Check equation for $N=1 \rightarrow$ linear elastic

$$N=1 \rightarrow \tilde{\Phi}'''' + \frac{3}{4}\tilde{\Phi}'' + \frac{3}{4}\tilde{\Phi}'' + \frac{9}{16}\tilde{\Phi} + \tilde{\Phi}'' = 0$$

$$\tilde{\Phi}'''' + \frac{5}{2}\tilde{\Phi}'' + \frac{9}{16}\tilde{\Phi} = 0$$

Mode I elastic solution $\rightarrow$ $\tilde{\Phi} = A\left(\cos\frac{\theta}{2} + \frac{1}{3}\cos\frac{3\theta}{2}\right)$

$$\tilde{\Phi}'' = A\left(-\frac{1}{4}\cos\frac{\theta}{2} - \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$\tilde{\Phi}'''' = A\left(\frac{1}{16}\cos\frac{\theta}{2} + \frac{27}{16}\cos\frac{3\theta}{2}\right)$$

check ODE: $\frac{1}{16}\cancel{\cos\frac{\theta}{2}} + \frac{27}{16}\cancel{\cos\frac{3\theta}{2}} - \frac{10}{16}\cancel{\cos\frac{\theta}{2}} - \frac{30}{16}\cancel{\cos\frac{3\theta}{2}} + \frac{9}{16}\cancel{\cos\frac{\theta}{2}} + \frac{3}{16}\cancel{\cos\frac{3\theta}{2}}$
$= 0$ ✓

$\tilde{\Phi}'(\theta=0) = 0$ ✓

$\tilde{\Phi}'''(\theta=0) = 0$ ✓

$\tilde{\Phi}(\theta=\pi) = 0$ ✓

$\tilde{\Phi}'(\theta=\pi) = A\left(-\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{3}\frac{3}{2}\sin\frac{3\pi}{2}\right) = A\left(-\frac{1}{2}+\frac{1}{2}\right) = 0$ ✓

$\therefore$ We get what we expect for $N=1$.

* Solution procedure for general N

1) Use an initial value ODE solver like 4th order R-K
(The equation on page 96 must be reorganized to get $\tilde{\Phi}'''' = f(\tilde{\Phi}, \tilde{\Phi}', \tilde{\Phi}'', \tilde{\Phi}''')$, & note that $\tilde{\sigma}''$ contains $\tilde{\Phi}''''$)
2) Initial conditions: $\tilde{\Phi}(0) = 1$, $\tilde{\Phi}'(0) = 0$
$\tilde{\Phi}''(0) = c$, $\tilde{\Phi}'''(0) = 0$
3) Solve over the interval from $0 \le \theta \le \pi$
4) Check $\tilde{\Phi}(\pi)$ and $\tilde{\Phi}'(\pi)$
5) Adjust c and repeat procedure until $\tilde{\Phi}(\pi) = 0$ and $\tilde{\Phi}'(\pi) = 0$

Next consider the limit as $N \to 0$, i.e. perfect plasticity. To do this, multiply the equation on page 96 by N^2 and then set $N=0$. The only term that survives is,

$$\boxed{4\left(\frac{\tilde{\sigma}'}{\tilde{\sigma}}\right)^2 \tilde{\phi}'' = 0 \quad \text{for} \quad N=0}$$

$\therefore$ either $\quad \tilde{\sigma}' = 0 \quad$ or $\quad \tilde{\phi}'' = 0$

$$\tilde{\sigma}' = \left(\tfrac{3}{2}\tilde{\phi}''' + 6\tilde{\phi}'\right)\tilde{\phi}'' = 0$$

$\hookrightarrow$ again $\quad \tilde{\phi}'' = 0 \quad$ or $\quad \tilde{\phi}''' + 4\tilde{\phi}' = 0$

$$\tilde{\phi}'' = 0 \quad \rightarrow \quad \begin{aligned} \phi &= A\theta + B \\ \tilde{\sigma}_{\theta\theta} &= 2A\theta + 2B \\ \tilde{\sigma}_{r\theta} &= -A \\ \tilde{\sigma}_{rr} &= 2A\theta + 2B \end{aligned}$$

This is known as a "centered fan" in slip-line theory

$$\tilde{\phi}''' + 4\tilde{\phi}' = 0 \quad \rightarrow \quad \begin{aligned} \phi &= C\cos 2\theta + D\sin 2\theta + E \\ \tilde{\sigma}_{\theta\theta} &= 2C\cos 2\theta + 2D\sin 2\theta + 2E \\ \tilde{\sigma}_{r\theta} &= 2C\sin 2\theta - 2D\cos 2\theta \\ \tilde{\sigma}_{rr} &= -2C\cos 2\theta - 2D\sin 2\theta + 2E \end{aligned}$$

$$\text{Tensor transformation} \rightarrow \quad \begin{aligned} \tilde{\sigma}_{xx} &= 2E - 2C \\ \tilde{\sigma}_{yy} &= 2E + 2C. \\ \tilde{\sigma}_{xy} &= -2D \end{aligned}$$

This is known as a "constant stress" solution in slip-line theory. [redacted]

In order to obtain a full solution we must piece together these solutions over a given θ range. Note that for $N=0$ the governing ODE is hyperbolic (instead of elliptic) and certain quantities (associated with higher order derivatives of $\hat{\phi}$) need not be continuous over the entire θ range.

Consider a radial line : 

Tractions must be continuous across any such line,
$\rightarrow$ $\tilde{\sigma}_{\theta\theta}$ & $\tilde{\sigma}_{r\theta}$ are continuous $\rightarrow$ $\tilde{\phi}$ and $\tilde{\phi}'$ are continuous

The solution satisfying these continuity conditions and the crack BC's is not unique. However the HRR solution approaches the following solution for small N.

$0 \le \theta \le \frac{\pi}{4}$

$$\tilde{\sigma}_{rr} = \frac{\sigma_0}{\sqrt{3}}\pi + \frac{2\sigma_0}{\sqrt{3}}\sin^2\theta$$
$$\tilde{\sigma}_{\theta\theta} = \frac{\sigma_0}{\sqrt{3}}\pi + \frac{2\sigma_0}{\sqrt{3}}\cos^2\theta$$
$$\tilde{\sigma}_{r\theta} = \frac{\sigma_0}{\sqrt{3}}\sin 2\theta$$

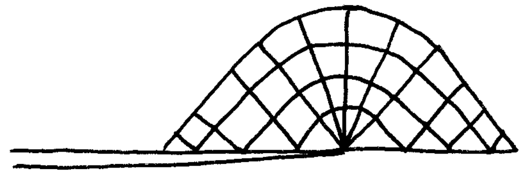
$\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

$$\tilde{\sigma}_{rr} = \tilde{\sigma}_{\theta\theta} = \frac{\sigma_0}{\sqrt{3}}\left(1 + \frac{3\pi}{2} - 2\theta\right)$$
$$\tilde{\sigma}_{r\theta} = \sigma_0/\sqrt{3}$$

$\frac{3\pi}{4} \le \theta \le \pi$

$$\tilde{\sigma}_{rr} = \frac{2\sigma_0}{\sqrt{3}}\cos^2\theta$$
$$\tilde{\sigma}_{\theta\theta} = \frac{2\sigma_0}{\sqrt{3}}\sin^2\theta$$
$$\tilde{\sigma}_{r\theta} = -\frac{\sigma_0}{\sqrt{3}}\sin 2\theta$$

The slip-line field looks like



On the next page there are other plots of $\tilde{\sigma}(\theta)$, $\hat{\sigma}_{rr}(\theta)$, $\hat{\sigma}_{\theta\theta}(\theta)$ and $\hat{\sigma}_{r\theta}(\theta)$ for $N = 0.0001, 0.1, 0.25, 0.5$ and 1. In all cases the results are normalized such that the maximum value of $\tilde{\sigma}$ is 1. Other interesting features of the solution can be found in the HRR papers.

For example, the relationship between J and the scaling factor K is determined. Note that the K used throughout this solution has <u>not</u> been the linear elastic stress intensity factor.

R-R give approximate shapes and sizes of the elastic-plastic boundary and crack opening profiles for a range of N.

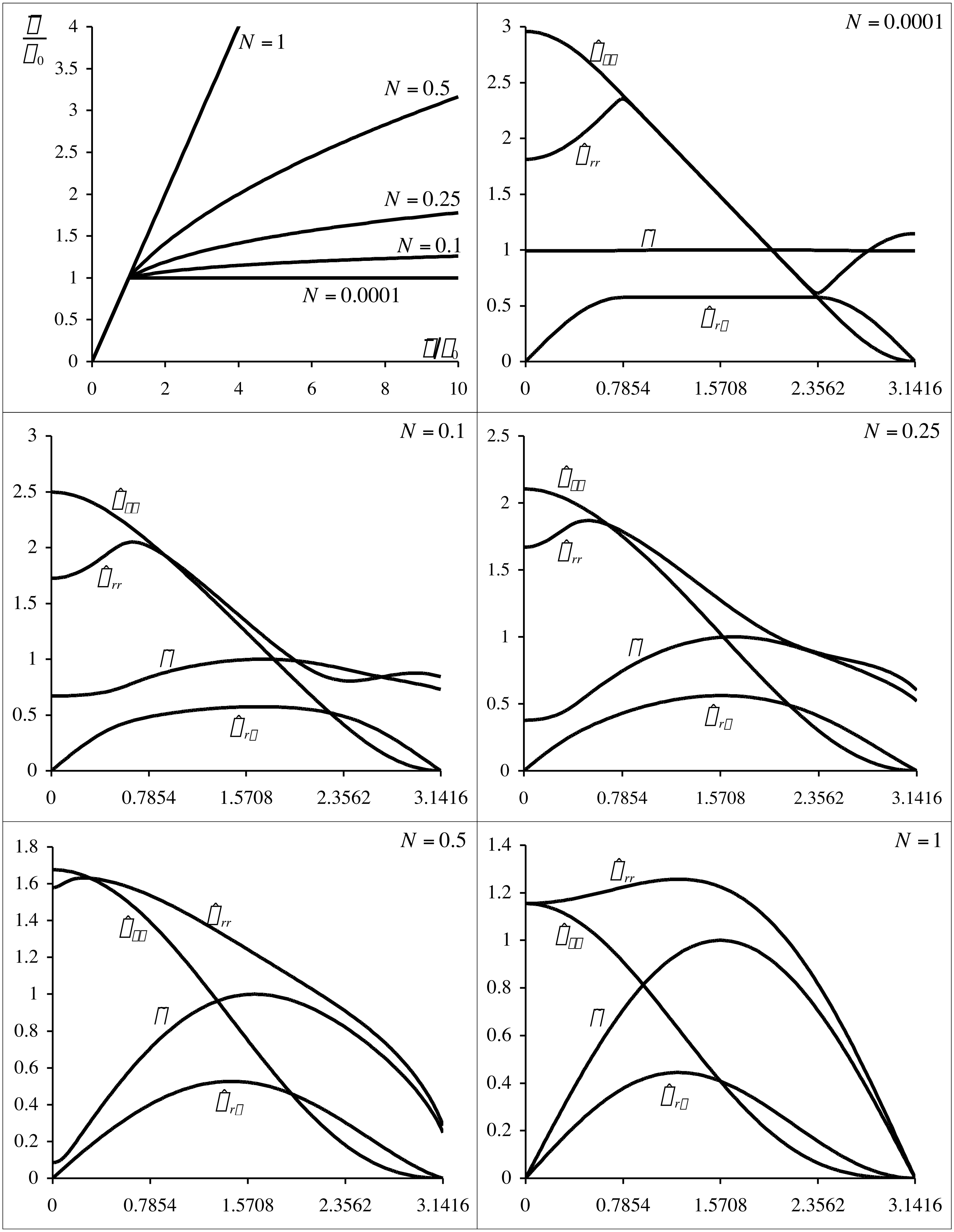
Note: $$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{3}{2}\varepsilon_0 \left(\frac{K}{\sigma_0}\right)^{1/N} \left(\frac{1}{r}\right)^{1/N+1} \tilde{\sigma}^{\frac{1-N}{N}} \frac{1}{2}\left[\frac{N(N+2)}{(N+1)^2}\tilde{\phi} + \tilde{\phi}''\right]$$

$$\therefore \boxed{u_r = \frac{3}{2}\varepsilon_0 \left(\frac{K}{\sigma_0}\right)^{1/N} r^{N/N+1} \underbrace{\tilde{\sigma}^{\frac{1-N}{N}} \frac{1}{2}\left[\frac{N+2}{N+1}\tilde{\phi} + \frac{N+1}{N}\tilde{\phi}''\right]}_{\tilde{u}_r(\theta)}}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{3}{2}\varepsilon_0 \left(\frac{K}{\sigma_0}\right)^{1/N} \left(\frac{1}{r}\right)^{1/N+1} \tilde{\sigma}^{\frac{1-N}{N}} \frac{1}{2}\left[-\frac{N(N+2)}{(N+1)^2}\tilde{\phi} - \tilde{\phi}''\right]$$

$$\therefore \boxed{u_\theta = \frac{3}{2}\varepsilon_0 \left(\frac{K}{\sigma_0}\right)^{1/N} r^{N/N+1} \tilde{u}_\theta(\theta) \text{ where } \tilde{u}_\theta' = -\frac{1+2N}{N+1}\tilde{u}_r}$$

Note: Hutchinson uses $\bar{\varepsilon} \sim \bar{\sigma}^n \rightarrow n = \frac{1}{N}$. Hutchinson also solves the plane stress proplem which is important for the fracture of thin plates.


N = 1
N = 0.5
N = 0.25
N = 0.1
N = 0.0001
0 2 4 6 8 10
0 0.5 1 1.5 2 2.5 3 3.5 4
N = 0.0001
N = 0.1
N = 0.25
N = 0.5
N = 1
0 0.7854 1.5708 2.3562 3.1416

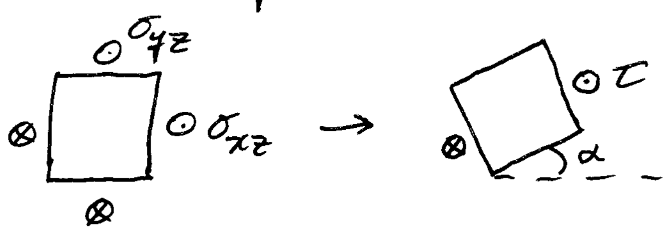
Mode III Elastic-Perfectly Plastic

Mode III Plasticity – Yield surface: $\frac{3}{2} s_{ij} s_{ij} = \sigma_0^2$

$$\rightarrow 3\sigma_{xz}^2 + 3\sigma_{yz}^2 = \sigma_0^2$$

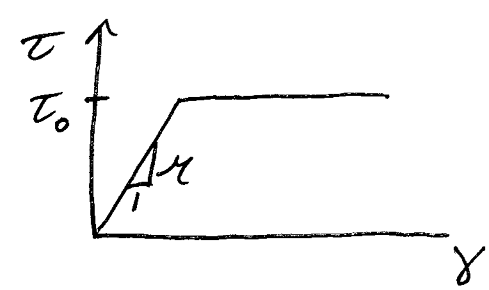
$$\tau_0 = \frac{\sigma_0}{\sqrt{3}} \rightarrow \boxed{\sigma_{xz}^2 + \sigma_{yz}^2 = \tau_0^2}$$

For any anti-plane shear stress state, there exists some angle α such that



where
$$\left.\begin{aligned} \sigma_{xz} &= \tau \cos\alpha \\ \sigma_{yz} &= \tau \sin\alpha \end{aligned}\right\} \sigma_{xz}^2 + \sigma_{yz}^2 = \tau^2$$

Elastic-perfectly plastic $\rightarrow$



$\therefore$ in plastic regions

$$\sigma_{xz} = \tau_0 \cos\alpha$$

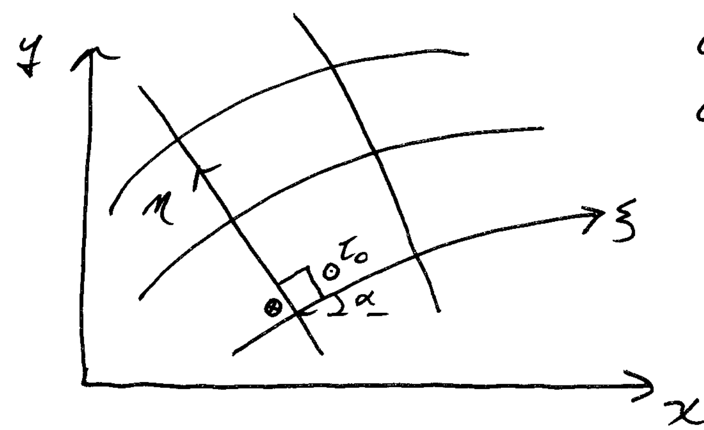
$$\sigma_{yz} = \tau_0 \sin\alpha$$

Equilibrium $\rightarrow$ $\dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} = 0$ (no body force)

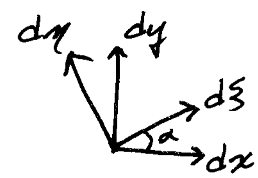
$$\therefore \frac{\partial}{\partial x}(\tau_0 \cos\alpha) + \frac{\partial}{\partial y}(\tau_0 \sin\alpha) = 0 \quad \text{in plastic regions}$$

$$-\sin\alpha \frac{\partial \alpha}{\partial x} + \cos\alpha \frac{\partial \alpha}{\partial y} = 0$$

Next, consider an orthogonal curvilinear coordinate system (η, ξ) such that at any given point the ξ-direction is at an angle α above the x-axis.



$\sigma_{\xi z} = \tau_0$
$\sigma_{\eta z} = 0$



The ξ-direction is the direction of maximum shear.

In general, if we know α at every point in the plastic region then we know the stress field. Hence, we can think of α as $\alpha = \alpha(x, y)$ or $\alpha = \alpha(\eta, \xi)$.

Consider the incremental relationships.

$$d\xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy = \cos\alpha \, dx + \sin\alpha \, dy$$

$$d\eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy = -\sin\alpha \, dx + \cos\alpha \, dy$$

or the inverse: $dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$
$= \cos\alpha \, d\xi - \sin\alpha \, d\eta$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$
$$= \sin\alpha \, d\xi + \cos\alpha \, d\eta$$

∴ We can use these differential and geometric relationships to determine the partial derivatives

Consider $\alpha = \alpha(\xi, \eta)$ as $\alpha = \alpha(x(\xi,\eta), y(\xi,\eta))$

then $d\alpha = \frac{\partial \alpha}{\partial \eta} d\eta + \frac{\partial \alpha}{\partial \xi} d\xi = \frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy$

$$= \frac{\partial \alpha}{\partial x}\left[\frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta\right]$$
$$+ \frac{\partial \alpha}{\partial y}\left[\frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta\right]$$
$$= \left(\frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \alpha}{\partial y}\frac{\partial y}{\partial \xi}\right) d\xi$$
$$+ \left(\frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \alpha}{\partial y}\frac{\partial y}{\partial \eta}\right) d\eta$$

$$\therefore \frac{\partial \alpha}{\partial \eta} = \frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \alpha}{\partial y}\frac{\partial y}{\partial \eta} = -\sin\alpha \frac{\partial \alpha}{\partial x} + \cos\alpha \frac{\partial \alpha}{\partial y}$$
$$\frac{\partial \alpha}{\partial \xi} = \frac{\partial \alpha}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \alpha}{\partial y}\frac{\partial y}{\partial \xi} = \cos\alpha \frac{\partial \alpha}{\partial x} + \sin\alpha \frac{\partial \alpha}{\partial y}$$

∴ The equilibrium equation in the η, ξ coordinate system is $\Rightarrow$ $\boxed{\frac{\partial \alpha}{\partial \eta} = 0}$

This implies $\boxed{\alpha = f(\xi)}$ i.e. no η dependence

or in other words the angle α only depends on the ξ position. Hence, for constant ξ values the angle α is constant. This implies that the $\boxed{\eta \text{ grid lines are straight.}}$

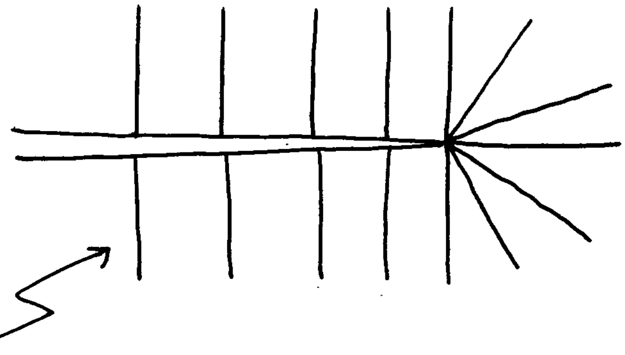
The η-lines are called characteristics, and for anti-plane shear in the plastic region we have found that these characteristics must be straight.

Now consider boundary conditions. Usually free surfaces are the most interesting features so we will focus on these, but in general the traction BC surfaces will have the α angle specified by the traction.

Recall that $\sigma_{\xi z} = \tau_0$ and $\sigma_{\eta z} = 0$.

Therefore, on a traction free boundary, the boundary curve is a ξ-line and the η-lines are perpendicular to the boundary.

To the crack problem



This is what the characteristics, i.e. η-lines must look like if the plastic zone extends back to the crack faces.

Elastic solution

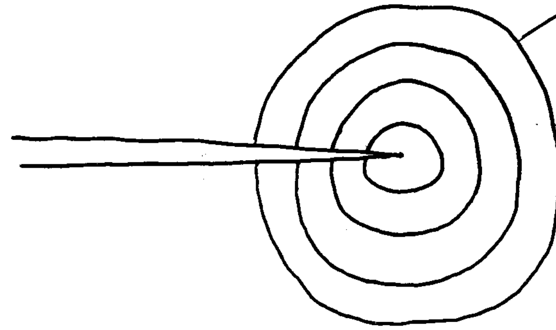
$$\sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}$$

$$\sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\underbrace{\sigma_{rz} = \frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \qquad \sigma_{\theta z} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}}$$

For SSY this solution is valid as $r \to \infty$.

Contours of constant τ from the elastic solution.



$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \text{constant}$

circles centered on the crack tip.

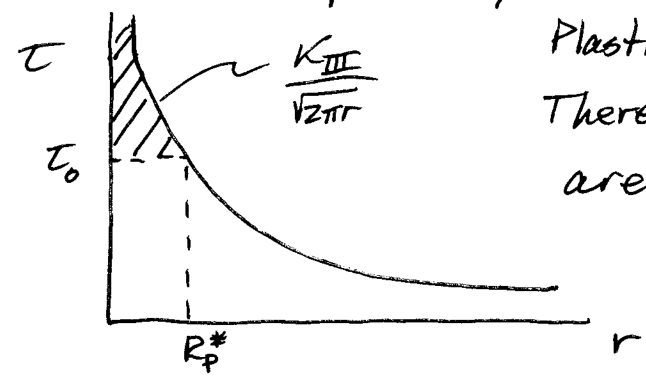
$$\tau^2 = \sigma_{xz}^2 + \sigma_{yz}^2 = \frac{K_{III}^2}{2\pi r}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) = \frac{K_{III}^2}{2\pi r}$$

$$\therefore \quad \tau = \frac{K_{III}}{\sqrt{2\pi r}}$$

Approximate $R_p^* \Rightarrow \dfrac{K_{III}}{\sqrt{2\pi R_p^*}} = \tau_0 \rightarrow R_p^* = \dfrac{1}{2\pi}\left(\dfrac{K_{III}}{\tau_0}\right)^2$

Our goal is to find elastic and plastic solutions that satisfy appropriate continuity conditions at the elastic-plastic boundary.

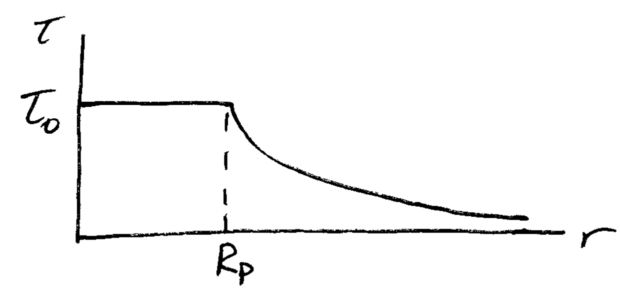
First, let's consider Irwin's correction to the plastic zone size. Consider stresses ahead of the crack tip only.



Plasticity limits τ to τ_0. Therefore, the hatched area must be cut off.

However, the hatched area carried a finite amount of force on the plane ahead of the crack. We must find a way to add this force back to the stress distribution while maintaining $\tau \leq \tau_0$ and $\lim_{r \to \infty} \tau = \frac{K_{III}}{\sqrt{2\pi r}}$.

These conditions imply that we cannot shift the $1/\sqrt{r}$ distribution up, so instead we will shift it to the right.



The force carried by the distributions to the right of the dashed lines is the same in both figures. So, to equate total forces, the areas under the curves to the left must be equal.

$$\therefore \quad \int_0^{R_p^*} \frac{K_{III}}{\sqrt{2\pi r}}\, dr = \tau_0 R_p$$

$$\frac{K_{III}}{\sqrt{2\pi}}\, 2 r^{1/2} \Big]_0^{\frac{1}{2\pi}\left(\frac{K_{III}}{\tau_0}\right)^2} = \tau_0 R_p$$

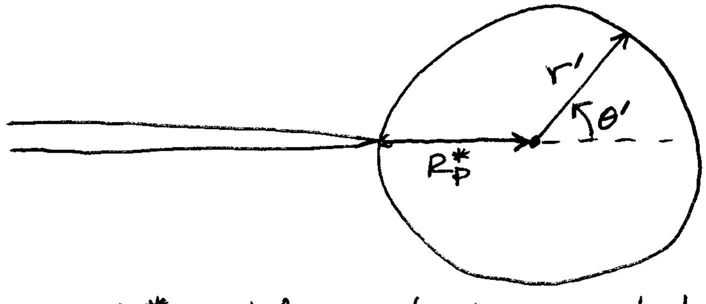
$$\frac{K_{III}}{\sqrt{2\pi}}\, 2 \frac{1}{\sqrt{2\pi}} \frac{K_{III}}{\tau_0} = \tau_0 R_p$$

$$\therefore \quad R_p = \frac{1}{\pi}\left(\frac{K_{III}}{\tau_0}\right)^2 = 2R_p^*$$

distance *ahead* of the crack tip where plasticity occurs.

Back to the full field solution. In general, it is difficult to find an elastic-plastic boundary, however for the mode III asymptotic crack tip field it turns out to be surprisingly easy.

Consider an "artifical" crack tip located ahead of the actual crack tip.

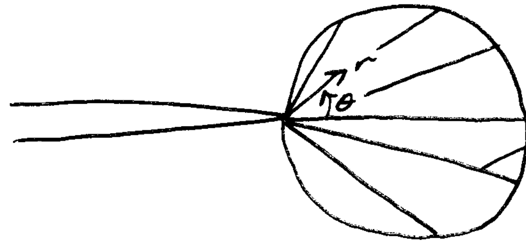


for $r' > R_p^*$ the elastic solution is

$$\sigma_{r'z} = \frac{K_{III}}{\sqrt{2\pi r'}} \sin\frac{\theta'}{2}$$

$$\sigma_{\theta' z} = \frac{K_{III}}{\sqrt{2\pi r'}} \cos\frac{\theta'}{2}$$

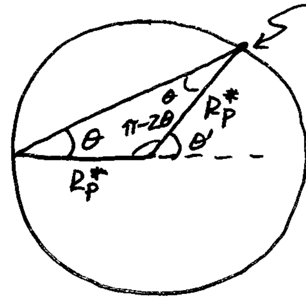
The plastic solution is:



for $r' < R_p^*$

$$\sigma_{\theta z} = \tau_0$$
$$\sigma_{rz} = 0$$

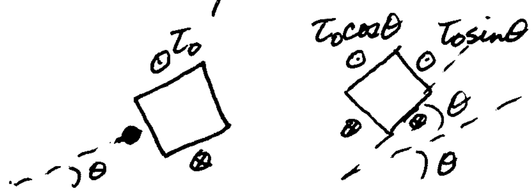
On the boundary $r' = R_p^*$ $\sigma_{r'z}$ must be continuous. To see if this is true we must first do a little geometry.



Some arbitrary point on the elastic-plastic boundary.

For points on the boundary $\theta' = 2\theta$.

Stress transformation



$\therefore$ On the e-p boudary $\sigma_{\theta' z} = \tau_0 \cos\theta$ from
$\sigma_{r'z} = \tau_0 \sin\theta$ plastic solution

Check: $\sigma_{\theta' z}^{elastic}(r' = R_p^*) = \sigma_{\theta' z}^{plastic}(r' = R_p^*)$

$$\frac{K_{III}}{\sqrt{2\pi R_p^*}} \cos\frac{\theta'}{2} \overset{?}{=} \tau_0 \cos\theta \qquad \text{recall } \theta = \frac{\theta'}{2} \text{ on } r' = R_p^*$$

$$\frac{\cancel{K_{III}}}{\cancel{\sqrt{2\pi}}} \cancel{\sqrt{2\pi}} \frac{\tau_0}{\cancel{K_{III}}} \cancel{\cos}\frac{\theta'}{2} \overset{\checkmark}{=} \tau_0 \cancel{\cos}\frac{\theta'}{2}$$