu)}\varepsilon_0$$

$$b_i = -\sigma_{ij,j} = 0$$

$$t_i = \sigma_{ij} n_j = -\frac{E}{3(1-2\nu)} \varepsilon_0 \underbrace{\delta_{ij}}_{i,j = 1,2 \text{ only}} n_j = -\frac{E}{3(1-2\nu)} \varepsilon_0 n_i$$

4) → Apply $b_i = 0$ and $t_i = \frac{E}{3(1-2\nu)} \varepsilon_0 n_i$ to cracked geometry.

5) → $$K_I^{\text{Trans.}} = \int_A h_i \cancelto{0}{b_i} \, dA + \int_{S_A} h_i t_i \, dS_A$$

$$= \int_{S_A} \frac{E}{3(1-2\nu)} \varepsilon_0 h_i n_i \, dS_A$$

$$= \frac{E}{3(1-2\nu)} \varepsilon_0 \int_A h_{i,i} \, dA$$

$$h_{i,i} = h_{x,x} + h_{y,y}$$

$$h_{x,x} = \frac{\partial h_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h_x}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta \frac{\partial h_x}{\partial r} + \frac{-\sin\theta}{r}\frac{\partial h_x}{\partial \theta}$$

$$= \frac{-1}{\kappa+1}\frac{1}{2\sqrt{2\pi}} r^{-3/2} \Big[(1-\kappa)\overbrace{\left(\cos^3\tfrac{\theta}{2} - 3\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)}^{\cos\frac{3\theta}{2}}$$
$$+ 3\cos\theta\sin\theta\sin\tfrac{3\theta}{2}$$
$$+ 3\sin\theta\sin\theta\cos\tfrac{3\theta}{2} \Big]$$

$$h_{y,y} = \frac{\partial h_y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h_y}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial h_y}{\partial r} + \frac{\cos\theta}{r}\frac{\partial h_y}{\partial \theta}$$

$$= \frac{-1}{\kappa+1}\frac{1}{2\sqrt{2\pi}} r^{-3/2} \Big[(\kappa+1)\left(-\cos\tfrac{3\theta}{2}\right)$$
$$- \sin\theta\sin\theta\cos\tfrac{3\theta}{2} + 2\cos\theta\cos\theta\cos\tfrac{3\theta}{2}$$
$$- 3\cos\theta\sin\theta\sin\tfrac{3\theta}{2} \Big]$$

$$\therefore\ h_{x,x}+h_{y,y} = \frac{-1}{\chi+1}\frac{1}{2\sqrt{2\pi}}r^{-3/2}\left[\cos\frac{3\theta}{2}(1-\chi-\chi-1) + 2(\sin^2\theta+\cos^2\theta)\cos\frac{3\theta}{2}\right]$$

$$= \frac{-1}{\chi+1}\frac{1}{2\sqrt{2\pi}}r^{-3/2}\cos\frac{3\theta}{2}(2-2\chi)$$

$$= \frac{\chi-1}{\chi+1}\frac{1}{\sqrt{2\pi}}r^{-3/2}\cos\frac{3\theta}{2}$$

plane strain

$\chi = 3-4\nu$

$\rightarrow \frac{\chi-1}{\chi+1} = \frac{2(1-2\nu)}{4(1-\nu)} = \frac{1-2\nu}{2(1-\nu)}$

$$= \frac{1-2\nu}{1-\nu}\frac{1}{2\sqrt{2\pi}}r^{-3/2}\cos\frac{3\theta}{2}$$

$$\therefore\ \boxed{K_I^{Trans} = \frac{E}{3(1-\nu)}\frac{\varepsilon_o}{2\sqrt{2\pi}}\int_A r^{-3/2}\cos\frac{3\theta}{2}\,dA}$$

For particles that undergo complete transformation

Stationary Case

$$K_I^{Trans} = \frac{E}{3(1-\nu)}\frac{\varepsilon_o}{2\sqrt{2\pi}}\,2\int_0^{\pi}\int_0^{R_T(\theta)} r^{-3/2}\cos\frac{3\theta}{2}\,r\,dr\,d\theta$$

$$= \frac{E\varepsilon_o}{3\sqrt{2\pi}(1-\nu)}\int_0^{\pi} 2r^{1/2}\Big]_0^{R_T}\cos\frac{3\theta}{2}\,d\theta$$

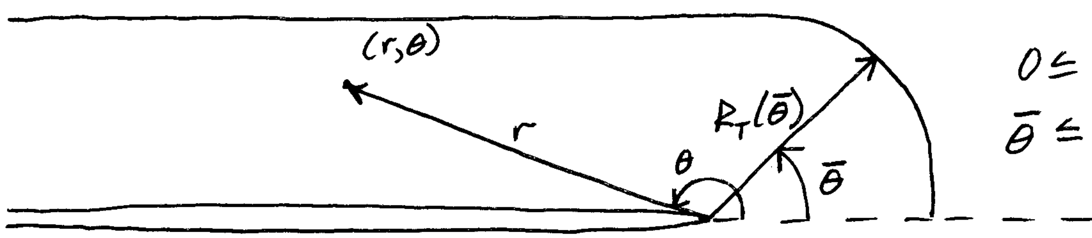
$$= \frac{2E\varepsilon_o}{3\sqrt{2\pi}(1-\nu)}\frac{\sqrt{2}(1+\nu)}{3\sqrt{\pi}}\frac{K_I}{\sigma_o}\underbrace{\int_0^{\pi}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\,d\theta}_{=\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta\Big]_0^{\pi} = 0}$$

$$\therefore\ \boxed{K_I^{Trans} = 0 \quad \text{for the stationary crack}}$$

∴ For the stationary crack $K_{Tip} = K_I$, i.e. there is no shielding of the crack tip stress intensity from the transformation zone. This has to do with path-independence of J (however, application of J to this type of "supercritically transforming" material must be done with care).

Steady-State Case

Consider the upper half-plane



$$0 \le \bar\theta \le \frac{\pi}{3}$$
$$\bar\theta \le \theta \le \pi$$

$$K_I^{Trans} = \int_A \frac{E}{3(1-\nu)} \frac{\varepsilon_0}{2\sqrt{2\pi}} r^{-3/2} \cos\frac{3\theta}{2}\, dA$$

$$dA = dx\, dy \qquad y = R_T(\bar\theta)\sin\bar\theta$$

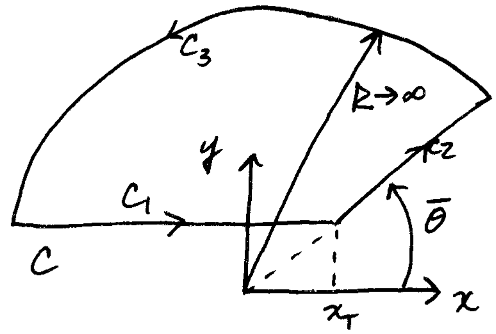
$$dy = \left(\frac{dR_T}{d\bar\theta}\sin\bar\theta + R_T\cos\bar\theta\right)d\bar\theta$$

$$= \frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2\left(\cos\bar\theta\cos^2\frac{\bar\theta}{2} - \sin\bar\theta\sin\frac{\bar\theta}{2}\cos\frac{\bar\theta}{2}\right)d\bar\theta$$

$$dy = \frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2\cos\frac{\bar\theta}{2}\cos\frac{3\bar\theta}{2}\, d\bar\theta$$

$$\therefore \boxed{K_I^{Trans} = \frac{E}{3(1-\nu)}\frac{\varepsilon_0}{2\sqrt{2\pi}}\frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2 2\int_0^{\pi/3}\cos\frac{\bar\theta}{2}\cos\frac{3\bar\theta}{2}\left(\int_{-\infty}^{x_T} r^{-3/2}\cos\frac{3\theta}{2}\,dx\right)d\bar\theta}$$

$$\int_{-\infty}^{x_T} r^{-3/2}\cos\frac{3\theta}{2}\,dx \quad \text{by contour integration}$$



$$\oint_C z^{-3/2}\,dz = 0$$

$C_1:\ z = x + iy = re^{i\theta}$

$dz = dx \qquad x_T = R_T\cos\bar{\theta}$

$C_2:\ z = re^{i\bar{\theta}}$

$dz = e^{i\bar{\theta}}\,dr$

$C_3:\ z = Re^{i\theta}$

$dz = iRe^{i\theta}\,d\theta$

$$\int_{-\infty}^{R_T\cos\bar{\theta}} r^{-3/2}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right)dx$$

$$+ \int_{R_T}^{\infty} r^{-3/2}\overbrace{\left(\cos\frac{3\bar{\theta}}{2} - i\sin\frac{3\bar{\theta}}{2}\right)}^{e^{-i\frac{3\bar{\theta}}{2}}}\overbrace{\left(\cos\bar{\theta} + i\sin\bar{\theta}\right)}^{e^{i\bar{\theta}}}dr$$

$$+ \int_{\bar{\theta}}^{\pi} i\,\underbrace{R^{-1/2}}_{\to\, 0 \text{ as } R\to\infty} e^{i\theta}e^{-i\frac{3\theta}{2}}\,d\theta \;=\; 0$$

$$\int_{R_T}^{\infty} r^{-3/2}e^{-i\frac{\bar{\theta}}{2}}\,dr = -2r^{-1/2}e^{-i\frac{\bar{\theta}}{2}}\Big]_{R_T}^{\infty}$$

$$= \frac{2}{\sqrt{R_T}}\left(\cos\frac{\bar{\theta}}{2} - i\sin\frac{\bar{\theta}}{2}\right)$$

But, both the real and imaginary parts of the sum must be zero.

$$\therefore \int_{-\infty}^{x_T} r^{-3/2} \cos\frac{3\theta}{2}\,dx = -\frac{2}{\sqrt{R_T}}\cos\frac{\bar\theta}{2}$$

$$= -\frac{2\cdot 3\sqrt{\pi}}{\sqrt{2}(1+\nu)}\frac{\sigma_0}{K_I}\cancel{\frac{\cos\frac{\bar\theta}{2}}{\cos\frac{\bar\theta}{2}}}$$

$$\rightarrow K_I^{Trans} = \frac{-E\varepsilon_0}{3(1-\nu)}\frac{2(1+\nu)^2}{9\pi\sqrt{2\pi}}\left(\frac{K_I}{\sigma_0}\right)^2\frac{2\cdot 3\sqrt{\pi}}{\sqrt{2}(1+\nu)}\frac{\sigma_0}{K_I}\int_0^{\pi/3}\cos\frac{\bar\theta}{2}\cos\frac{3\bar\theta}{2}\,d\bar\theta$$

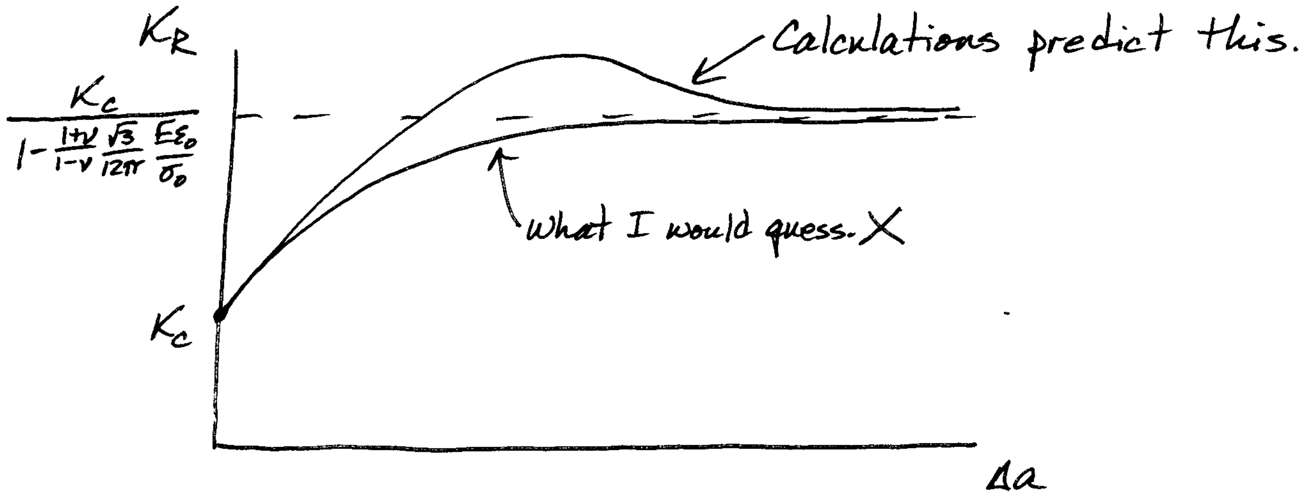
$$= -\frac{1+\nu}{1-\nu}\frac{2}{9\pi}\frac{E\varepsilon_0}{\sigma_0}K_I\int_0^{\pi/3}\frac{1}{2}\left(\cos 2\bar\theta + \cos\bar\theta\right)d\bar\theta$$

$$\frac{1}{4}\sin 2\bar\theta + \frac{1}{2}\sin\bar\theta\Big]_0^{\pi/3}$$

$$\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{8}$$

$$\boxed{\begin{aligned} K_I^{Trans} &= -\frac{1+\nu}{1-\nu}\frac{3\sqrt{3}}{8}\frac{2}{9\pi}\frac{E\varepsilon_0}{\sigma_0}K_I \\ &= -\frac{1+\nu}{1-\nu}\frac{\sqrt{3}}{12\pi}\frac{E\varepsilon_0}{\sigma_0}K_I \end{aligned}}$$

$$\therefore \boxed{K_{Tip} = K_I\left(1 - \frac{1+\nu}{1-\nu}\frac{\sqrt{3}}{12\pi}\frac{E\varepsilon_0}{\sigma_0}\right) \quad \text{to first order in } \varepsilon_0}$$



We actually did not have to do the y-integration using the transformation to $\bar{\theta}$.

$$K_I^{Trans} = \frac{E}{3(1-\nu)}\,\frac{\varepsilon_0}{2\sqrt{2\pi}}\,2\int_0^h\int_{-\infty}^{x_T} r^{-3/2}\cos\frac{3\theta}{2}\,dx\,dy$$

$$= \frac{E}{\cancel{3}(1-\nu)}\,\frac{\varepsilon_0}{\sqrt{\cancel{2\pi}}}\,\frac{-\cancel{2}\cdot\cancel{3}\sqrt{\cancel{\pi}}}{\sqrt{\cancel{2}}\,(1+\nu)}\,\frac{\sigma_0}{K_I}\int_0^h dy$$

$$= \frac{-E\varepsilon_0}{(1-\nu)(1+\nu)}\,\frac{\sigma_0}{K_I}\,h$$

$$= \frac{-E\varepsilon_0}{(1-\nu)\cancel{(1+\nu)}}\,\frac{\cancel{\sigma_0}}{\cancel{K_I}}\,\frac{\cancel{2}(1+\nu)^{\cancel{2}}}{3\cancel{4}\pi}\left(\frac{K_I}{\sigma_0}\right)^{\cancel{2}}\frac{\cancel{3}\sqrt{3}}{84}$$

$$K_I^{Trans} = -\frac{1+\nu}{1-\nu}\,\frac{\sqrt{3}}{12\pi}\,\frac{E\varepsilon_0}{\sigma_0}\,K_I$$

$$\therefore\ \boxed{K_{Tip} = K_I\left(1-\frac{1+\nu}{1-\nu}\,\frac{\sqrt{3}}{12\pi}\,\frac{E\varepsilon_0}{\sigma_0}\right)}$$

Note the ratio $\frac{E\varepsilon_0}{\sigma_0} = \frac{\sigma_0\varepsilon_0}{\sigma_0^2/E}$

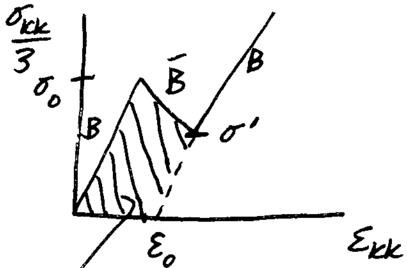
$$= \frac{\text{dissipated energy}}{\text{stored elastic energy}}$$

The energy approach for Steady-State

$$\frac{K_I^2}{E'} = \mathcal{G}_c + \int_{-h}^{h} W(x\to-\infty, y)\,dy$$

$$\frac{K_I^2(1-\nu^2)}{E} = \frac{K_{Tip}^2(1-\nu^2)}{E} + 2\int_0^h W(x\to-\infty, y)\,dy$$

How do we determine $W(x\to-\infty, y)$?
Two parts, dissipation due to transformation and elastic energy stored in wake.



For reasons we will not discuss $\bar{B} = \frac{-4\mu}{3} = -\frac{2E}{3(1+\nu)}$ for the case where transformation is always complete.

$$\text{Dissipation} = \sigma_0\varepsilon_0 + \frac{1}{2}(\sigma' - \sigma_0)\varepsilon_0$$
$$= \frac{1}{2}\sigma_0\varepsilon_0 + \frac{1}{2}\sigma'\varepsilon_0$$

$$\sigma' = B(\varepsilon_{kk} - \varepsilon_0) = \sigma_0 + \bar{B}\left(\varepsilon_{kk} - \frac{\sigma_0}{B}\right)$$
$$\varepsilon_{kk}(B - \bar{B}) = \sigma_0\left(1 - \frac{\bar{B}}{B}\right) + B\varepsilon_0$$
$$\varepsilon_{kk} = \sigma_0\frac{B-\bar{B}}{B(B-\bar{B})} + \frac{B}{B-\bar{B}}\varepsilon_0 = \frac{\sigma_0}{B} + \frac{B}{B-\bar{B}}\varepsilon_0$$

$$\therefore\ \sigma' = \sigma_0 + \frac{B^2}{B-\bar{B}}\varepsilon_0 - \frac{B(B-\bar{B})}{B-\bar{B}}\varepsilon_0 = \sigma_0 + \frac{B\bar{B}}{B-\bar{B}}\varepsilon_0$$

$$\therefore\ \text{Dissipation} = \sigma_0\varepsilon_0 + \frac{1}{2}\frac{B\bar{B}}{B-\bar{B}}\varepsilon_0^2$$
$$= \sigma_0\varepsilon_0 + \frac{1}{2}\frac{E}{3(1-2\nu)}\frac{-2E}{3(1+\nu)}\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\varepsilon_0^2$$

$$= \sigma_0\varepsilon_0 - \frac{1}{9}\frac{E\varepsilon_0^2}{1-\nu}$$

Stored elastic energy in wake $= SE = \frac{1}{2}\sigma_{ij}\,\underbrace{\varepsilon_{ij}^{e}}_{\text{elastic part of strain}}$

In wake as $x \to -\infty$

$\varepsilon_{xx} = \varepsilon_{zz} = 0 \qquad \sigma_{yy} = 0$ (crack BC)

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{zz} + \frac{\varepsilon_0}{3} = 0 \\ \varepsilon_{zz} &= \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}\sigma_{xx} + \frac{\varepsilon_0}{3} = 0 \end{aligned}\right\} \begin{aligned} \sigma_{xx} &= \sigma_{zz} = -\frac{E\varepsilon_0}{3(1-\nu)} \\ \varepsilon_{xx}^{e} &= \varepsilon_{zz}^{e} = \frac{-\varepsilon_0}{3} \end{aligned}$$

$$\therefore\ SE = \frac{1}{2}\left(\frac{E\varepsilon_0^2}{9(1-\nu)} + \frac{E\varepsilon_0^2}{9(1-\nu)}\right) = \frac{E\varepsilon_0^2}{9(1-\nu)}$$

$$W = \text{Dissipation} + SE = \sigma_0\varepsilon_0 - \frac{E\varepsilon_0^2}{9(1-\nu)} + \frac{E\varepsilon_0^2}{9(1-\nu)} = \sigma_0\varepsilon_0$$

$$\therefore\ 2\int_0^h W(x\to-\infty, y)\,dy = 2\sigma_0\varepsilon_0 h$$

$$h = R_T\left(\frac{\pi}{3}\right)\sin\frac{\pi}{3} = \frac{2(1+\nu)^2}{9\pi}\left(\frac{K_I}{\sigma_0}\right)^2 \frac{3}{4}\frac{\sqrt{3}}{2}$$

Recall this is the 1st order approximation valid up to $O(\varepsilon_0^2)$

$$\therefore\ \frac{K_I^2(1-\nu^2)}{E} = \frac{K_{Tip}^2(1-\nu^2)}{E} + \frac{2\sqrt{3}(1+\nu)^2}{12\pi}\left(\frac{K_I}{\sigma_0}\right)^2\sigma_0\varepsilon_0$$

$$K_{Tip}^2 = K_I^2 - \frac{2E}{(1-\nu)\cancel{(1+\nu)}}\frac{\sqrt{3}}{12\pi}(1+\nu)^{\cancel{2}}\frac{\varepsilon_0}{\sigma_0}K_I^2$$

$$K_{Tip} = K_I\sqrt{1 - \frac{1+\nu}{1-\nu}\frac{\sqrt{3}}{6\pi}\frac{E\varepsilon_0}{\sigma_0}} \qquad \text{Recall: } \sqrt{1-x} \approx 1 - \tfrac{1}{2}x + O(x^2)$$

$$\approx K_I\left(1 - \frac{1+\nu}{1-\nu}\frac{\sqrt{3}}{12\pi}\frac{E\varepsilon_0}{\sigma_0}\right) \checkmark \quad \text{To 1st order in } \varepsilon_0$$