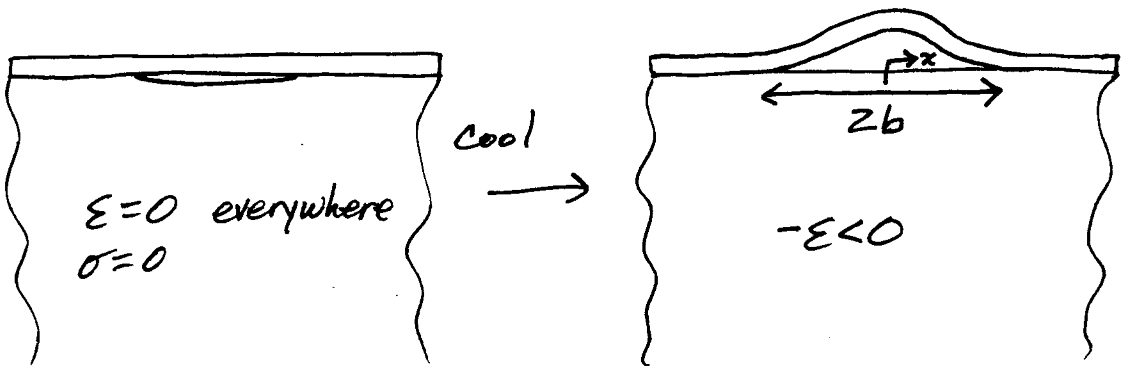
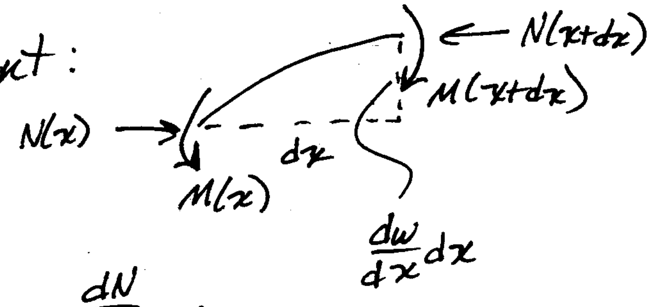
Buckle Delamination



What is the energy release rate?

First we need to analyze the buckled configuration.

Consider a differential element:



$$\Sigma F_x = N(x) - N(x+dx) = 0 \rightarrow \frac{dN}{dx} = 0$$

$$\rightarrow N = \text{constant}$$

$$\Sigma M_z^{(dx)} = N(x)\frac{dw}{dx}dx + M(x) - M(x+dx) = 0$$

$$\rightarrow \; -\frac{dM}{dx} + N\frac{dw}{dx} = 0$$

Sign convention for M and w $\rightarrow$ $M = -D\chi = -D\frac{d^2w}{dx^2}$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$ = Plane strain bending stiffness per unit thickness into page

$$\rightarrow \quad D\frac{d^3w}{dx^3} + N\frac{dw}{dx} = 0$$

General solution symmetric about $x=0$

$$\rightarrow \quad w = A + B\cos\sqrt{\tfrac{N}{D}}\,x \qquad w(\pm b)=0 \rightarrow B=A$$

$$w' = -B\sqrt{\tfrac{N}{D}}\sin\sqrt{\tfrac{N}{D}}\,x$$

$$w'(\pm b) = \mp B\sqrt{\tfrac{N}{D}}\sin\sqrt{\tfrac{N}{D}}\,b = 0$$

$$\rightarrow \quad \sqrt{\tfrac{N}{D}}\,b = \pi \quad \rightarrow \quad \boxed{N = \frac{\pi^2 D}{b^2}}$$

This tells us the force in the buckled region. To determine A we need to describe the axial strain of the <u>centerline</u> ε^A.

This mildly non-linear kinematics theory $\rightarrow \varepsilon^A = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$

The film is bonded to the rigid substrate and so $\quad u(x=b) - u(x=-b) = -2b\varepsilon \quad$ (Note $\varepsilon > 0$ for a contraction)

The axial material response of the film is linear elastic such that $\quad \varepsilon^A = \frac{\sigma^A}{E'} = \frac{N}{E'h}$

$$\rightarrow \varepsilon^A = \frac{-N}{E'h} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

$$\int_{-b}^{b}\frac{du}{dx}\,dx = u(b) - u(-b) = -2b\varepsilon = \int_{-b}^{b}\frac{-N}{E'h} - \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

$$-2b\varepsilon = \frac{-N}{E'h}(2b) - \frac{1}{2}A^2\frac{\pi^2}{b^2}\underbrace{\int_{-b}^{b}\sin^2\left(\frac{\pi x}{b}\right)dx}_{b}$$

$$\varepsilon_c = \frac{\pi^2}{12}\left(\frac{h}{b}\right)^2$$

$$A^2 = \frac{4b^2}{\pi^2}(\varepsilon - \varepsilon_c) \qquad \varepsilon_c = \text{critical strain for buckling}$$

Recall: $w = A\left(1 + \cos\frac{\pi x}{b}\right)$

So now we know the solution for the buckled film. Let's compute $\mathcal{G}$.

$$U_{unbuckled} = \frac{1}{2}E'\varepsilon^2 h(L-2b) \quad \text{where } L = \text{total length of film}$$

$$U_{buckled} = \int_{-b}^{b}\frac{D}{2}(w'')^2 + \frac{E'h}{2}(\varepsilon^A)^2\,dx$$

$$= \frac{D}{2}A^2\frac{\pi^4}{b^4}\underbrace{\int_{-b}^{b}\cos^2\left(\frac{\pi x}{b}\right)dx}_{b} + \frac{E'h}{2}\frac{\pi^4}{144}\frac{h^4}{b^4}(2b)$$

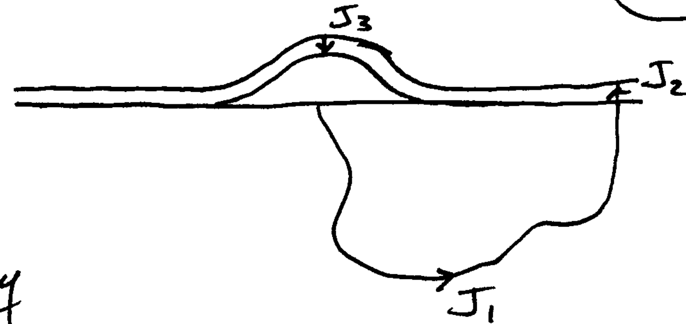
$$= \frac{E'h^3}{24}\frac{4b^2}{\pi^2}(\varepsilon-\varepsilon_c)\frac{\pi^4}{b^4}b + \frac{E'h}{2}\frac{\pi^4}{144}\frac{h^4}{b^4}(2b)$$

$$U = \frac{E'h}{2}\left[(L-2b)\varepsilon^2 + \frac{\pi^2 h^2}{3}\frac{\varepsilon}{b}\underbrace{- \frac{\pi^4}{36}\frac{h^4}{b^3} + \frac{\pi^4}{72}\frac{h^4}{b^3}}_{-\frac{\pi^4}{72}\frac{h^4}{b^3}}\right]$$

$$\mathcal{G} = -\frac{\partial U}{\partial(2b)} = -\frac{1}{2}\frac{\partial U}{\partial b} = \frac{E'h}{2}\left[\varepsilon^2 + \underbrace{\frac{\pi^2\varepsilon h^2}{6b^2}}_{2\varepsilon\varepsilon_c} - \underbrace{\frac{\pi^4}{48}\frac{h^4}{b^4}}_{3\varepsilon_c^2}\right]$$

$$\boxed{\mathcal{G} = \frac{E'h}{2}(\varepsilon + 3\varepsilon_c)(\varepsilon - \varepsilon_c)}$$

Next, let's try J.



$$J_2 = \int_0^h W - \sigma_{xx}\varepsilon_{xx}\, dy$$

$$W = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} \rightarrow J_2 = -\frac{E'h}{2}\varepsilon^2$$

$$J_3 = \int_h^0 -W + \sigma_{xx}\varepsilon_{xx}\,(-dy) \qquad \text{note } \varepsilon_{xx} \neq \varepsilon^A$$

$$\varepsilon_{xx} = \varepsilon^A + \kappa y$$

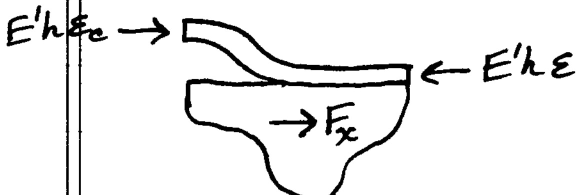
$$= \int_0^h \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx}\, dy \qquad \text{again } W = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx}$$

$$= \frac{D}{2}(w'')^2 + \frac{E'h}{2}(\varepsilon^A)^2 \qquad \varepsilon^A = \varepsilon_c \text{ on } \Gamma_3$$

$$= \frac{E'h^3}{24}\frac{4b^2}{\pi^2}(\varepsilon-\varepsilon_c)\frac{\pi^4}{b^4} + \frac{E'h}{2}\varepsilon_c^2 = \frac{E'h}{2}(4\varepsilon\varepsilon_c - 3\varepsilon_c^2)$$

$$J_1 = \int_{\Gamma_1} \cancelto{0 \text{ rigid}}{W n_x} - t_x \cancelto{-\varepsilon}{\varepsilon_{xx}} - t_y \cancelto{0 \text{ rigid}}{u_{y,x}}\, d\Gamma$$

$$J_1 = \varepsilon \int_{\Gamma_1} t_x\, d\Gamma = \varepsilon F_x = \frac{E'h}{2}(2\varepsilon^2 - 2\varepsilon\varepsilon_c)$$



$$F_x = E'h\varepsilon - E'h\varepsilon_c$$

$$\rightarrow J = J_1 + J_2 + J_3 = \frac{E'h}{2}\left[-\varepsilon^2 + 4\varepsilon\varepsilon_c - 3\varepsilon_c^2 + 2\varepsilon^2 - 2\varepsilon\varepsilon_c\right]$$

$$= \frac{E'h}{2}\left[\varepsilon^2 + 2\varepsilon\varepsilon_c - 3\varepsilon_c^2\right]$$

$$J = \mathcal{G} = \frac{E'h}{2}(\varepsilon - \varepsilon_c)(\varepsilon + 3\varepsilon_c)$$