

Review of Planar Rigid Body Kinetics

198

$$\vec{F} = m \vec{a}_{cm}$$

$$\vec{M}_A = \vec{r}_{cm/A} \times m \vec{a}_{cm} + I_{cm} \alpha \vec{k} \quad \begin{array}{l} A \text{ is fixed} \\ \text{or } A = CM \end{array}$$

Planar motion

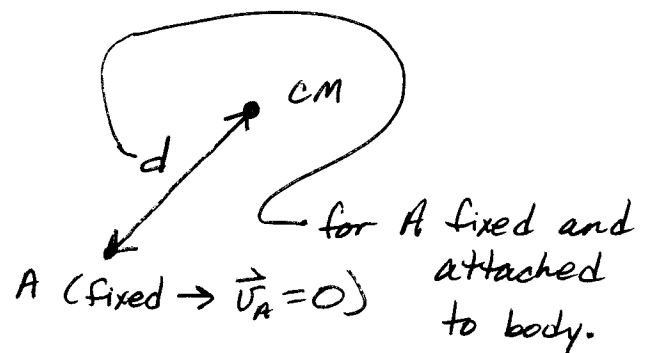
$$F_x = m a_{cmx}$$

$$F_y = m a_{cmy}$$

$$M_z^A = (I_{cm} + m d^2) \alpha = I_A \alpha$$

or $M_z^{cm} = I_{cm} \alpha$ $A = CM$

Use this if you can.



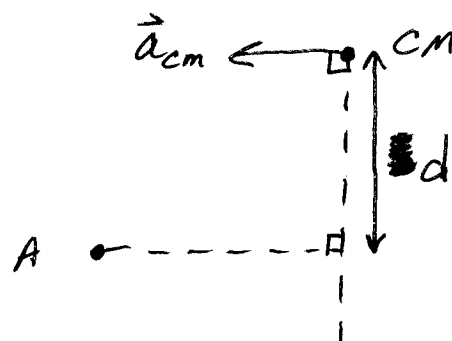
A fixed and attached to the body.

2nd choice. Sometimes easiest to apply.

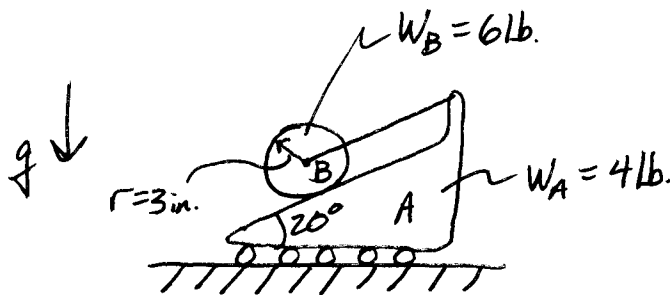
If A is fixed, but not attached to the body then

$$M_z^A = I_{cm} \alpha + m a_{cm} d \quad A \text{ fixed in space}$$

Last resort use this.



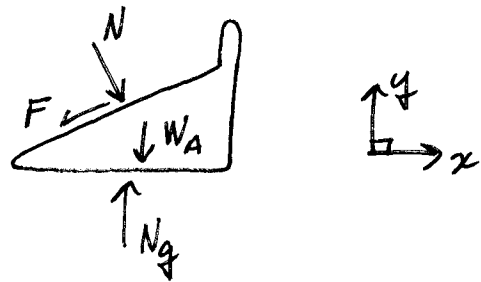
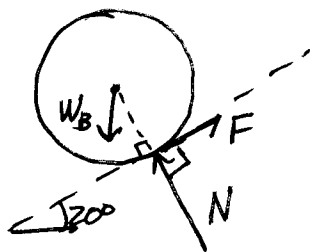
Example Problem



Initially at rest.
Rolls without slipping.
Neglect friction between
wedge & ground.

Determine the acceleration of the wedge and the angular acceleration of the cylinder immediately after the cord is cut.

Draw free body diagrams.



Cylinder

$$① \sum F_x = F \cos 20^\circ - N \sin 20^\circ = m_B a_{Bx} = \frac{W_B}{g} a_{Bx}$$

$$② \sum F_y = F \sin 20^\circ + N \cos 20^\circ - W_B = m_B a_{By} = \frac{W_B}{g} a_{By}$$

$$③ \sum M_z^{cm} = Fr = I_{cm} \alpha_B = \frac{1}{2} m_B r^2 \alpha_B = \frac{1}{2} \frac{W_B}{g} r^2 \alpha_B$$

Wedge

$$\textcircled{4} \sum F_x = -F \cos 20^\circ + N \sin 20^\circ = m_A a_{Ax} = \frac{W_A}{g} a_{Ax}$$

$$\textcircled{5} \sum F_y = -F \sin 20^\circ - N \cos 20^\circ - W_A + N_g = m_A a_{Ay} \quad \text{0}$$

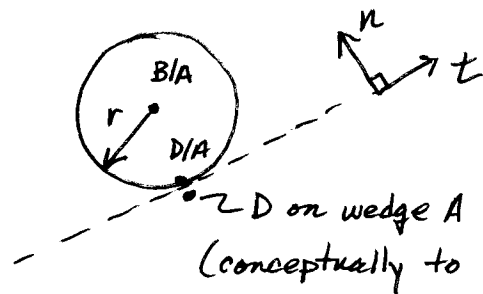
5 equations for $\underbrace{F, N, a_{Bx}, a_{By}, \alpha_B, a_{Ax}, N_g}_{7 \text{ unknowns}}$

Note that we could have analyzed the cylinder on an inclined coordinate system. Also note that analyzing the system as a whole will not yield any new equations.

We must now use relative acceleration and velocity equations to find relationships between a_{Bx}, a_{By}, α_B and a_{Ax} .

First, $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$

To analyze $\vec{a}_{B/A}$ we can think of the wedge as being fixed and the cylinder rolling down it. Let's be general and analyze both velocity and acceleration.



$t \equiv$ tangential direction
 $n \equiv$ normal direction

(conceptually to analyze $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$ both $\vec{v}_D=0$ and $\vec{a}_D=0$)

$$\vec{v}_{D/A} = \vec{v}_{B/A} + \vec{v}_{D/B} = \vec{v}_{B/A} + \vec{\omega}_B \times \vec{r}_{D/B}$$

Since D/A is coincident with D ~~and~~ and there is no slipping, both the normal and tangential components of $\vec{v}_{D/A}$ equal those of $\vec{v}_D = 0$.

$$\therefore \vec{v}_{B/A} = -\vec{\omega}_B \times \vec{r}_{D/B} \text{ for rolling w/o slipping}$$

$$\text{or } \boxed{\vec{v}_{B/A} = -\omega_B r \vec{e}_t} \text{ rolling w/o slipping}$$

$$\text{acceleration: } \vec{a}_{D/A} = \vec{a}_{B/A} + \vec{\alpha}_B \times \vec{r}_{D/B} + \vec{\omega}_B \times (\vec{\omega}_B \times \vec{r}_{D/B})$$

$$\vec{a}_{D/A} = \vec{a}_{B/A} + \alpha_B r \vec{e}_t + \omega_B^2 r \vec{e}_n$$

$$\text{for rolling without slipping } \vec{a}_{D/A} = a_{D/A} \vec{e}_n$$

$$(\text{i.e. } a_{D/A}^{\text{tangential}} = 0) \quad \text{and} \quad \vec{a}_{B/A} = a_{B/A} \vec{e}_t$$

$$\therefore \boxed{a_{B/A} = -\alpha_B r} \text{ for rolling w/o slipping}$$

↑ In tangential direction.

$$\therefore \vec{a}_B = \vec{a}_A + (-\alpha_B r) \vec{e}_t$$

$$a_{Bx} \vec{i} + a_{By} \vec{j} = a_{Ax} \vec{i} - \alpha_B r (\cos 20^\circ \vec{i} + \sin 20^\circ \vec{j})$$

$$\boxed{\begin{aligned} a_{Bx} &= a_{Ax} - \alpha_B r \cos 20^\circ \\ a_{By} &= -\alpha_B r \sin 20^\circ \end{aligned}}$$

$$\textcircled{3} \rightarrow F = \frac{1}{2} \frac{W_B}{g} r \alpha_B$$

$$\textcircled{4} \rightarrow N \sin 20^\circ = \frac{W_A}{g} a_{Ax} + F \cos 20^\circ = \frac{W_A}{g} a_{Ax} + \frac{1}{2} \frac{W_B}{g} r \alpha_B \cos 20^\circ$$

$$\textcircled{1} \rightarrow \frac{1}{2} \frac{W_B}{g} r \alpha_B \cos 20^\circ - \frac{W_A}{g} a_{Ax} - \frac{1}{2} \frac{W_B}{g} r \alpha_B \cos 20^\circ = \frac{W_B}{g} a_{Bx}$$

$$- \frac{W_A}{g} a_{Ax} = \frac{W_B}{g} (a_{Ax} - \alpha_B r \cos 20^\circ)$$

$$\therefore a_{Ax} = \frac{W_B}{W_A + W_B} \alpha_B r \cos 20^\circ$$

$$\textcircled{2} \rightarrow \frac{1}{2} \frac{W_B}{g} \alpha_B r \sin 20^\circ + \left(\frac{W_A}{g} a_{Ax} + \frac{1}{2} \frac{W_B}{g} \alpha_B r \cos 20^\circ \right) \frac{\cos 20^\circ}{\sin 20^\circ} - W_B = \frac{W_B}{g} a_{By}$$

Multiply by $\sin 20^\circ$ and substitute in for a_{Ax} and a_{By}

$$\frac{1}{2} \frac{W_B}{g} \alpha_B r + \frac{W_A W_B}{W_A + W_B} \alpha_B r \cos^2 20^\circ - W_B \sin 20^\circ = -\frac{W_B}{g} \alpha_B r \sin^2 20^\circ$$

$$\therefore \alpha_B = \left[\frac{\frac{1}{2} \frac{W_B}{g} r + \frac{W_A W_B}{g(W_A + W_B)} r \cos^2 20^\circ + \frac{W_B}{g} r \sin^2 20^\circ}{W_B \sin 20^\circ} \right]^{-1}$$

(Sorry, I solved for $1/\alpha_B$ accidentally.)

Plugging in $W_A = 4 \text{ lb.}$, $W_B = 6 \text{ lb.}$, $r = 3 \text{ in.}$, $g = 32.2 \frac{\text{ft}}{\text{s}^2}$
 $\rightarrow r = \frac{1}{4} \text{ ft.}$

We get $\alpha_B = 45.4 \frac{\text{rad}}{\text{s}^2}$

$$\vec{\alpha}_B = 45.4 \vec{k} \frac{\text{rad}}{\text{s}^2}$$

$$\therefore a_{Ax} = \frac{W_B}{W_A + W_B} (\alpha_B r) \cos 20^\circ = 6.4 \text{ ft/s}^2$$

$$\vec{a}_A = 6.4 \vec{i} \frac{\text{ft}}{\text{s}^2}$$

Work-Energy

204

Work of a couple in planar motion.

$$W^c = \int_{\theta_1}^{\theta_2} C d\theta$$

where C is a couple in the z -direction and θ increases in the counterclockwise direction about the z -axis.

Kinetic energy: Particle system - $KE = \sum_{i=1}^N \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$

$$\rightarrow \text{Rigid body} - KE = \int_V \frac{1}{2} \rho \vec{v} \cdot \vec{v} dV$$

Take A to be on the rigid body, then

$$\vec{v} = \vec{v}_A + \vec{\omega} \times \vec{r}_{IA}$$

$$KE = \frac{1}{2} \int_V \rho [\vec{v}_A \cdot \vec{v}_A + 2 \vec{v}_A \cdot \vec{\omega} \times \vec{r}_{IA} + \vec{\omega} \times \vec{r}_{IA} \cdot \vec{\omega} \times \vec{r}_{IA}] dV$$

$$= \frac{1}{2} \vec{v}_A \cdot \vec{v}_A \int_V \rho dV$$

$$+ \frac{1}{2} \cdot 2 \vec{v}_A \cdot \vec{\omega} \times \int_V \rho \vec{r}_{IA} dV$$

$$+ \frac{1}{2} \int_V \rho (\underbrace{\vec{\omega} \times \vec{r}_{IA}}_{\vec{A}}) \cdot (\underbrace{\vec{\omega} \times \vec{r}_{IA}}_{\vec{B} \cdot \vec{C}}) dV$$

$$\text{Recall } \vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A}$$

$$\begin{aligned}
\therefore \frac{1}{2} \int_V \rho (\vec{\omega} \times \vec{r}_{IA}) \cdot (\vec{\omega} \times \vec{r}_{IA}) dV \\
= \frac{1}{2} \int_V \rho \vec{\omega} \cdot \vec{r}_{IA} \times (\vec{\omega} \times \vec{r}_{IA}) dV \\
= \frac{1}{2} \vec{\omega} \cdot \underbrace{\int_V \rho \vec{r}_{IA} \times (\vec{\omega} \times \vec{r}_{IA}) dV}_{\vec{h}_A}
\end{aligned}$$

$$\begin{aligned}
\therefore KE = \frac{1}{2} m \underbrace{\vec{v}_A \cdot \vec{v}_A}_{v_A^2} + \vec{v}_A \cdot \vec{\omega} \times \int_V \rho \vec{r}_{IA} dV \\
+ \frac{1}{2} \vec{\omega} \cdot \vec{h}_A
\end{aligned}$$

For planar motion either choose $A = CM$ or choose A to be the instant center, i.e. $\vec{v}_A = 0$.

$$A = CM \rightarrow \int_V \rho \vec{r}_{IA} dV = \int_V \rho \vec{r}_{cm} dV = \int_V \rho \vec{r}' dV = 0$$

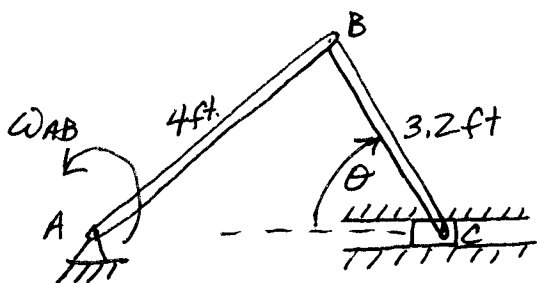
$$\vec{h}_A = \vec{h}_{cm} = I_{cm} \omega \vec{k}$$

$$\therefore KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$\text{or } A \text{ fixed} \rightarrow \vec{v}_A = 0 \rightarrow \vec{h}_A = I_A \omega \vec{k}$$

$$\therefore KE = \frac{1}{2} I_A \omega^2 \quad \underline{A \text{ fixed}}$$

Example Problem 18.21



$$m_{AB} = \frac{30 \text{ lb.}}{32.2 \text{ ft/s}^2}$$

$$m_{BC} = \frac{10 \text{ lb.}}{32.2 \text{ ft/s}^2}$$

$$m_C = \frac{2 \text{ lb.}}{32.2 \text{ ft/s}^2}$$

$$\omega_{AB} = 2.4 \frac{\text{rad}}{\text{s}}$$

Determine the kinetic energy of the entire system when $\theta = 50^\circ$.

$$KE = \frac{1}{2} m_C v_C^2 + \underbrace{\frac{1}{2} I_A^{AB} \omega_{AB}^2}_{A \text{ is fixed}} + \frac{1}{2} m_{BC} v_{cm}^2 + \frac{1}{2} I_{cm}^{BC} \omega_{BC}^2$$

Recall for a rod $I_{cm} = \frac{1}{12} m L^2$

$$\therefore I_{cm}^{BC} = \frac{1}{12} m_{BC} L_{BC}^2 = \frac{1}{12} \frac{10}{32.2} (3.2)^2 = 0.265$$

$$I_A^{AB} = \cancel{\frac{1}{12} m_{AB} L_{AB}^2} + m_{AB} d^2 = \frac{1}{3} m_{AB} L_{AB}^2 = \frac{1}{3} \frac{30}{32.2} 4^2 = 4.969$$

$$\therefore \frac{1}{2} I_A^{AB} \omega_{AB}^2 = \frac{1}{2} 4.969 (2.4)^2 = 14.31 \text{ ft}\cdot\text{lb}$$

We still need v_C , v_{cm}^{BC} , ω_{BC} .

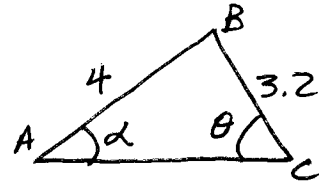
Use relative velocity equations.

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$$

$$\vec{v}_B = \omega_{AB} \vec{k} \times r_{B/A} (\cos \alpha \vec{i} + \sin \alpha \vec{j})$$

$$\vec{v}_B = \omega_{AB} r_{B/A} (-\sin \alpha \vec{i} + \cos \alpha \vec{j})$$

$$\begin{aligned} \vec{v}_B &= -2.4(4) \sin 37.8^\circ \vec{i} + 2.4(4) \cos 37.8^\circ \vec{j} \\ &= (-5.883 \vec{i} + 7.586 \vec{j}) \text{ ft/s} \end{aligned}$$



$$\frac{4}{\sin \theta} = \frac{3.2}{\sin \alpha}$$

$$\rightarrow \alpha = \arcsin\left(\frac{3.2}{4} \sin 50^\circ\right)$$

$$\alpha = 37.795^\circ$$

$$\text{Then } \vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$$

$$\vec{v}_C = v_C \vec{i} = \omega_{AB} r_{B/A} (-\sin \alpha \vec{i} + \cos \alpha \vec{j})$$

$$+ \omega_{BC} \vec{k} \times r_{C/B} (\cos \theta \vec{i} - \sin \theta \vec{j})$$

$$= \omega_{AB} r_{B/A} (-\sin \alpha \vec{i} + \cos \alpha \vec{j})$$

$$+ \omega_{BC} r_{C/B} (\sin \theta \vec{i} + \cos \theta \vec{j})$$

$$\therefore \vec{j} \cdot \vec{v}_C = 0 = \omega_{AB} r_{B/A} \cos \alpha + \omega_{BC} r_{C/B} \cos \theta$$

$$\omega_{BC} = \frac{-\omega_{AB} r_{B/A} \cos \alpha}{r_{C/B} \cos \theta} = -3.688 \frac{\text{rad}}{\text{s}}$$

$$\vec{i} \rightarrow v_C = -\omega_{AB} r_{B/A} \sin \alpha + \omega_{BC} r_{C/B} \sin \theta$$

$$v_C = -14.924 \text{ ft/s}$$

Finally we need \vec{v}_{cm}^{BC} . We can get this by three different approaches.

$$\vec{v}_{cm}^{BC} = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{cm/B} \leftarrow (1)$$

$$\vec{v}_{cm}^{BC} = \vec{v}_C + \vec{\omega}_{BC} \times \vec{r}_{cm/C} \leftarrow (2)$$

$$2\vec{v}_{cm}^{BC} = \vec{v}_B + \vec{v}_C + \vec{\omega}_{BC} \times (\vec{r}_{cm/B} + \vec{r}_{cm/C})$$

Note $\vec{r}_{cm/B} = -\vec{r}_{cm/C}$
 $\rightarrow \vec{r}_{cm/B} + \vec{r}_{cm/C} = 0$

$$\therefore \vec{v}_{cm}^{BC} = \frac{\vec{v}_B + \vec{v}_C}{2} \leftarrow (3)$$

Any of these 3 equations will give \vec{v}_{cm}^{BC} .

$$\vec{v}_{cm}^{BC} = \frac{1}{2} [(-5.883 - 14.924)\vec{i} + 7.586\vec{j}]$$

$$= -10.4035\vec{i} + 3.793\vec{j}$$

$$(\vec{v}_{cm}^{BC})^2 = 122.62 \frac{\text{ft}^2}{\text{s}^2}$$

$$\therefore KE = \underbrace{14.31}_{\frac{1}{2} I_A^{AB} \omega_{AB}^2} + \underbrace{\frac{1}{2} \frac{2}{32.2} (14.924)^2}_{\frac{1}{2} m_C v_C^2} + \underbrace{\frac{1}{2} \frac{(0.265)}{32.2} (3.688)^2}_{\frac{1}{2} I_{cm}^{BC} \omega_{BC}^2} + \underbrace{\frac{1}{2} \frac{10}{32.2} \cdot 122.62}_{\frac{1}{2} m_{BC} (\vec{v}_{cm}^{BC})^2}$$

$$KE = 42.07 \text{ ft}\cdot\text{lb}$$