Some notation: σ_{ij} are Cartesian components of the Cauchy stress tensor

ε_{ij} are Cartesian components of the infinitesimal strain tensor

u_i are Cartesian components of the material displacement vector

T_i are Cartesian components of the traction vector

Repeated indicies imply summation, e.g.

$$\sigma_{ij}\varepsilon_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\varepsilon_{ij}$$

A comma followed by one or more indicies implies differentiation wrt the indicies after the comma,

e.g. $$u_{i,j} = \frac{\partial u_i}{\partial x_j} \quad \text{or} \quad u_{i,jk} = \frac{\partial^2 u_i}{\partial x_j \partial x_k}$$

We can have both differentiation and summation

e.g. $$\sigma_{ji,j} = \sum_{j=1}^{3}\frac{\partial \sigma_{ji}}{\partial x_j}$$

Our previous considerations gave us way to compute $\mathcal{G}$ in simple situations or measure it for a given sample geometry. How can we compute $\mathcal{G}$ in general?

Answer: Solve a boudary value problem.

Let's consider the predictions of linear elasticity theory for the stress fields near a crack tip.

For general elasticity problems we must solve the following equations.

Equilibrium: $\sigma_{ji,j} = 0$ in V (no body forces)

$\sigma_{ji} n_j = T_i$ on S
unit normal out of S

$\sigma_{ij} = \sigma_{ji}$ in V and on S

Compatibility: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

Hooke's law: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

or $\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl}$

First we will consider the in-plane loading modes,

$\therefore$ Equilibrium $\rightarrow$
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Compatibility $\rightarrow$
$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$
$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Isotropic elasticity $\rightarrow$
$$\varepsilon_{11} = \frac{1}{E'}\sigma_{11} - \frac{\nu'}{E'}\sigma_{22}$$
$$\varepsilon_{22} = \frac{1}{E'}\sigma_{22} - \frac{\nu'}{E'}\sigma_{11}$$
$$\varepsilon_{12} = \frac{1+\nu'}{E'}\sigma_{12}$$

$$E' = \begin{cases} E & \text{plane stress} \\ \dfrac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

$$\nu' = \begin{cases} \nu & \text{plane stress} \\ \dfrac{\nu}{1-\nu} & \text{plane strain} \end{cases}$$

To solve these equations we introduce Airy's stress function ϕ such that

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2}, \quad \sigma_{12} = \sigma_{21} = \frac{-\partial^2 \phi}{\partial x_1 \partial x_2}$$

then $$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} = \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2} - \frac{\partial^3 \phi}{\partial x_1 \partial x_2^2} = 0 \checkmark$$

and $$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = -\frac{\partial^3 \phi}{\partial x_1^2 \partial x_2} + \frac{\partial^3 \phi}{\partial x_1^2 \partial x_2} = 0 \checkmark$$

$\therefore$ Equilibrium is automatically satisfied.

Compatibility can be written as

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} - \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} - \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} = 0$$

inserting Hooke's law gives

$$\frac{1}{E'}\frac{\partial^2 \sigma_{11}}{\partial x_2^2} - \frac{\nu'}{E'}\frac{\partial^2 \sigma_{22}}{\partial x_2^2} + \frac{1}{E'}\frac{\partial^2 \sigma_{22}}{\partial x_1^2} - \frac{\nu'}{E'}\frac{\partial^2 \sigma_{11}}{\partial x_1^2} - \frac{2}{E'}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} - \frac{2\nu'}{E'}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = 0$$

Now insert ϕ and we can multiply by E'

$$\phi_{,2222} - \nu' \phi_{,1122} + \phi_{,1111} - \nu' \phi_{,1122} + 2\phi_{,1122} + 2\nu' \phi_{,1122} = 0$$

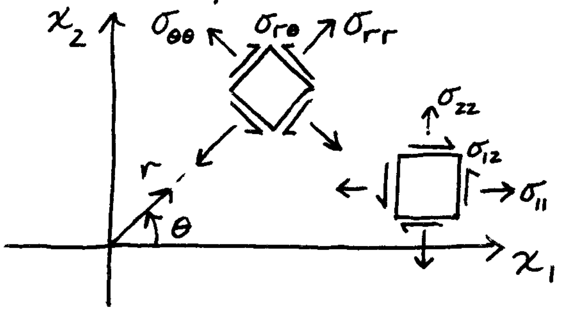
$$\therefore \quad \phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0$$

or $\nabla^4 \phi = \nabla^2(\nabla^2 \phi) = \dfrac{\partial^4 \phi}{\partial x_1^4} + 2\dfrac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \dfrac{\partial^4 \phi}{\partial x_2^4} = 0$

This is the biharmonic equation.

Note in polar coordinates $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$

Also in polar coordinates $\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$



$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

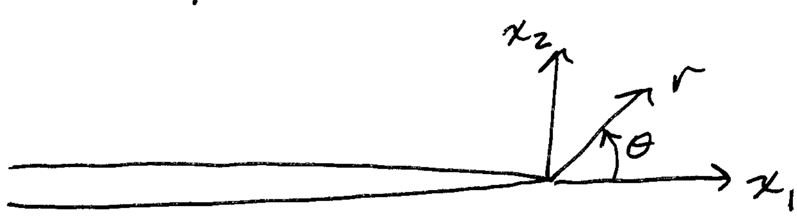
$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

It can be shown (and you can verify) that

$$\phi = \sum_p r^{p+2}\left[A_p \cos p\theta + B_p \cos(p+2)\theta + C_p \sin p\theta + D_p \sin(p+2)\theta\right]$$

satisfies $\nabla^4 \phi = 0$

For our problem we want to investigate the fields very close to the crack tip.



The crack faces are traction free so we have the following boundary conditions.

at $\theta = \pm\pi$ for all r $\sigma_{22} = 0$ and $\sigma_{12} = 0$

or in polar coordinates $\sigma_{\theta\theta} = 0$ and $\sigma_{r\theta} = 0$
at $\theta = \pm\pi$ and any r

From ϕ we can derive $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ as

$$\sigma_{\theta\theta} = \sum_p (p+1)(p+2) r^p \left[A_p \cos p\theta + B_p \cos(p+2)\theta + C_p \sin p\theta + D_p \sin(p+2)\theta\right]$$

$$\sigma_{r\theta} = \sum_p (p+1) r^p \left[p A_p \sin p\theta + (p+2) B_p \sin(p+2)\theta - p C_p \cos p\theta - (p+2) D_p \cos(p+2)\theta\right]$$

at $\theta = \pm\pi$ we must have $\sigma_{\theta\theta}(r, \pm\pi) = 0$
and $\sigma_{r\theta}(r, \pm\pi) = 0$

$\rightarrow$ $A_p \cos p\pi + B_p \cos(p+2)\pi \pm C_p \sin p\pi \pm D_p \sin(p+2)\pi = 0$
and
$\pm p A_p \sin p\pi \pm (p+2) B_p \sin(p+2)\pi - p C_p \cos p\pi - (p+2) D_p \cos(p+2)\pi = 0$

These equations can be satisfied 2 ways

1) $\sin p\pi = 0$, $\rightarrow \cos p\pi = \pm 1 \rightarrow A_p + B_p = 0$
$\rightarrow p =$ integers and $p C_p + (p+2) D_p = 0$

2) $\cos p\pi = 0 \rightarrow \sin p\pi = \pm 1 \rightarrow C_p + D_p = 0$
$\rightarrow p = \frac{\text{odd integers}}{2}$ and $p A_p + (p+2) B_p = 0$

Let's consider the most singular term, we can write

$$\sigma_{\theta\theta} = r^p \, \tilde{\sigma}_{\theta\theta/p}(\theta)$$

$$\sigma_{rr} = r^p \, \tilde{\sigma}_{rr/p}(\theta)$$

$$\sigma_{r\theta} = r^p \, \tilde{\sigma}_{r\theta/p}(\theta)$$

or for Cartesian components $\sigma_{ij} = r^p \tilde{\sigma}_{ij/p}(\theta)$

Then strains are $\varepsilon_{ij} = s_{ijkl} \, r^p \tilde{\sigma}_{kl/p}(\theta)$

and the strain energy density is

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2} r^{2p} \underbrace{s_{ijkl}\,\tilde{\sigma}_{ij/p}(\theta)\,\tilde{\sigma}_{kl/p}(\theta)}_{2\tilde{U}_p(\theta)}$$

$$\therefore \; U = r^{2p}\,\tilde{U}_p(\theta)$$

most singular term

Now the total strain energy in some finite region near the crack tip is

$$\int_V U \, dV = \int_{-\pi}^{\pi}\int_0^R r^{2p}\,\tilde{U}_p(\theta)\, r\,dr\,d\theta$$

$$= \int_{-\pi}^{\pi} \tilde{U}_p(\theta)\,d\theta \int_0^R r^{2p+1}\,dr$$

the integral $\int_{-\pi}^{\pi} \tilde{U}_p(\theta)\,d\theta$ will be finite

but $\int_0^R r^{2p+1}\,dr$

$$= \frac{1}{2p+2} r^{2p+2} \Big]_0^R \quad \text{for } p \neq -1$$

$$= \ln r \Big]_0^R \quad \text{for } p = -1$$

if $p = -1$ $\ln r$ diverges as $r \to 0$

if $p \neq -1$ r^{2p+2} diverges ar $r \to 0$ if $p < -1$

Therefore, in order to have a finite amount of energy stored in a finite region near the crack tip $p > -1$.

$$\therefore \; p = -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

Let's consider the most dominant term, i.e. $p = -\frac{1}{2}$.

Then as $r \to 0$ $\quad \sigma_{ij} \to r^{-1/2}\,\tilde{\sigma}_{ij}(\theta)$

For $p = -\frac{1}{2}$ $\to$ $C_{-1/2} = -D_{-1/2}$

$$B_{-1/2} = \frac{1}{3} A_{-1/2}$$

$$\therefore \quad \sigma_{\theta\theta} = A r^{-1/2}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + C r^{-1/2}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = A r^{-1/2}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + C r^{-1/2}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{rr} = A r^{-1/2}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + C r^{-1/2}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

A convention due to Irwin calls for (Actually, Irwin used $\frac{K}{\sqrt{2r}}$, the π showed up later)

$\sigma_{yy} = \dfrac{K_I}{\sqrt{2\pi r}}$ and $\sigma_{xy} = \dfrac{K_{II}}{\sqrt{2\pi r}}$ on the plane ahead of the crack, i.e. on $\theta = 0$.

$$\sigma_{yy}(r, \theta=0) = \sigma_{\theta\theta}(r, \theta=0) = \frac{K_I}{\sqrt{2\pi r}} \rightarrow A = \frac{K_I}{\sqrt{2\pi}}$$

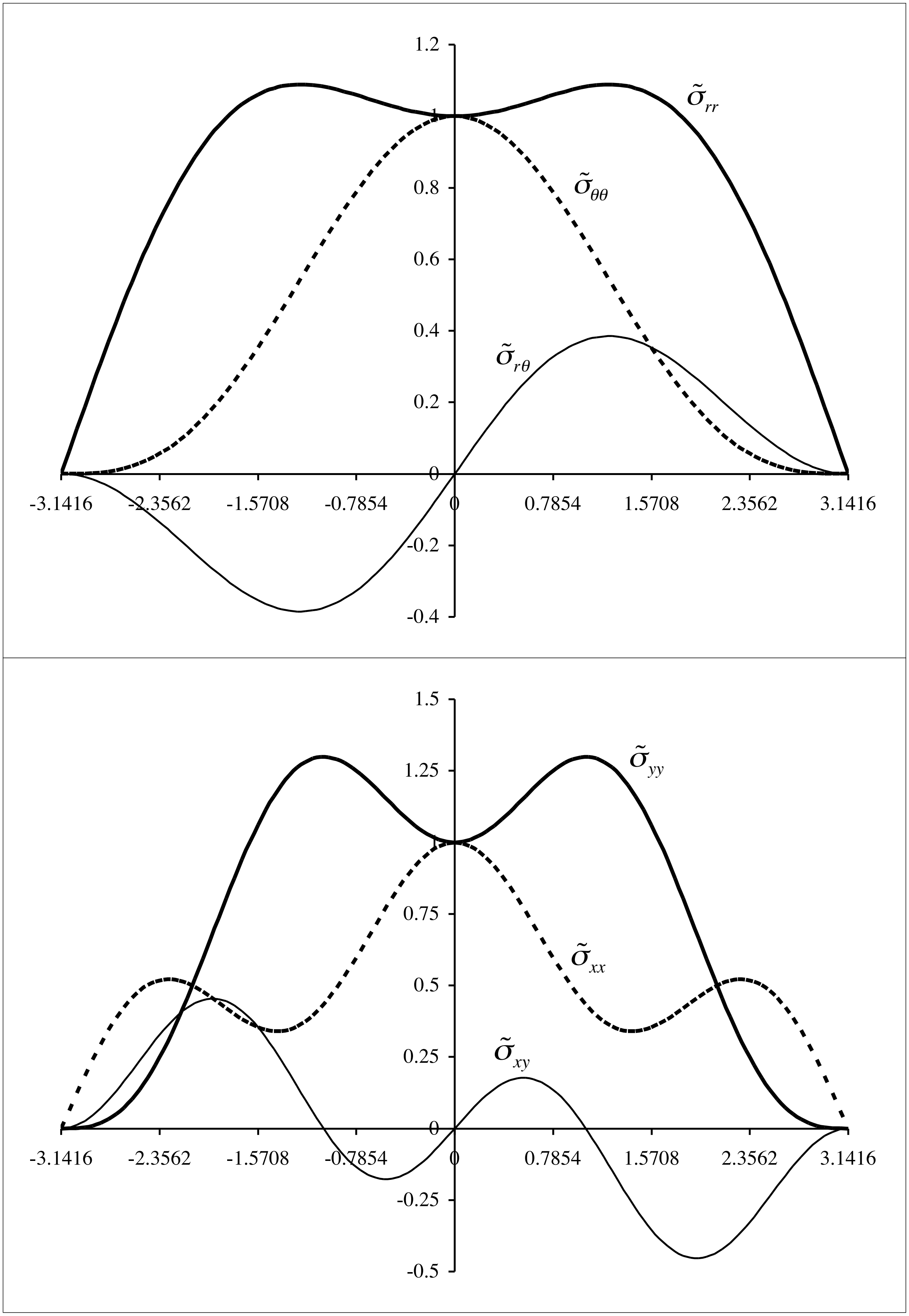
$$\sigma_{xy}(r, \theta=0) = \sigma_{r\theta}(r, \theta=0) = \frac{K_{II}}{\sqrt{2\pi r}} \rightarrow C = \frac{K_{II}}{\sqrt{2\pi}}$$

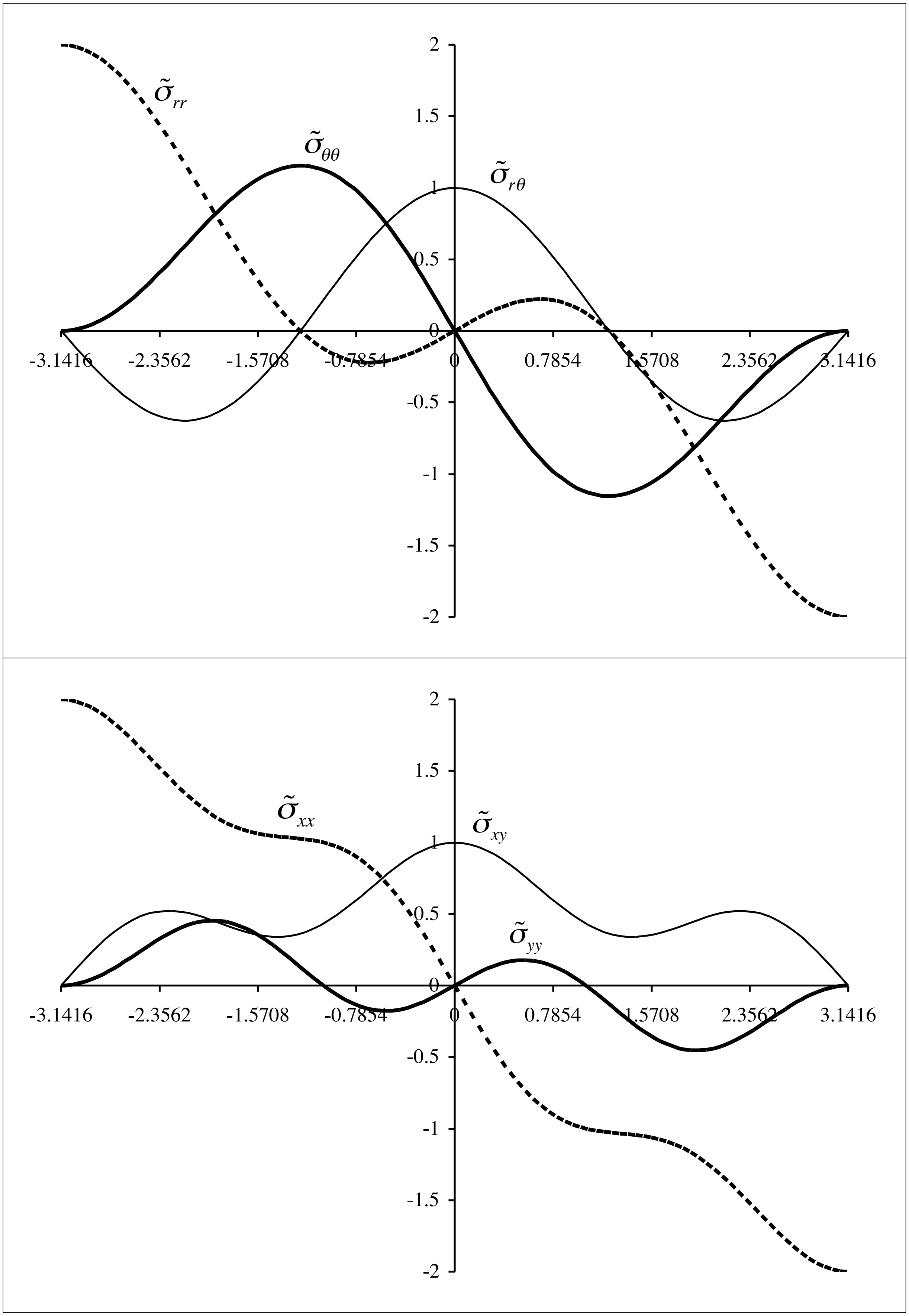
$$\boxed{\begin{aligned} \therefore \quad \sigma_{\theta\theta} &= \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right) \\ \sigma_{r\theta} &= \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) \\ \sigma_{rr} &= \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right) \end{aligned}}$$

In Cartesian components these can be written as

$$\boxed{\begin{aligned}
\sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right] \\
&\quad + \frac{K_{II}}{\sqrt{2\pi r}}\left[-\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\right] \\
\sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right] \\
&\quad + \frac{K_{II}}{\sqrt{2\pi r}}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \\
\sigma_{xy} &= \frac{K_I}{\sqrt{2\pi r}}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \\
&\quad + \frac{K_{II}}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right]
\end{aligned}}$$

K_I and K_{II} have dimensions of $\sigma\sqrt{L}$ and are called the mode I and mode II stress intensity factors. In general, these depend on the specific loading and geometry of the specimen.


1.2
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-3.1416
-2.3562
-1.5708
-0.7854
0
0.7854
1.5708
2.3562
3.1416
$\tilde{\sigma}_{rr}$
$\tilde{\sigma}_{\theta\theta}$
$\tilde{\sigma}_{r\theta}$
1.5
1.25
0.75
0.5
0.25
0
-0.25
-0.5
$\tilde{\sigma}_{yy}$
$\tilde{\sigma}_{xx}$
$\tilde{\sigma}_{xy}$


$\tilde{\sigma}_{rr}$
$\tilde{\sigma}_{\theta\theta}$
$\tilde{\sigma}_{r\theta}$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.1416
-2.3562
-1.5708
-0.7854
0
0.7854
1.5708
2.3562
3.1416
$\tilde{\sigma}_{xx}$
$\tilde{\sigma}_{xy}$
$\tilde{\sigma}_{yy}$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3.1416
-2.3562
-1.5708
-0.7854
0
0.7854
1.5708
2.3562
3.1416

The corresponding displacement fields are

$$u_r = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[(2\chi-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$
$$+ \frac{K_{II}}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi-1)\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\right]$$

$$u_\theta = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]$$
$$+ \frac{K_{II}}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi+1)\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right]$$

$$u_x = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[(2\chi-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$
$$+ \frac{K_{II}}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[(2\chi+3)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]$$

$$u_y = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[(2\chi+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right]$$
$$+ \frac{K_{II}}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi-3)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$

where $\chi = \begin{cases} 3-4\nu & \text{plane strain} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$

Finally, consider the $p = 0$ term.

$$\rightarrow \quad \phi_0 = \left(A_0 + B_0 \cos 2\theta + D_0 \sin 2\theta\right) r^2$$

with the BCs $\rightarrow$

$$A_0 = -B_0$$

$$2D_0 = 0 \rightarrow D_0 = 0$$

$$\therefore \quad \phi_0 = A_0 \left(1 - \cos 2\theta\right) r^2$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi_0}{\partial r^2} = 2A_0\left(1 - \cos 2\theta\right)$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 2A_0\left(1 - \cos 2\theta\right) + 4A_0 \cos 2\theta$$
$$= 2A_0\left(1 + \cos 2\theta\right)$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} = 2A_0 \sin 2\theta - 4A_0 \sin 2\theta$$
$$= -2A_0 \sin 2\theta$$

$4A_0 = T$ by convention & is called the "T stress"

Transforming to Cartesian coordinates gives

$$\boxed{\left.\begin{aligned} \sigma_{xx} &= T_{xx} \\ \sigma_{yy} &= 0 \\ \sigma_{xy} &= 0 \end{aligned}\right\} \text{T stress terms}}$$

Also note that a uniaxial stress in the x_3 direction can be applied and the BCs will still be satisfied, hence there is a T_{33} (T_{zz}) T-stress term as well.

Finally, there exists another mode of crack loading called mode III. This mode is a "tearing" mode and results from anti-plane/longitudinal shear.

Longitudinal shear in isotropic elasticity is governed by the following equations:

Equilibrium: $$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0$$

Compatibility: $$\varepsilon_{13} = \frac{1}{2}\frac{\partial u_3}{\partial x_1}$$

$$\varepsilon_{23} = \frac{1}{2}\frac{\partial u_3}{\partial x_2}$$

Hooke's law: $$\sigma_{13} = 2\mu\,\varepsilon_{13} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$\sigma_{23} = 2\mu\,\varepsilon_{23}$$

For HW you will be in charge of finding the asymptotic K_{III} field and any T-stresses.

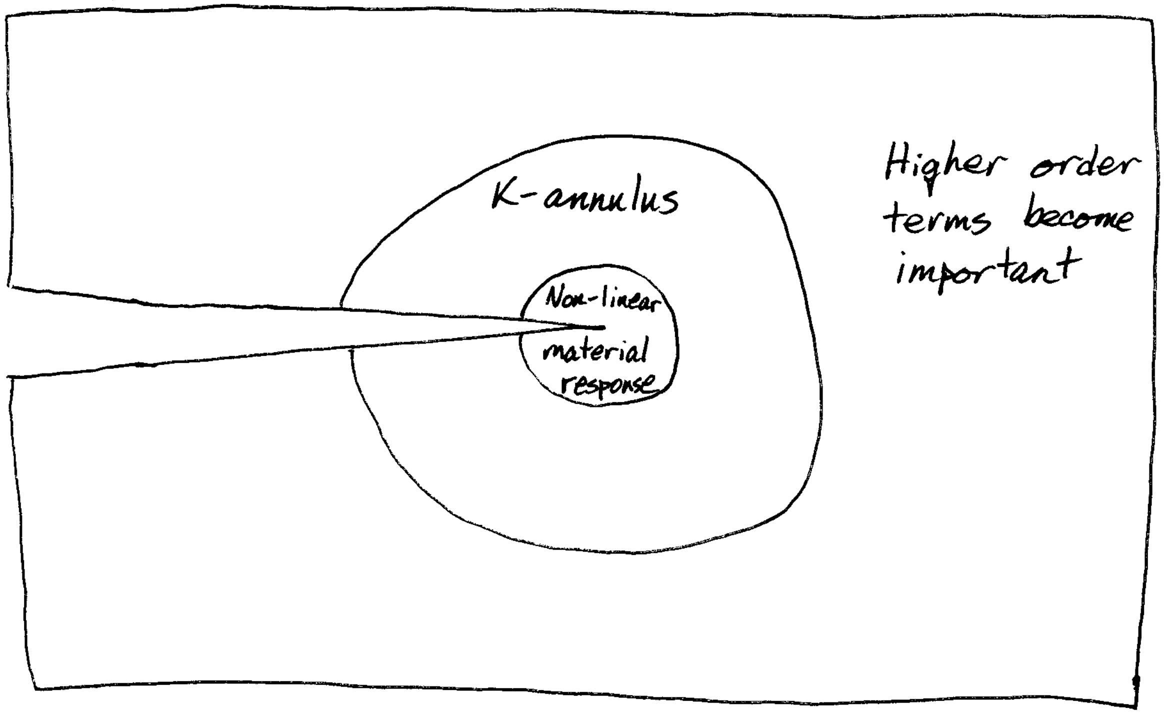
So the stress field near a crack tip can be expanded in the following way

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}}\tilde{\sigma}^{I}_{ij}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}}\tilde{\sigma}^{II}_{ij}(\theta) + \frac{K_{III}}{\sqrt{2\pi r}}\tilde{\sigma}^{III}_{ij}(\theta)$$

$$+ T_{11}\,\delta_{i1}\delta_{j1} + T_{33}\,\delta_{i3}\delta_{j3} + T_{13}(\delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1})$$

$$+ O(r^{1/2}) + O(r) + \ldots$$

This gives rise to the idea of a "K-annulus"



There usually exists a region between R_m and R_G where the dominant K terms are valid. In the region inside R_m the assumptions of linear elasticity break down, i.e. physically stresses do not $\to \infty$. This is usually manifested in some type of material non-linearity.

In the "perfectly brittle" materials that we considered in the first lecture the atomic forces were limited to some peak values. For ductile metals, yielding will occur at high stress. The size of R_m can be estimated as follows,

$$\sigma \propto \frac{K}{\sqrt{2\pi r}} \leq \sigma_c \quad (\sigma_c = \text{peak stress or yield strength etc.})$$

$$\therefore \quad R_m \approx \frac{1}{2\pi}\left(\frac{K}{\sigma_c}\right)^2$$

In the region outside R_G higher order terms (T stresses and above) arising due to the specimen <u>geometry</u> become important. Therefore

R_G is at least an order of magnitude smaller that any characteristic specimen dimension including the length of the crack.

$$\therefore \quad R_G \approx \frac{1}{10} \min(a, L, \dots)$$

a = crack length

L = specimen dimension, e.g. ligament length

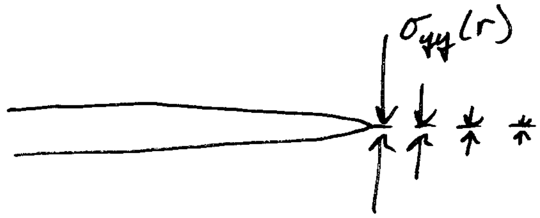
In order to determine K for a given geometry and loading we still need to solve a (usually complicated) boundary value problem.

Before we get to that let's determine the relationship between K and $\mathcal{G}$.

<u>Mode I</u>

Irwin performed the following "crack closure" integral to determine how much energy is required to close the crack by the increment δa.

In order to close the crack we must apply the tractions



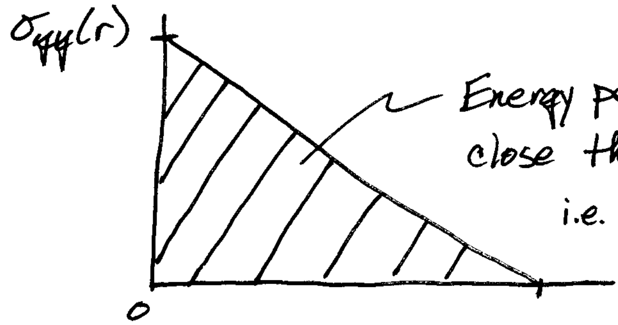
$$\sigma_{yy}(r) = \frac{K_I}{\sqrt{2\pi r}}$$

As we apply these tractions the crack opening displacement goes from

$$COD = u_y(r', \pi) - u_y(r', -\pi) = \frac{K_I}{E}\sqrt{\frac{r'}{2\pi}}\,(1+\nu)(2\chi+2)$$

to zero.

Consider some point along the closing region, and the energy per unit area required to close that point



Energy per area to close the point at r, i.e. at $r' = \delta a - r$ $= \frac{1}{2}\sigma_{yy}(r)\,COD(\delta a - r)$

$COD(r') = COD(\delta a - r)$

Now the total energy per unit thickness required to close the crack is

$$\delta W = \int_0^{\delta a} \frac{1}{2}\sigma_{yy}(r)\,COD(\delta a - r)\,dr$$

$$= \frac{1}{2}\frac{K_I}{\sqrt{2\pi}}\frac{K_I}{E}\frac{(1+\nu)\,2(\chi+1)}{\sqrt{2\pi}}\int_0^{\delta a}\sqrt{\frac{\delta a - r}{r}}\,dr$$

take $r = \delta a \sin^2 u$

$dr = 2\,\delta a \sin u \cos u \, du$

then $$\delta W = \frac{K_I^2}{E}\frac{(1+\nu)(\chi+1)}{2\pi}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\delta a-\delta a\sin^2 u}{\delta a\sin^2 u}}\,2\delta a\sin u\cos u\,du$$

$$= \frac{K_I^2}{E}\frac{(1+\nu)(\chi+1)}{2\pi}2\delta a\int_0^{\pi/2}\cos^2 u\,du$$

$$= \frac{K_I^2}{E}\frac{(1+\nu)(\chi+1)}{\pi}\delta a\int_0^{\pi/2}\frac{1}{2}(1+\cos 2u)\,du$$

$$= \frac{K_I^2}{E}\frac{(1+\nu)(\chi+1)}{2\pi}\delta a\left(u+\frac{1}{2}\sin 2u\right)\Big]_0^{\pi/2}$$

$$= \frac{K_I^2}{E}\frac{(1+\nu)(\chi+1)}{2\pi}\delta a\left(\frac{\pi}{2}-0\right)$$

$$\delta W = \frac{K_I^2}{E}\frac{(1+\nu)(\chi+1)}{4}\delta a = \mathcal{G}\,\delta a$$

$$\therefore \boxed{\mathcal{G} = \begin{cases}\dfrac{K_I^2}{E} & \text{plane stress}\\[2ex] \dfrac{K_I^2(1-\nu^2)}{E} & \text{plane strain}\end{cases}}$$

We will find similar results for Mode II and Mode III, i.e.

for homework

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

True for isotropic elasticity, i.e. modes decouple

Note that stresses and displacements are linear in K but non-linear in $\mathcal{G}$. Therefore K values can be added for two superposed elasticity problems/solutions, but $\mathcal{G}$ values <u>cannot</u> be added in general.

One exception to this rule is the decoupling of modes I, II and III in isotropic elasticity.

For anisotropic elasticity

$$\mathcal{G} = \sum_{i=I}^{III} \sum_{j=I}^{III} K_i H_{ij} K_j$$

where for isotropic elasticity $\underset{\approx}{H} = \begin{bmatrix} 1/E' & 0 & 0 \\ 0 & 1/E' & 0 \\ 0 & 0 & 1/2\mu \end{bmatrix}$

Back to the superposition argument,

$$\mathcal{G}^{(1)} = \frac{(K_I^{(1)})^2}{E'} \quad , \quad \mathcal{G}^{(2)} = \frac{(K_I^{(2)})^2}{E'}$$

$$K_I = K_I^{(1)} + K_I^{(2)}$$

$$\mathcal{G} = \frac{K_I^2}{E'} = \frac{(K_I^{(1)} + K_I^{(2)})^2}{E'} \neq \mathcal{G}^{(1)} + \mathcal{G}^{(2)}$$