Computing/Modeling Features of the R-curve

Up to now we have focused our attention on stationary cracks. We have been interested in identifying rational parameters that can be used to predict when a crack will propagate. These parameters have been K and $\mathcal{G}$ for linear elastic solids and J and $\mathcal{G}$ for non-linear elastic solids.

In addition to these fracture parameters we have also alluded to an interesting material response to crack growth called R-curve behavior. We would like to address the question, what is the source of R-curve behavior?

To answer this question it is informative to consider the "flow" of energy during crack propagation.

First consider the linear elastic case with some type of separation process occuring at the crack tip.

The cracked structure is loaded, and prior to crack grow work will be done by the applied loads through the displacements of the structure and elastic energy will be stored within the structure.

Now, if the crack length increases by some small amount δa the system of applied loads will do some work δW and the strain energy in the body will change by δU. In an elastic body where cracks are not growing we will always have $\delta U = \delta W$, however when there is crack growth we have,

$$\delta W - \delta U = \mathcal{G}\, \delta a$$

$\mathcal{G}$ tells us how much energy is lost into the crack tip per unit area of crack advance. From the cohesive zone perspective the fracture energy is the area under the cohesive traction law curve.

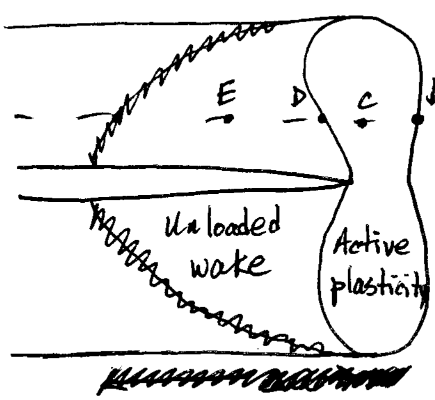
Once a given area of material traverses this curve to form new fracture surfaces, that energy cannot be returned to the system.

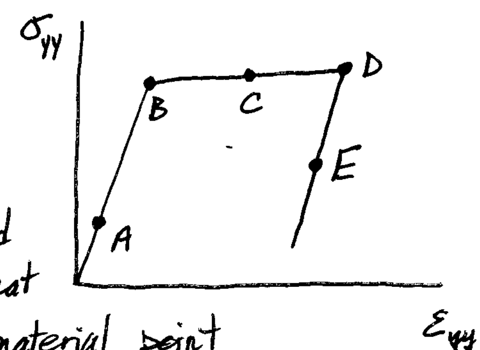
Hence we can think of energy "flowing" into the crack tip to "feed" the separation process. Once separation is complete this energy is lost, i.e. it has been dissipated at the crack tip.

These same arguments can be applied to non-linear elastic materials as well. The point to take away from this, at least with regard to R-curve behavior, is that as long as the cohesive traction law does not depend on the amount of crack advance, there should be no R-curve behavior.

Next, consider what happens in an elastic-plastic material. Prior to crack advance a plastic zone will develop around the crack tip. However, as long as there is no unloading all material points close to the tip experience very nearly proportional loading, a so, as far as the material just ahead of the tip that is almost about to separate is concerned, the amount of energy available to flow into the crack tip is still $\mathcal{G} = J$.

But when the crack grows material points near the tip experience non-proportional loading and even unloading





Very simplified picture of what happens to a material point as the crack passes.

As the crack grows material points begin to load non-proportionally, or unload on a different path from which they were loaded. This is dissipation. So now for an increment of crack growth our energy balance reads

$$\delta W - \delta U = \mathcal{G}\,\delta a + \underbrace{\delta W_D}$$

increment of energy dissipated by the material
This usually changes with crack growth.

So now our loading system has to feed energy into the dissipation due to the cohesive separation process at the crack tip, as well as the dissipation due to diffuse plastic deformation around the crack.

Let's take a look at a Mode III crack loaded by the linear asymptotic field growing under steady state conditions. This implies that a wake of plastically deformed material is fully developed and the plateau on the R-curve has been reached.

During steady propagation in the x-direction increments of Cartesian components of field variables are given as:
(origin move w/ crack tip) e.g. $\dot{\varepsilon}_{xz} = -\dot{a}\,\dfrac{\partial \varepsilon_{xz}}{\partial x}$

Under conditions of non-proportional loading, our constitutive law must now be cast in incremental or "rate" form.

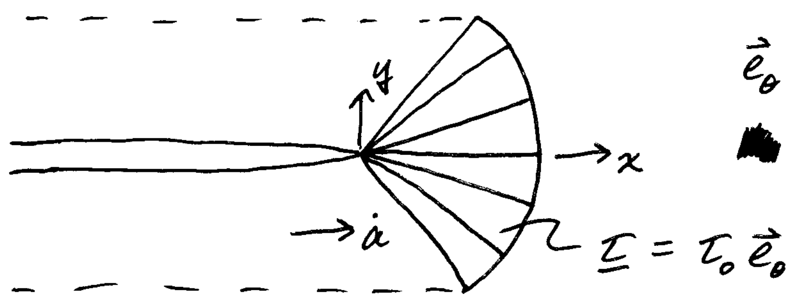
Following Rice's notation:

$$\underline{\dot{\gamma}} = \underline{\dot{\tau}}/\mu + \lambda \underline{\tau} \quad \leftarrow \text{Note the } \lambda \text{ depends on position in the body}$$

$$\underline{\tau} = \sigma_{xz}\,\vec{i} + \sigma_{yz}\,\vec{j}$$

$$\underline{\gamma} = \gamma_{xz}\,\vec{i} + \gamma_{yz}\,\vec{j} = \underline{\nabla}\, w$$

Equilibrium and the yield condition still imply that there is a centered fan of slip lines radiating from the crack tip.



$$\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$$

$$\frac{\partial \vec{e}_\theta}{\partial x} = -\cos\theta \frac{\partial\theta}{\partial x}\vec{i} - \sin\theta\frac{\partial\theta}{\partial x}\vec{j}$$

$$= -\vec{e}_r \frac{\partial\theta}{\partial x} = \vec{e}_r \frac{\sin\theta}{r}$$

$$\underline{\dot{\gamma}} = \underline{\nabla}\,\dot{w} = -\dot{a}\,\underline{\nabla}\frac{\partial w}{\partial x} = \frac{\dot{\tau}_0 \vec{e}_\theta}{\mu} + \lambda \tau_0 \vec{e}_\theta$$

$$= \frac{\tau_0}{\mu}\left(-\dot{a}\frac{\partial \vec{e}_\theta}{\partial x}\right) + \lambda \tau_0 \vec{e}_\theta$$

$$-\dot{a}\left[\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right)\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial x}\right)\vec{e}_\theta\right] = -\gamma_0\,\dot{a}\,\frac{\sin\theta}{r}\vec{e}_r + \lambda\tau_0\vec{e}_\theta$$

$\vec{e}_r$ components: $-\dot{a}\,\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) = -\dot{a}\,\gamma_0 \frac{\sin\theta}{r}$

$$\rightarrow \frac{\partial \gamma_{xz}}{\partial r} = \gamma_0 \frac{\sin\theta}{r}$$

$$\therefore \gamma_{xz} = \gamma_0 \sin\theta \ln r + f(\theta)$$

To determine $f(\theta)$, note that on the elastic-plastic boundary, i.e. at $r = R(\theta)$, $\underline{\gamma} = \gamma_0 \vec{e}_\theta \rightarrow \gamma_{xz} = -\gamma_0 \sin\theta$.

$$\therefore \gamma_{xz}(r = R(\theta)) = \gamma_0 \sin\theta \ln R(\theta) + f(\theta) = -\gamma_0 \sin\theta$$

$$\therefore f(\theta) = -\gamma_0 \sin\theta \left[1 + \ln R(\theta)\right]$$

$$\rightarrow \boxed{\gamma_{xz} = -\gamma_0 \sin\theta \left[1 + \ln \frac{R(\theta)}{r}\right]}$$

now $\frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial \gamma_{xz}}{\partial y} = -\gamma_0 \cos\theta \frac{\partial \theta}{\partial y}\left[1 + \ln \frac{R(\theta)}{r}\right]$

$$- \gamma_0 \sin\theta \left[\frac{R'(\theta)}{R(\theta)}\frac{\partial \theta}{\partial y} - \frac{1}{r}\frac{\partial r}{\partial y}\right]$$

$$= -\gamma_0 \frac{\cos^2\theta}{r}\left[1 + \ln \frac{R(\theta)}{r}\right]$$

$$- \gamma_0 \frac{\sin\theta\cos\theta}{r}\frac{R'(\theta)}{R(\theta)} + \gamma_0 \frac{\sin^2\theta}{r}$$

$$\frac{\partial \gamma_{yz}}{\partial x} = \frac{\gamma_0}{r}\left[\sin^2\theta - \cos^2\theta\left[1 + \ln \frac{R(\theta)}{r}\right] - \sin\theta\cos\theta \frac{R'(\theta)}{R(\theta)}\right]$$

Then γ_{yz} is determined by integrating wrt x from the e-p boundary where we know $\gamma_{yz} = \gamma_0 \cos\theta$

This is a difficult procedure in general, but simplifies on the line ahead of the crack, i.e.

$$\left.\frac{\partial \gamma_{yz}}{\partial x}\right|_{\theta=0} = -\frac{\gamma_0}{x}\left[1+\ln\frac{R(0)}{x}\right] = -\frac{\gamma_0}{x}\left[1+\ln\frac{R_0}{x}\right]$$

$$\therefore\ \gamma_{yz}(x,0) = -\gamma_0 \ln x + \frac{\gamma_0}{2}(\ln x)^2 - \gamma_0 \ln R_0 \ln x + C$$

but $\gamma_{yz}(R_0,0) = \gamma_0 = -\gamma_0 \ln R_0 + \frac{\gamma_0}{2}(\ln R_0)^2 - \gamma_0(\ln R_0)^2 + C$

$$\therefore\ C = \gamma_0 \ln R_0 + \frac{\gamma_0}{2}(\ln R_0)^2 + \gamma_0$$

$$\rightarrow \gamma_{yz}(x,0) = \gamma_0\Big\{1+\ln\frac{R_0}{x} + \underbrace{\tfrac{1}{2}[\ln x]^2 - \ln R_0 \ln x + \tfrac{1}{2}[\ln R_0]^2}_{\frac{1}{2}[\ln x - \ln R_0]^2}\Big\}$$

$$\therefore\ \boxed{\gamma_{yz}(x,0) = \gamma_0\left\{1+\ln\frac{R_0}{x} + \frac{1}{2}\left[\ln\frac{R_0}{x}\right]^2\right\}}$$

We will discuss later why we cannot use an energetic fracture criterion with this solution. Instead we will assume the crack grows when a critical plastic strain γ_c^p is achieved at a specified distance r_c ahead of the crack on $\theta=0$.

Steady State: $\gamma^p(r_c,\theta=0) = \gamma_{yz}(r_c,\theta=0) - \gamma_0 = \gamma_0\left\{\ln\frac{R_0}{r_c} + \frac{1}{2}\left[\ln\frac{R_0}{r_c}\right]^2\right\} = \gamma_c^p$

$$\rightarrow \ln\frac{R_0}{r_c}\left(1+\frac{1}{2}\ln\frac{R_0}{r_c}\right) = \frac{\gamma_c^p}{\gamma_0}$$

or better yet: $\frac{1}{2}\left[\ln\frac{R_0}{r_c}\right]^2 + \ln\frac{R_0}{r_c} + \frac{1}{2} = \frac{\gamma_c^p}{\gamma_0} + \frac{1}{2}$

$$\left\{ \ln \frac{R_o}{r_c} + 1 \right\}^2 = 2 \frac{\gamma_c^p}{\gamma_o} + 1$$

$$\ln \frac{R_o}{r_c} = \sqrt{2 \frac{\gamma_c^p}{\gamma_o} + 1} \; - 1$$

$$\therefore \boxed{R_o^{ss} = r_c \exp\left[\sqrt{2 \frac{\gamma_c^p}{\gamma_o} + 1} \; - 1\right]}$$

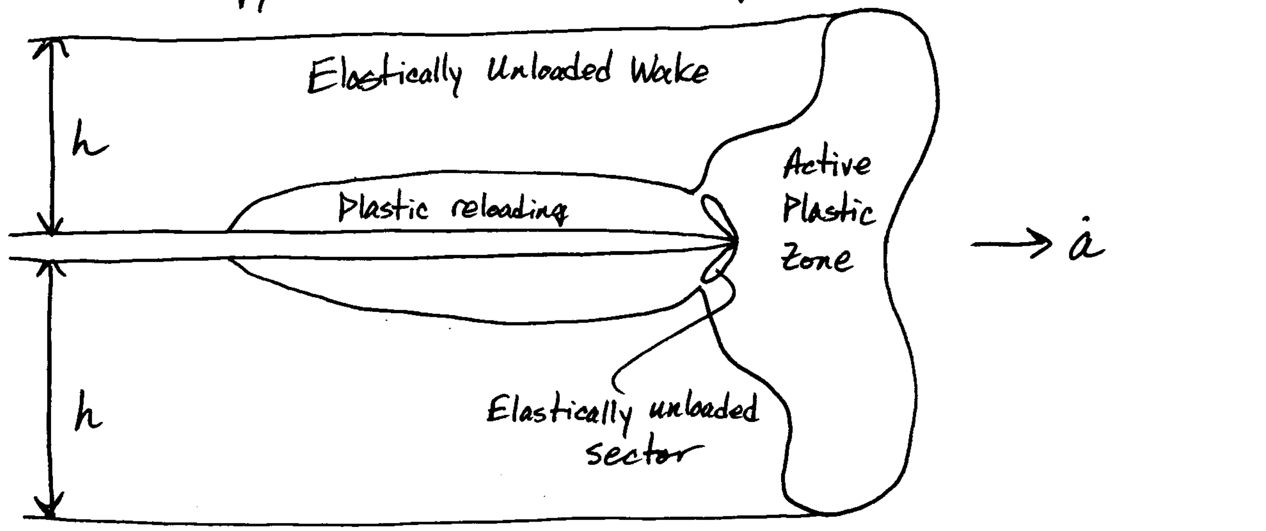
For the stationary crack: $\gamma^p(r_c, \theta = 0) = \gamma_o \frac{R_o}{r_c} - \gamma_o = \gamma_c^p$

$$\therefore \boxed{R_o^{initiation} = r_c \left(\frac{\gamma_c^p}{\gamma_o} + 1 \right)}$$

Note : $\dfrac{R_o^{ss}}{R_o^{init}} \approx \left(\dfrac{K^{ss}}{K^{init}} \right)^2 \approx \dfrac{\mathcal{G}_{ss}}{\mathcal{G}_{init}}$

$$\frac{R_o^{ss}}{R_o^{init}} = \frac{\exp\left[\sqrt{2 \frac{\gamma_c^p}{\gamma_o} + 1} \; - 1\right]}{\frac{\gamma_c^p}{\gamma_o} + 1} = \begin{cases} 1.04 & \text{for } \frac{\gamma_c^p}{\gamma_o} = 1 \\ 3 & 10 \\ 169 & 50 \\ 5100 & 100 \end{cases}$$

The Energy Balance for Steady-State Crack Growth



To perform an energy balance we will allow the crack to grow by an increment δa. In words the balance reads as:

Energy from applied K field (both work done and changes in stored energy) goes into Dissipation at crack tip due to cohesive zone separation + Dissipation and stored energy in the wake of plastically deformed material

$$\frac{K_I^2}{E'}\delta a = \mathcal{G}_c\,\delta a + \delta a\int_{-h}^{h} W(x_1 \to -\infty, x_2)\,dx_2$$

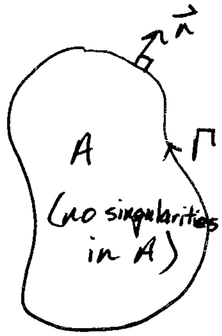
W is the <u>path-dependent</u> or history-dependent work density at a material point (x_1, x_2) given as

$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij}\,d\varepsilon_{ij}\ , \quad \sigma_{ij} \neq \frac{\partial W}{\partial \varepsilon_{ij}}$$

We can also obtain this energy balance by applying the I-integral under <u>steady-state</u> conditions.

$$I = \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

First, we will prove path-independence of I.



$$W = \int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij} = \int_\infty^{x_1} \sigma_{ij} \, d\varepsilon_{ij}$$

$$W = \int_\infty^{x_1} \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1'} dx_1'$$

only true under ss conditions

$$\frac{\partial W}{\partial x_1} = \frac{\partial}{\partial x_1} \int_\infty^{x_1} \sigma_{ij}(x_1', x_2) \frac{\partial \varepsilon_{ij}(x_1', x_2)}{\partial x_1'} dx_1'$$

$$= \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} \quad \text{for steady state conditions}$$

$$\therefore \int_\Gamma W n_1 \, d\Gamma = \int_A \frac{\partial W}{\partial x_1} dA = \int_A \sigma_{ij} \varepsilon_{ij,1} \, dA$$

$$= \int_A \sigma_{ij} \tfrac{1}{2}(u_{i,j} + u_{j,i})_{,1} \, dA$$

$$= \int_A \tfrac{1}{2} \sigma_{ij} u_{i,j1} + \tfrac{1}{2} \sigma_{ij} u_{j,i1} \, dA$$

$$= \int_A \tfrac{1}{2} \sigma_{ij} u_{i,j1} + \tfrac{1}{2} \underbrace{\sigma_{ji} u_{j,i1}}_{= \sigma_{ij} u_{i,j1} \text{ b/c change dummy indices } i \to j,\ j \to i} dA \quad \text{b/c } \sigma_{ji} = \sigma_{ij}$$

$$\therefore \int_\Gamma W n_1 \, d\Gamma = \int_A \sigma_{ij} u_{i,j1} \, dA$$

$$= \int_A (\sigma_{ij} u_{i,1})_{,j} - \cancel{\sigma_{ij,j}} u_{i,1} \, dA$$

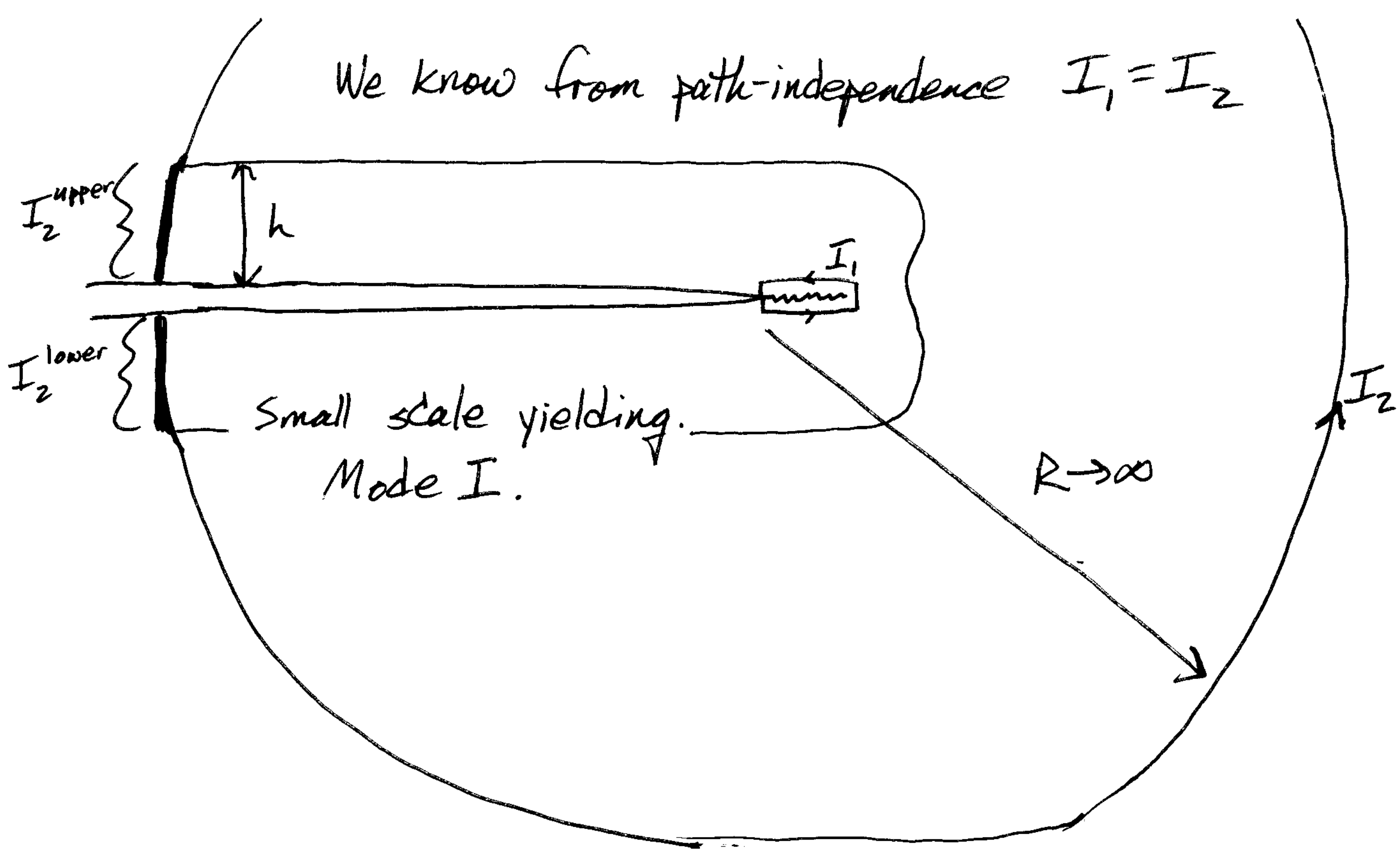
0 due to Equilibrium

$$= \int_\Gamma \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

$$\rightarrow \int_\Gamma W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma = 0$$ on a closed contour that contains no singularities

$\therefore$ I is a path-independent integral when taken counterclockwise around a crack tip.

Now, apply I to determine the energy balance.



$$I_1 : n_1 = 0 \rightarrow I_1 = \int_0^b -\sigma_{ij} n_j^- u_{i,1}^- \, dx_1$$

$$+ \int_b^0 -\sigma_{ij} n_j^+ u_{i,1}^+ \, (-dx_1)$$

$$= \int_0^b -\sigma_{22} \underbrace{(u_{2,1}^+ - u_{2,1}^-)}_{\delta_{,1}} dx_1$$

$$= \int_0^{\delta_c} \sigma_{22} \, d\delta = \mathcal{G}_c$$

$$\boxed{I_1 = \mathcal{G}_c}$$

I_2 : break into 2 parts, an arc as $R \to \infty$ but does not include $-h \le y \le h$ and 2 straight segments that include the plastic wake.

$$I_2 = I_2^{arc} + I_2^{lower} + I_2^{upper}$$

$$= \underbrace{\int_{-\pi}^{\pi} (K_I \text{ terms}) R \, d\theta}_{\frac{K_I^2}{E'}} + \int_0^{-h} (\quad)(-dx_2) + \int_h^0 (\quad)(-dx_2)$$

$$I_2^{upper} = \int_h^0 W(-1) + \sigma_{11} u_{1,1} + \sigma_{21} u_{2,1} \, (-dx_2)$$

$$= - \int_0^h W - \sigma_{11} u_{1,1} - \sigma_{21} u_{2,1} \, dx_2$$

Due to the massiveness of the elastic material attached to the wake, ε_{11}, ε_{33} and ε_{13} in the wake as $x_1 \to -\infty$ must be equal to that in the elastic material as $x_1 \to -\infty$.

From the K-field we know $\varepsilon_{11} \to 0$ as $x_1 \to -\infty$.

Furthermore σ_{12} must also vanish in the wake as $x_1 \to -\infty$ b/c

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\text{as } x_1 \to -\infty \quad \frac{\partial \sigma_{11}}{\partial x_1} \to 0 \quad \to \quad \frac{\partial \sigma_{12}}{\partial x_2} = 0 \to \sigma_{12} = \sigma_{12}(x_1)$$

as $x_1 \to -\infty$

but on the crack faces $\sigma_{12} = 0$,
i.e. $\sigma_{12}(x_1, x_2 = 0) = 0$

$\therefore$ $\sigma_{12} = 0$ in wake as $x_1 \to -\infty$

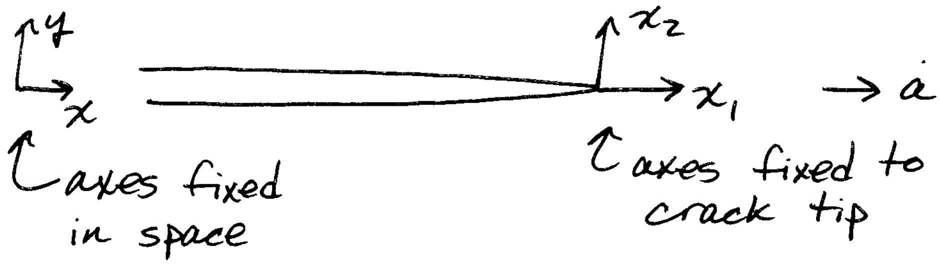
$$\therefore \quad I_2^{upper} = -\int_0^h W(x_1 \to -\infty, x_2)\, dx_2$$

A similar case is made for I_2^{lower}.

$$\therefore \quad I_2 = \frac{K_I^2}{E'} - \int_{-h}^h W(x_1 \to -\infty, x_2)\, dx_2$$

$$I_1 = I_2 \to \boxed{\frac{K_I^2}{E'} = \mathcal{G}_c + \int_{-h}^h W(x_1 \to -\infty, x_2)\, dx_2}$$

* A note about increments of field variables under steady state conditions.



We are interested in increments of field quantities at a given material point which can be located by its coordinates in the fixed system (x, y). Under steady-state conditions it is easier to solve problems with respect to the moving system (x_1, x_2). We want material increments (a.k.a. the material time derivative) in this moving coordinate system.

Note: $x_1 = x - a$ and $x_2 = y$

Consider some field quantity $f(x, y, t)$

$$\frac{d}{dt} \equiv \left.\frac{\partial}{\partial t}\right|_{x,y \text{ fixed}}$$

f can also be written as $f(x_1, x_2, t)$.

$$\therefore \frac{df}{dt} = \left.\frac{\partial f}{\partial t}\right|_{\substack{x_1, x_2 \\ \text{fixed}}} + \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$$

total time derivatives

Under steady-state conditions an observer moving along with the crack tip sees no changes in the fields around him. This is essentially the definition of steady-state.

i.e. $$\left.\frac{\partial f}{\partial t}\right|_{\substack{x_1, x_2 \\ \text{fixed}}} = 0 \quad \text{under steady-state conditions}$$

$$\frac{\partial x_1}{\partial t} = \cancel{\frac{dx}{dt}}\nearrow 0 - \frac{da}{dt} = -\dot{a}$$

$$\frac{dx_2}{dt} = \cancel{\frac{dy}{dt}}\nearrow 0 = 0$$

$$\therefore \quad \boxed{\frac{df}{dt} = -\dot{a}\,\frac{\partial f}{\partial x_1} \quad \text{under steady state conditions}}$$

In other words, for the (x_1, x_2, t) description

$$f = f(x_1, x_2) \quad \text{during steady-state.}$$

Hence for any field quantity (Cartesian components)

$$\boxed{\dot{(\;)} = -\dot{a}\,\frac{\partial (\;)}{\partial x_1}} \quad \text{Steady-State only}$$

If we attempt to implement an energetic fracture criterion with our steady-state solution, we run into a contradiction.

The I-integral tells us how much energy is fed into the crack tip per unit of crack advance under ss conditions.

$$I = \int_{\Gamma} W n_1 - \sigma_{ij} n_j u_{i,1} \, d\Gamma$$

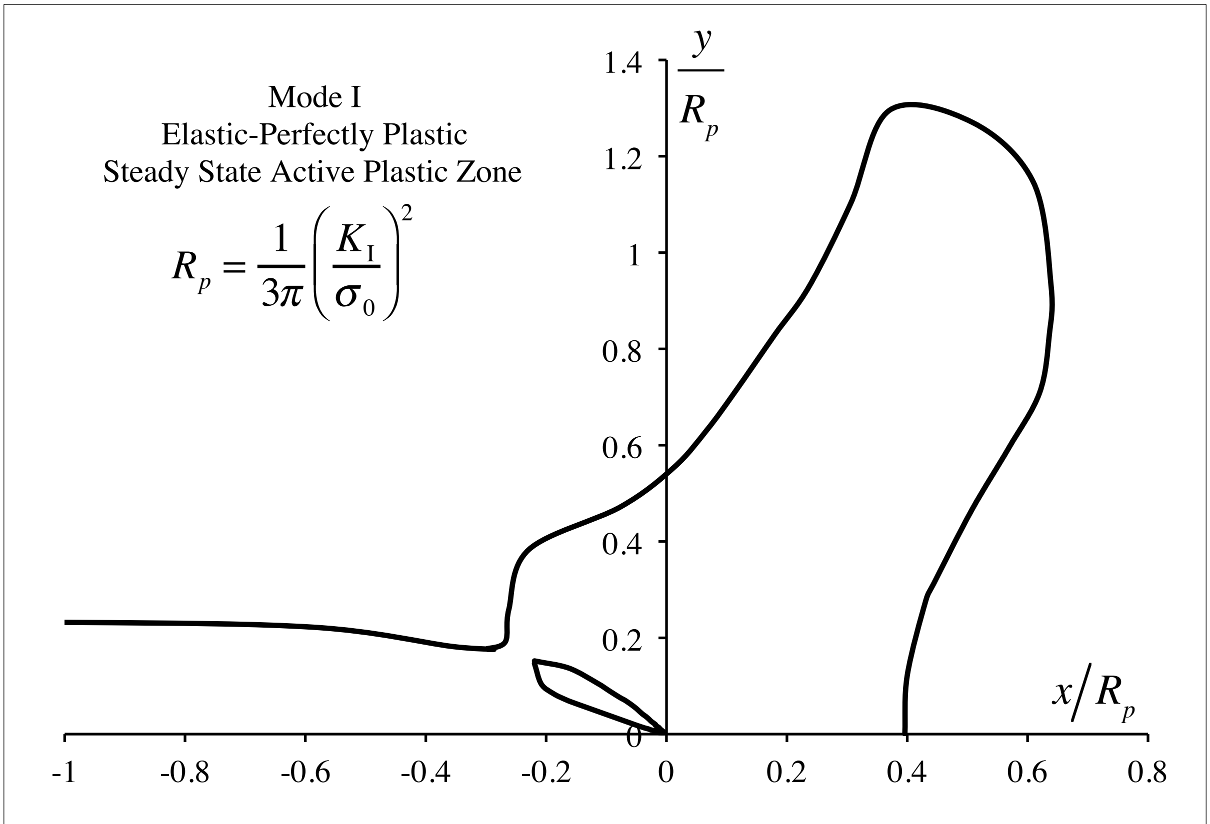
$$SS \rightarrow \quad \frac{\partial W}{\partial x_1} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_1} \sim -\frac{\tau_0 \gamma_0}{x}\left[1 + \ln \frac{R_0}{x}\right] \text{ from Mode III Solution on } \theta = 0$$

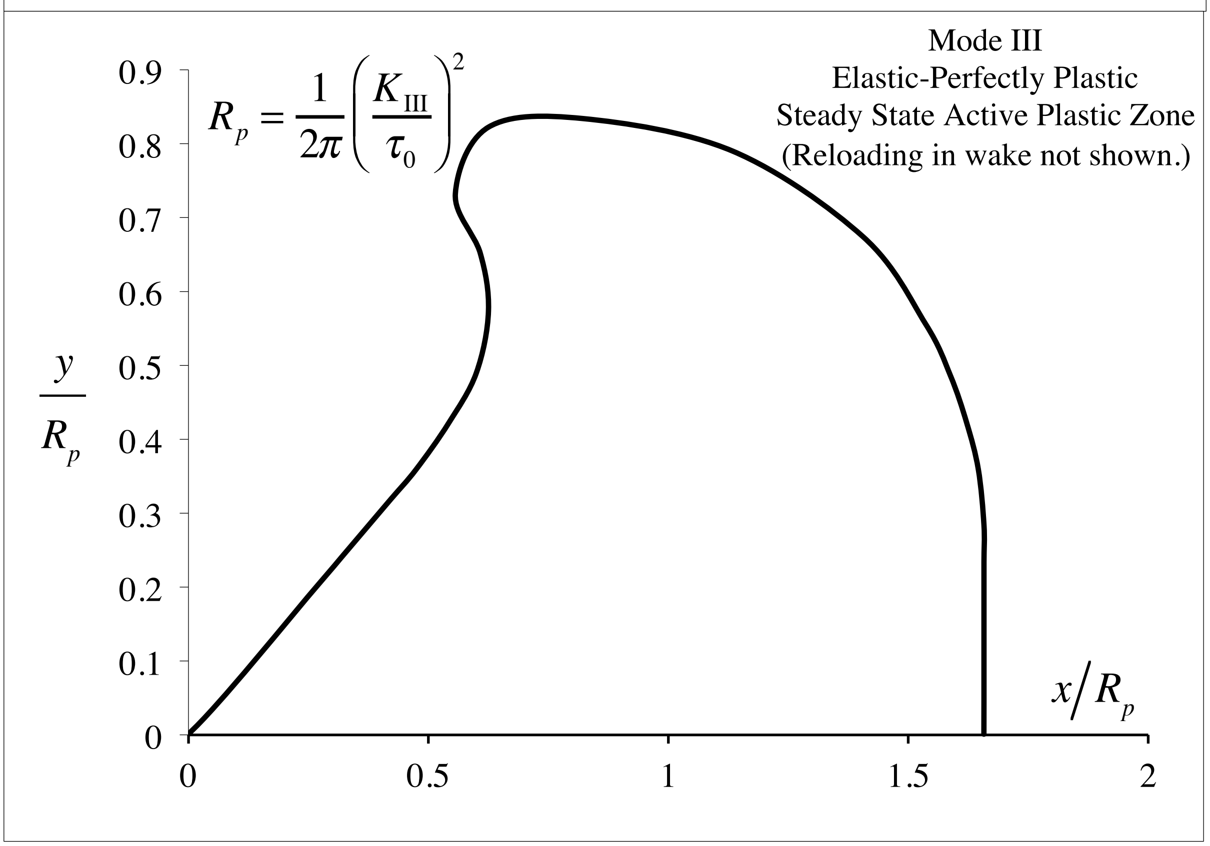
$$\rightarrow W \sim O\left[\left(\ln \frac{R_0}{x}\right)^2\right]$$

$$\text{also} \quad \sigma_{ij} n_j u_{i,1} \sim O\left[\left(\ln \frac{R_0}{x}\right)^2\right]$$

In order for a finite amount of energy to feed into the crack W and/or $\sigma_{ij} n_j u_{i,1}$ must be $O\left(\frac{1}{r}\right)$. But $O\left[\ln^2\left(\frac{R_0}{r}\right)\right] < O\left[\frac{1}{r}\right]$, therefore $I = 0$ for this solution. Hence, this solution claims that there is no energy available at the crack tip to drive propagation. This feature holds in Mode I as well.

This contradiction can be "fixed" by explicitly modeling the crack tip cohesive zone process. Analogous to the Dugdale model but including diffuse plasticity in the bulk. Solutions of such a model must be carried out numerically.


Mode I
Elastic-Perfectly Plastic
Steady State Active Plastic Zone
$R_p = \frac{1}{3\pi}\left(\frac{K_{\mathrm{I}}}{\sigma_0}\right)^2$
$\frac{y}{R_p}$
x/R_p
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8


Mode III
Elastic-Perfectly Plastic
Steady State Active Plastic Zone
(Reloading in wake not shown.)
$R_p = \frac{1}{2\pi}\left(\frac{K_{\mathrm{III}}}{\tau_0}\right)^2$
$\frac{y}{R_p}$
x/R_p
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2