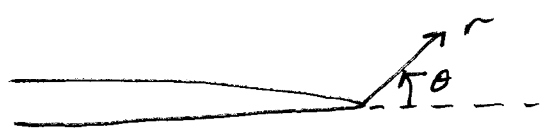
Now we know the relationship for $\mathcal{G}$ in terms of K. Given an elasticity solution, how do we dermine K?

The two most common procedures are to use the stresses in front of the crack tip or the crack opening/sliding displacements behind the crack tip.



$$\left.\begin{aligned}
K_I &= \lim_{r\to 0} \sqrt{2\pi r}\;\sigma_{yy}(r, \theta=0) \\
K_{II} &= \lim_{r\to 0} \sqrt{2\pi r}\;\sigma_{xy}(r, \theta=0) \\
K_{III} &= \lim_{r\to 0} \sqrt{2\pi r}\;\sigma_{yz}(r, \theta=0)
\end{aligned}\right\}$$

True for isotropic <u>and</u> anisotropic materials.

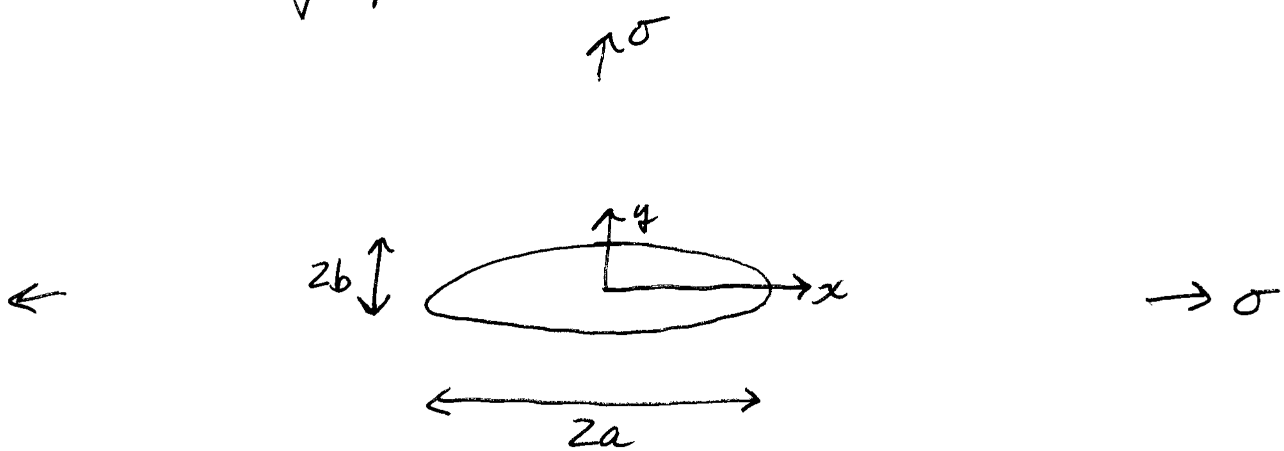
or

$$\left.\begin{aligned}
K_I &= \lim_{r\to 0} \frac{E'}{8}\sqrt{\frac{2\pi}{r}}\,\underbrace{\left[u_y(r, \theta=\pi) - u_y(r, \theta=-\pi)\right]}_{COD(r)} \\
K_{II} &= \lim_{r\to 0} \frac{E'}{8}\sqrt{\frac{2\pi}{r}}\,\underbrace{\left[u_x(r, \theta=\pi) - u_x(r, \theta=-\pi)\right]}_{CSD(r)} \\
K_{III} &= \lim_{r\to 0} \frac{\mu}{4}\sqrt{\frac{2\pi}{r}}\,\underbrace{\left[u_z(r, \theta=\pi) - u_z(r, \theta=-\pi)\right]}_{CSD(r)}
\end{aligned}\right\}$$

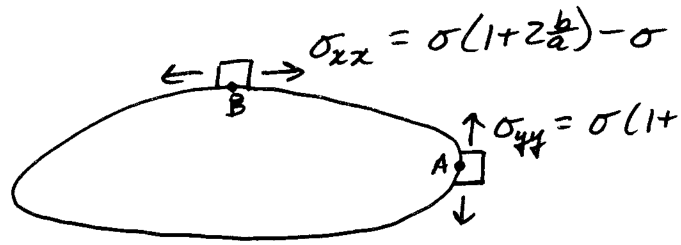
True <u>only</u> for isotropic materials.

The trick is to either get the stresses or displacements.

Before we get to complex variable methods, consider the following problem.



From elasticity we know the stress concentrations at 2 key points.



And we also know, from Eshelby, that the strain in an ellipsoidal inclusion embedded in an infinite matrix loaded at infinity is <u>uniform</u>.

How can we use this information to determine K_I for a center crack?

First, think of the void as an "inclusion" with shear modulus $\mu \to 0$.

Using strain continuity conditions at the edge of the void we can determine the "strain" in the void.

i.e. $\varepsilon_{yy}^{A/void} = \varepsilon_{yy}^{A/matrix}$ and $\varepsilon_{xx}^{B/void} = \varepsilon_{xx}^{B/matrix}$

when we have the symmetric loading shown, we know $\varepsilon_{xy}^{void} = 0$.

at A $\sigma_{xx} = 0 \rightarrow \varepsilon_{yy}^{A} = \frac{\sigma}{E'}\left[2\frac{a}{b}\right] = \varepsilon_{yy}^{void}$

at B $\sigma_{yy} = 0 \rightarrow \varepsilon_{xx}^{B} = \frac{\sigma}{E'}\left[2\frac{b}{a}\right] = \varepsilon_{xx}^{void}$

then the displacement field in the void is

$$u_x = \varepsilon_{xx}^{void}\, x = \frac{2\sigma}{E'}\frac{b}{a}x$$

$$u_y = \varepsilon_{yy}^{void}\, y = \frac{2\sigma}{E'}\frac{a}{b}y$$

To model a crack we take $b \to 0$, but only after we determine the displacements on the surface.

i.e. the surface is $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$

$$\therefore \quad \frac{y}{b} = \pm\sqrt{1-\left(\frac{x}{a}\right)^2}$$

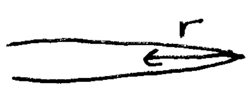
$\therefore$ on the surface $$u_y = \frac{2\sigma}{E'}\frac{a}{b}\left(\pm b\sqrt{1-\left(\frac{x}{a}\right)^2}\right)$$

$$= \pm\frac{2\sigma}{E'}\sqrt{a^2-x^2}$$

$$u_x = \frac{2\sigma}{E'}\frac{b}{a}x$$

Now take $\lim\limits_{b\to 0}$ $\rightarrow$ $$u_x = 0$$ $$u_y = \pm\frac{2\sigma}{E'}\sqrt{a^2-x^2}$$

$$\therefore \quad COD(x) = \frac{4\sigma}{E'}\sqrt{a^2-x^2}$$



$r = a - x$ on right crack tip

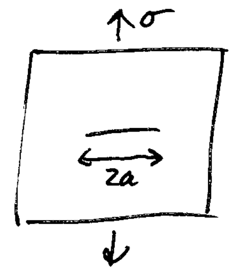
$$\therefore COD(r) = \frac{4\sigma}{E'}\sqrt{(a+x)(a-x)}$$

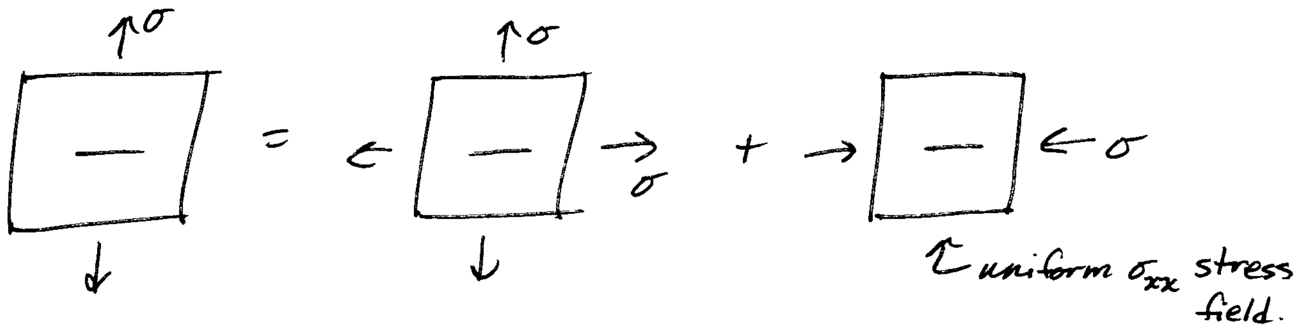
$$= \frac{4\sigma}{E'}\sqrt{(2a-r)r}$$

$$K_I = \lim_{r\to 0}\frac{E'}{8}\sqrt{\frac{2\pi}{r}}\,COD(r) = \lim_{r\to 0}\frac{\cancel{E'}}{8}\sqrt{\frac{2\pi}{\cancel{r}}}\,\frac{4\sigma}{\cancel{E'}}\sqrt{(2a-r)\cancel{r}}$$

$$= \lim_{r\to 0}\frac{\sigma\sqrt{2\pi}}{2}\sqrt{2a-r} = \sigma\sqrt{\pi a}$$

$$\boxed{K_I = \sigma\sqrt{\pi a}}$$

What about:  ?



$$K_I \quad = \quad K_I = \sigma\sqrt{\pi a} \quad + \quad K_I = 0$$

$$\therefore \quad K_I = \sigma\sqrt{\pi a} \quad \text{for this problem too.}$$

This is nice for a simple problem, but what about more general methods?

Westergaard's Stress Functions

Recall standard complex potentials $\phi(z)$ and $\psi(z)$.

$$z = x + iy = re^{i\theta}$$

$$\bar{z} = x - iy = re^{-i\theta}$$

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}\left[\phi'(z)\right]$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right]$$

or $\quad \sigma_{xx} = \mathrm{Re}\left[2\phi'(z) - \bar{z}\phi''(z) - \psi'(z)\right]$

$$\sigma_{yy} = \mathrm{Re}\left[2\phi'(z) + \bar{z}\phi''(z) + \psi'(z)\right]$$

$$\sigma_{xy} = \mathrm{Im}\left[\bar{z}\phi''(z) + \psi'(z)\right]$$

Consider cracks on the x-axis with symmetric loading such that

$$\sigma_{xx}(x,y) = \sigma_{xx}(x,-y)$$
$$\sigma_{yy}(x,y) = \sigma_{yy}(x,-y)$$
$$\sigma_{xy}(x,y) = -\sigma_{xy}(x,-y)$$
$$\rightarrow \sigma_{xy}(x, y=0) = 0$$

Reorganize: $\sigma_{xx} = \mathrm{Re}\left[2\phi'(z) + (z-\bar{z})\phi''(z) - z\phi''(z) - \psi'(z)\right]$

$$\sigma_{yy} = \mathrm{Re}\left[2\phi'(z) - (z-\bar{z})\phi''(z) + z\phi''(z) + \psi'(z)\right]$$

$$\sigma_{xy} = \mathrm{Im}\left[-(z-\bar{z})\phi''(z) + z\phi''(z) + \psi'(z)\right]$$

Now, since $z\phi''(z)$ is an analytic function we can always take $\psi'(z) = -z\phi''(z)$ and we are ensured that the equations of elasticity are satisfied. However, the solutions that these functions generate only satisfy a limited set of boundary conditions.

These solutions have $\sigma_{xy}(x, y=0) = 0$ and $\sigma_{xx}(x, y=0) = \sigma_{yy}(x, y=0)$.

Define the mode I Westergaard stress function as

$$Z_I(z) = 2\phi'(z)$$

and note that $z - \bar{z} = 2iy$

$$\therefore \quad \boxed{\begin{aligned} \sigma_{xx} &= Re\left[Z_I(z) + iy\,Z_I'(z)\right] \\ &= Re\left[Z_I(z)\right] - y\,Im\left[Z_I'(z)\right] \\ \sigma_{yy} &= Re\left[Z_I(z) - iy\,Z_I'(z)\right] \\ &= Re\left[Z_I(z)\right] + y\,Im\left[Z_I'(z)\right] \\ \sigma_{xy} &= Im\left[-iy\,Z_I'(z)\right] \\ &= -y\,Re\left[Z_I'(z)\right] \end{aligned}}$$

Useful for mode I solutions for cracks on the x-axis in infinite 2-D spaces.

Next consider mode II type loadings were the antisymmetry dictates

$$\sigma_{xy}(-x,y) = \sigma_{xy}(x,-y)$$

$$\sigma_{xx}(x,y) = -\sigma_{xx}(x,-y)$$

$$\sigma_{yy}(x,y) = -\sigma_{yy}(x,-y)$$

For traction free crack faces these conditions imply $\rightarrow$ $\sigma_{yy}(x, y=0) = 0$

but $\sigma_{xx}(x, y=0) \neq 0$ due to the possibility of discontinuity across the crack

Note that $\sigma_{xx}(x,y=0) = 0$ on <u>intact</u> regions.

$$\sigma_{yy} = \mathrm{Re}\left[2\phi'(z) - (z-\bar{z})\phi''(z) + z\phi''(z) + \psi'(z)\right]$$

$\therefore$ take $\quad \psi'(z) = -z\phi''(z) - 2\phi'(z)$

again $\rightarrow$ $\psi(z)$ is analytic $\rightarrow$ equations of elasticity are satisfied

Now define the mode II Westergaard stress function as

$$Z_{II}(z) = i2\phi'(z)$$

$$\therefore \quad \sigma_{yy} = -y \, Re[Z_{II}'(z)]$$

$$\sigma_{xx} = Re[-2i \, Z_{II}(z) + y \, Z_{II}'(z)]$$

$$= 2 \, Im[Z_{II}(z)] + y \, Re[Z_{II}'(z)]$$

$$\sigma_{xy} = Im[-y \, Z_{II}'(z) + i \, Z_{II}(z)]$$

$$= Re[Z_{II}(z)] - y \, Im[Z_{II}'(z)]$$

Again, these are useful for cracks on the x-axis in infinite 2-D spaces with Mode II type loading.

<u>Displacement fields</u>

$$2\mu u_x = \tfrac{1}{2}(\chi-1) Re[\hat{Z}_I(z)] - y \, Im[Z_I(z)] + \tfrac{1}{2}(\chi+1) Im[\hat{Z}_{II}(z)] + y \, Re[Z_{II}(z)]$$

$$2\mu u_y = \tfrac{1}{2}(\chi+1) Im[\hat{Z}_I(z)] - y \, Re[Z_I(z)] - \tfrac{1}{2}(\chi-1) Re[\hat{Z}_{II}(z)] - y \, Im[Z_{II}(z)]$$

where $Z_I(z) = \dfrac{d\hat{Z}_I}{dz}$ and $Z_{II}(z) = \dfrac{d\hat{Z}_{II}}{dz}$

Finally in Mode III we know that a single analytic function satisfies the equations of elasticity.

i.e.
$$u_z = \frac{1}{\mu}\,\mathrm{Im}[\omega(z)]$$
$$\sigma_{yz} + i\sigma_{xz} = \omega'(z)$$

Redefine : $Z_{III}(z) = \omega'(z)$, $\hat{Z}_{III}(z) = \omega(z)$

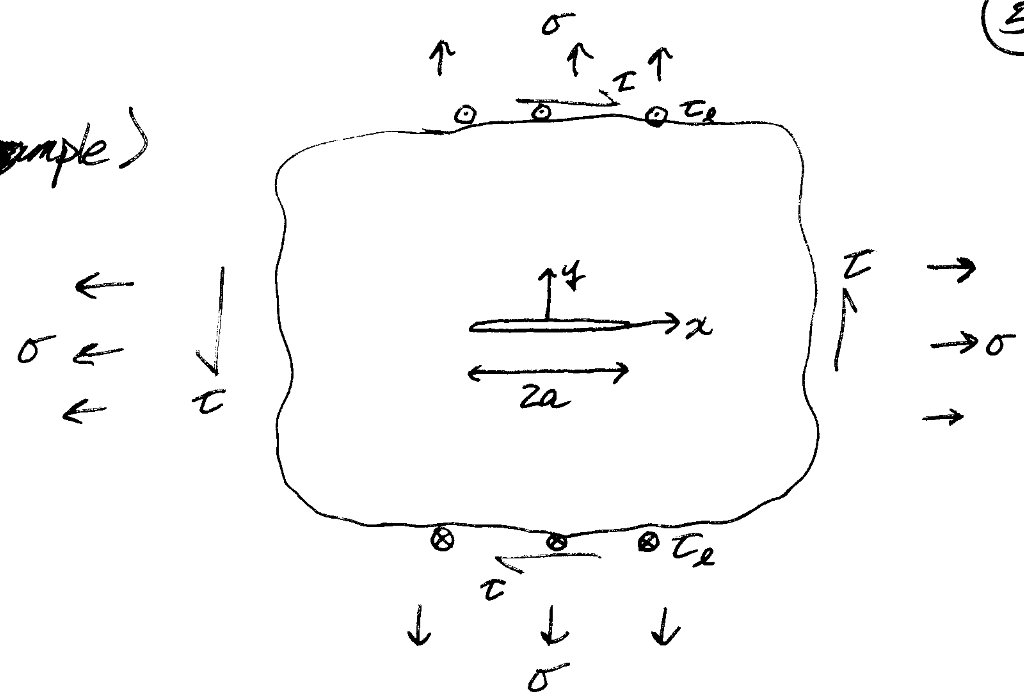
then
$$\boxed{\begin{aligned} \sigma_{yz} &= \mathrm{Re}[Z_{III}(z)] \\ \sigma_{xz} &= \mathrm{Im}[Z_{III}(z)] \\ u_z &= \frac{1}{\mu}\,\mathrm{Im}[\hat{Z}_{III}(z)] \end{aligned}}$$
Good for any Mode III problem

With these definitions for the Westergaard stress functions, it turns out that many problems have similar solutions in different modes.

Ex) The asymptotic solutions look like

$$\begin{Bmatrix} Z_I(z) \\ Z_{II}(z) \\ Z_{III}(z) \end{Bmatrix} = \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \frac{1}{\sqrt{2\pi z}}$$

Another example)



Let's look at Mode I first.

BCs $\rightarrow$ $\sigma_{xy}(|x|<a, y=0)=0$ $\leftarrow$ Satisfied automatically by Z_I

$$\sigma_{yy}(|x|<a, y=0)=0$$

$$\sigma_{yy} = \mathrm{Re}\, Z_I + y\, \mathrm{Im}\, Z_I'$$

$$\sigma_{yy}(|x|<a, y=0) = \mathrm{Re}\, Z_I \Big|_{\substack{|x|<a \\ y=0}} = 0$$

So, for $|x|<a$ Z_I must be imaginary.

This is a case where we need the quantity $\sqrt{x^2-a^2}$, or in terms of $z \rightarrow \sqrt{z^2-a^2}$

Also, as $r=|z|\rightarrow\infty$ we need $\sigma_{xx}=\sigma_{yy}=\sigma$.

Note that Z_I has dimensions of stress.

Also, as we approach the crack tips we expect $1/\sqrt{r}$ type singularities, $\therefore$ the $\sqrt{z^2-a^2}$ term should appear in the denominator.

The BCs as $|z|\to\infty$ imply $Z_I \to \sigma$ as $|z|\to\infty$

$$\boxed{\begin{aligned} &\therefore \quad Z_I = \frac{\sigma z}{\sqrt{z^2-a^2}} \\ &\text{we would also find} \\ &\qquad Z_{II} = \frac{\tau z}{\sqrt{z^2-a^2}} \\ &\qquad Z_{III} = \frac{\tau_e z}{\sqrt{z^2-a^2}} \end{aligned}}$$

Need to use the following branch cuts

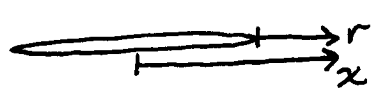
$z \to re^{i\theta}$

$z-a \to r_1 e^{i\theta_1}$

$z+a \to r_2 e^{i\theta_2}$

Determine K_I: $\quad \sigma_{yy}(x>a, y=0) = \operatorname{Re} Z_I \Big|_{\substack{x>a \\ y=0}}$

$$\sigma_{yy} = \frac{\sigma x}{\sqrt{x^2-a^2}} = \frac{\sigma x}{\sqrt{(x+a)(x-a)}}$$



$x = r + a$

$$\therefore \sigma_{yy} = \frac{\sigma(r+a)}{\sqrt{(r+2a)r}}$$

$$K_I = \lim_{r\to 0} \sigma_{yy}\sqrt{2\pi r} = \lim_{r\to 0} \frac{\sigma(r+a)}{\sqrt{(r+2a)\cancel{r}}}\sqrt{2\pi \cancel{r}} = \frac{\sigma a}{\sqrt{2a}}\sqrt{2\pi} = \sigma\sqrt{\pi a}$$

Similarly we would find $K_{II} = \tau\sqrt{\pi a}$

$K_{III} = \tau_e \sqrt{\pi a}$