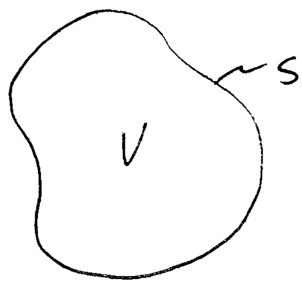
General Theorems

Uniqueness



Tractions prescribed on S_T
Displacements prescribed on S_u

Assume that we have two solutions $\vec{u}^{(1)}$ and $\vec{u}^{(2)}$ that satisfy all of the field equations and prescribed boundary conditions. i.e.

$$\begin{aligned} &\sigma^{(1)}_{ji,j} + b_i = 0 \\ &\tfrac{1}{2}(u^{(1)}_{i,j} + u^{(1)}_{j,i}) = \varepsilon^{(1)}_{ij} \\ &\sigma^{(1)}_{ij} = C_{ijkl}\varepsilon^{(1)}_{kl} \\ &\sigma^{(1)}_{ji} n_j = T_i \text{ on } S_T \\ &u^{(1)}_i = u^o_i \text{ on } S_u \end{aligned} \qquad\qquad \begin{aligned} &\sigma^{(2)}_{ji,j} + b_i = 0 \\ &\tfrac{1}{2}(u^{(2)}_{i,j} + u^{(2)}_{j,i}) = \varepsilon^{(2)}_{ij} \\ &\sigma^{(2)}_{ij} = C_{ijkl}\varepsilon^{(2)}_{kl} \\ &\sigma^{(2)}_{ji} n_j = T_i \text{ on } S_T \\ &u^{(2)}_i = u^o_i \text{ on } S_u \end{aligned}$$

Now let's take the difference of the solutions $\vec{u} = \vec{u}^{(2)} - \vec{u}^{(1)}$, and due to the linearity of all of the equations this solution satisfies

$$\sigma_{ij} = \sigma^{(2)}_{ij} - \sigma^{(1)}_{ij}, \quad \varepsilon_{ij} = \varepsilon^{(2)}_{ij} - \varepsilon^{(1)}_{ij}, \quad u_i = u^{(2)}_i - u^{(1)}_i$$

$$\rightarrow \quad \begin{aligned} &\sigma_{ji,j} = 0 \\ &\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \\ &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \\ &\sigma_{ji} n_j = 0 \text{ on } S_T \\ &u_i = 0 \text{ on } S_u \end{aligned}$$

Consider the integral $\int_V \sigma_{ji} u_{i,j} \, dV$

$$\int_V \sigma_{ji} u_{i,j} \, dV = \int_V (\sigma_{ji} u_i)_{,j} - \sigma_{ji,j} u_i \, dV$$

$$= \int_S \sigma_{ji} n_j u_i \, dS - \int_V \cancel{\sigma_{ji,j}}^{\,0} u_i \, dV$$

$$= \int_{S_T} \cancel{\sigma_{ji} n_j}^{\,0} u_i \, dS + \int_{S_u} \sigma_{ji} n_j \cancel{u_i}^{\,0} \, dS$$

$$\therefore \int_V \sigma_{ji} u_{i,j} \, dV = 0$$

$$\int_V \tfrac{1}{2}(\sigma_{ji} u_{i,j} + \sigma_{ij} u_{j,i}) \, dV = 0$$

$$\int_V \sigma_{ij} \tfrac{1}{2}(u_{i,j} + u_{j,i}) \, dV = 0$$

$$\int_V \sigma_{ij} \varepsilon_{ij} \, dV = 0$$

$$\int_V C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, dV = 0$$

if C_{ijkl} is positive definite, then $C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} > 0$ for any $\varepsilon_{ij} \neq 0$, hence

$\int_V C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, dV = 0$ if and only if

$\varepsilon_{ij} = 0$ at <u>every</u> point in the body and therefore $u_i = 0$. (For traction problems u_i = RBM)

Be specific Here. $\therefore$ $\boxed{u_i^{(1)} = u_i^{(2)} \text{ and the solution is unique.}}$

The Dynamic case (This could be better)

Assume the reference energy state at $t=0$ is zero. Displacements and tractions are applied to their specified surfaces as a function of time. Equilibrium now appears as:

$$\sigma^{(1)}_{ji,j} + b_i = \rho \ddot{u}^{(1)}_i \quad \text{and} \quad \sigma^{(2)}_{ji,j} + b_i = \rho \ddot{u}^{(2)}_i$$

$$\therefore \quad \sigma_{ji,j} = \rho \ddot{u}_i$$

Now we have $\int_V \sigma_{ji} \dot{u}_{i,j}\, dV = \int_S \cancelto{0}{\sigma_{ji} n_j} \dot{u}_i\, dS - \int_V \sigma_{ji,j} \dot{u}_i\, dV$

$$\therefore \quad \int_V C_{ijkl} \varepsilon_{ij} \dot{\varepsilon}_{kl}\, dV + \int_V \rho \ddot{u}_i \dot{u}_i\, dV = 0$$

$$\frac{d}{dt}(\dot{u}_i \dot{u}_i) = 2 \ddot{u}_i \dot{u}_i$$
$$\frac{d}{dt}(C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}) = 2 C_{ijkl} \dot{\varepsilon}_{ij} \varepsilon_{kl}$$

$$\therefore \frac{d}{dt} \int_V \tfrac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \tfrac{1}{2} \rho \dot{u}_i \dot{u}_i \, dV = 0$$

$$\int_V \underbrace{\tfrac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}}_{\text{positive definite}} + \underbrace{\tfrac{1}{2} \rho \dot{u}_i \dot{u}_i}_{\text{positive}} \, dV = \text{constant} = 0$$

b/c energy = 0 at t = 0

$$\therefore \boxed{u_i, \dot{u}_i = 0 \text{ for all } t \text{ at all points in } V}$$

Minimum Potential Energy

$$\Pi = \underbrace{U}_{\text{stored elastic energy}} - \underbrace{W}_{\text{work done by applied loads}}$$

Assume that the solution to a given boundary value problem of linear elasticity is u_i. Consider the set of displacements $u_i + \delta u_i$ that satisfy (all) any displacement boundary conditions.

$$\Pi(u_i) = \int_V \tfrac{1}{2} C_{ijkl} u_{i,j} u_{k,l} \, dV - \int_V b_i u_i \, dV - \int_{S_T} T_i u_i \, dS$$

$$\Pi(u_i + \delta u_i) = \int_V \tfrac{1}{2} C_{ijkl} (u_{i,j} + \delta u_{i,j})(u_{k,l} + \delta u_{k,l})$$

$$- \int_V b_i (u_i + \delta u_i) \, dV - \int_{S_T} T_i (u_i + \delta u_i) \, dS$$

$$= \int_V \tfrac{1}{2} C_{ijkl} u_{i,j} u_{k,l} \, dV - \int_V b_i u_i \, dV - \int_{S_T} T_i u_i \, dS$$

$$+ \int_V \tfrac{1}{2} C_{ijkl} u_{i,j} \delta u_{k,l} + \tfrac{1}{2} C_{ijkl} \delta u_{i,j} u_{k,l} \, dV$$

$$- \int_V b_i \delta u_i \, dV - \int_{S_T} T_i \delta u_i \, dS$$

$$+ \int_V \tfrac{1}{2} C_{ijkl} \delta u_{i,j} \delta u_{k,l} \, dV$$

$$= \Pi(u_i) + \int_V C_{ijkl} u_{i,j} \delta u_{k,l} \, dV$$

$$- \int_V b_i \delta u_i \, dV - \int_{S_T} T_i \delta u_i \, dS + \int_V \tfrac{1}{2} C_{ijkl} \delta u_{i,j} \delta u_{k,l} \, dV$$

$$\delta\pi = \pi(u_i+\delta u_i) - \pi(u_i)$$

$$= \int_V \sigma_{kl}\,\delta u_{k,l}\,dV - \int_V b_i\,\delta u_i\,dV - \int_{S_T} T_i\,\delta u_i\,dS + HOT$$

$$= \int_V (\sigma_{ij}\,\delta u_i)_{,j} - \sigma_{ij,j}\,\delta u_i\,dV - \int_V b_i\,\delta u_i\,dV - \int_{S_T} T_i\,\delta u_i\,dS + HOT$$

$$= \int_V -(\sigma_{ij,j}+b_i)\,\delta u_i\,dV + \int_{S_T}(\sigma_{ij}n_j - T_i)\,\delta u_i\,dS + HOT$$

$$\therefore\ \delta\pi = 0 + HOT(\delta u_i\,\delta u_j)$$

Note $\int_{S_u} \sigma_{ij} n_j\,\delta u_i\,dV = 0$ b/c $\delta u_i = 0$ on S_u

Note that $HOT = \int_V \frac{1}{2} C_{ijkl}\,\delta u_{i,j}\,\delta u_{k,l}\,dV > 0$

$$\therefore\ \boxed{\delta\pi = 0 \text{ and } \pi(u_i) \text{ is at a minimum.}}$$

$$\pi(u_i+\delta u_i) = \pi(u_i) + \delta\pi(u_i,\delta u_i) + \frac{1}{2}\delta^2\pi(u_i,\delta u_i)$$

Principle of Virtual Work (not limited to linear elasticity, only small strains and displacements)

Let δu_i be any displacement field that satisfies $\delta \varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$. δu_i is called a "virtual" displacement field.

Equilibrium $\rightarrow$ $\sigma_{ji,j} + b_i = 0$ ← Any equilibrium σ_{ij} field

$$\int_V \sigma_{ji,j}\, \delta u_i \, dV + \int_V b_i \delta u_i \, dV = 0$$

$$\int_V (\sigma_{ji}\,\delta u_i)_{,j} - \sigma_{ji}\,\delta u_{i,j}\, dV + \int_V b_i \delta u_i \, dV = 0$$

$$\int_V -\sigma_{ji}\left[\tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) + \tfrac{1}{2}(\delta u_{i,j} - \delta u_{j,i})\right] dV$$

$$+ \int_S \sigma_{ji} n_j \,\delta u_i \, dS + \int_V b_i \delta u_i \, dV = 0$$

$$\int_V \sigma_{ji}\,\delta\varepsilon_{ij} + \sigma_{ji}\,\delta\Omega_{ij}\, dV = \int_S \sigma_{ji} n_j \,\delta u_i\, dS + \int_V b_i \delta u_i \, dV$$

Note : $\sigma_{ji} = \sigma_{ij}$, $\delta\Omega_{ij} = -\delta\Omega_{ji}$

$\therefore \quad \sigma_{ji}\,\delta\Omega_{ij} = 0$

$$\boxed{\int_V \sigma_{ji}\,\delta\varepsilon_{ij}\, dV = \int_S T_i \,\delta u_i \, dS + \int_V b_i \,\delta u_i \, dV}$$

σ_{ij} is any equilibrium stress field and δu_i is any compatible displacement field.

Betti-Rayleigh Reciprocity Relations

Claim: For a linear elastic body subjected to two different loading systems, the work done by system one through the displacements due to system two is equal to the work done by system two through the displacements due to system one.

$$
\begin{aligned}
W^{1-2} &= \int_S T_i^{(1)} u_i^{(2)} dS + \int_V b_i^{(1)} u_i^{(2)} dV \\
&= \int_V \sigma_{ji}^{(1)} \varepsilon_{ij}^{(2)} dV \\
&= \int_V c_{ijkl} \varepsilon_{kl}^{(1)} \varepsilon_{ij}^{(2)} dV \\
&= \int_V \sigma_{kl}^{(2)} \varepsilon_{kl}^{(1)} dV \\
&= \int_S T_i^{(2)} u_i^{(1)} dS + \int_V b_i^{(2)} u_i^{(1)} dV \\
&= W^{2-1}
\end{aligned}
$$

QED

The Betti-Raleigh Relations are useful for determing average stresses or strains in a body.

Ex) $u_i^{(1)} = A x_i \rightarrow \varepsilon_{ij}^{(1)} = A\delta_{ij} \rightarrow \sigma_{ij}^{(1)} = \frac{E}{1-2\nu} A\,\delta_{ij}$

$$\rightarrow T_i^{(1)} = \frac{E}{1-2\nu} A\,\delta_{ij} n_j = \frac{E}{1-2\nu} A n_i \;,\; b_i^{(1)} = 0$$

Now let $u_i, \varepsilon_{ij}, \sigma_{ij}$ [redacted] solve any other boundary value problem with loading T_i and b_i.

$$\therefore \int_S \frac{E}{1-2\nu} A n_i u_i \, dS = \int_S T_i A x_i \, dS + \int_V b_i A x_i \, dV$$

$$\int_S \frac{E}{1-2\nu} u_i n_i \, dS = \int_S T_i x_i \, dS + \int_V b_i x_i \, dV$$

$$\int_V u_{i,i} \, dV = \frac{1-2\nu}{E} \int_S T_i x_i \, dS + \frac{1-2\nu}{E} \int_V b_i x_i \, dV$$

$$\int_V \varepsilon_{ii} \, dV = \frac{1-2\nu}{E} \left(\int_S T_i x_i dS + \int_V b_i x_i \, dV \right)$$

$$\rightarrow \boxed{\Delta V = \frac{1-2\nu}{E} \left(\int_S T_i x_i \, dS + \int_V b_i x_i \, dV \right)}$$

Ex) $\sigma_{11}^{(1)} = A \;,\; \varepsilon_{11}^{(1)} = \frac{A}{E} \;,\; \varepsilon_{22}^{(1)} = \varepsilon_{33}^{(1)} = \frac{-A\nu}{E}$

$$u_1^{(1)} = \frac{A}{E} x_1 \;,\; u_2^{(1)} = \frac{-A\nu}{E} x_2 \;,\; u_3^{(1)} = \frac{-A\nu}{E} x_3$$

$$T_1^{(1)} = A n_1 \;,\; T_2^{(1)} = T_3^{(1)} = 0$$

$$\therefore \int_S A n_1 u_1 \, dS = \int_S T_1 \frac{A}{E} x_1 - T_2 \frac{A\nu}{E} x_2 - T_3 \frac{A\nu}{E} x_3 \, dS$$

$$+ \int_V b_1 \frac{A}{E} x_1 - b_2 \frac{A\nu}{E} x_2 - b_3 \frac{A\nu}{E} x_3 \, dV$$

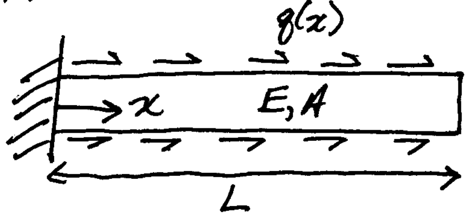
$$\therefore \int_V u_{1,1}\,dV = \frac{1}{E}\int_S T_1 x_1 - \nu(T_2 x_2 + T_3 x_3)\,dS + \frac{1}{E}\int_V b_1 x_1 - \nu(b_2 x_2 + b_3 x_3)\,dV$$

$$\boxed{\frac{1}{V}\int_V \varepsilon_{11}\,dV = \frac{1}{VE}\left[\int_S T_1 x_1 + \nu(-T_2 x_2 - T_3 x_3)\,dS + \int_V b_1 x_1 - \nu(T_2 x_2 + T_3 x_3)\,dV\right]}$$

Only loading needs to be known.

Average strain without knowing full solution.

Approximate Methods → Rayleigh-Ritz



Let's take $q(x) = C_n x^n$
$n \geq 0$

Exact Solution

$$\frac{d\sigma}{dx} + \frac{q(x)}{A} = 0 \quad , \quad \varepsilon = \frac{du}{dx} \quad , \quad \sigma = E\varepsilon$$

$$\therefore E\frac{d^2u}{dx^2} + \frac{C_n}{A}x^n = 0$$

$$\frac{du}{dx} = \frac{-C_n}{EA(n+1)}x^{n+1} + a_1$$

$$u = \frac{-C_n}{EA(n+1)(n+2)}x^{n+2} + a_1 x + a_0$$

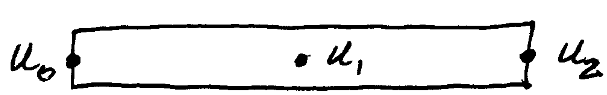
BCs: $u(x=0) = 0 \rightarrow a_0 = 0$

$\sigma(x=L) = 0 \rightarrow \frac{du}{dx}(x=L) = 0 \rightarrow a_1 = \frac{C_n}{EA(n+1)}L^{n+1}$

$$\boxed{u = \frac{C_n L^{n+2}}{A(n+1)E}\left[\frac{x}{L} - \frac{1}{n+2}\left(\frac{x}{L}\right)^{n+2}\right]}$$

Approximate Solution Implementing Minimum PE

Assume some displacement field in the bar. To illustrate the method we will assume a bilinear displacement field.



$$\therefore \quad u = \begin{cases} \dfrac{2(u_1-u_0)}{L}x + u_0 & \text{for } 0 \le x \le \frac{L}{2} \\ \dfrac{2(u_2-u_1)}{L}\left(x-\frac{L}{2}\right) + u_1 & \text{for } \frac{L}{2} \le x \le L \end{cases}$$

$$\pi = \int_V \tfrac{1}{2}E\varepsilon^2 dV - \int_V \frac{q(x)}{A} u\, dV - \int_S T u\, dS$$

$$= A\int_0^L \tfrac{1}{2}E\varepsilon^2 dx - A\frac{1}{A}\int_0^L q(x)\, u\, dx - \sigma_0 A u_0$$

$$= A\int_0^{\frac{L}{2}} \tfrac{1}{2}E\frac{4}{L^2}(u_1-u_0)^2 dx + \int_{\frac{L}{2}}^{L}\tfrac{1}{2}E\frac{4}{L^2}(u_2-u_1)^2 dx$$

$$- \int_0^{L/2} C_n x^n\left[\frac{2(u_1-u_0)}{L}x + u_0\right]dx$$

$$- \int_{L/2}^{L} C_n x^n\left[\frac{2(u_2-u_1)}{L}\left(x-\frac{L}{2}\right) + u_1\right]dx - \sigma_0 A u_0$$

$$\pi = \frac{EA}{L}\left[(u_1-u_0)^2 + (u_2-u_1)^2\right]$$

$$- \left(\frac{L}{2}\right)^{n+1}\frac{C_n}{(n+1)(n+2)}\left[u_0 + (2^{n+2}-2)\,u_1 + (1+2^{n+1}n)\,u_2\right]$$

$$- \sigma_0 A u_0$$

Minimize $\pi \rightarrow \dfrac{\partial\pi}{\partial u_0} = 0, \; \dfrac{\partial\pi}{\partial u_1} = 0, \; \dfrac{\partial\pi}{\partial u_2} = 0$

$$\frac{\partial\pi}{\partial u_0} = \frac{2EA}{L}(u_0-u_1) - \left(\frac{L}{2}\right)^{n+1}\frac{C_n}{(n+1)(n+2)} - \sigma_0 A = 0$$

$$\frac{\partial\pi}{\partial u_1} = \frac{2EA}{L}(2u_1-u_0-u_2) - \left(\frac{L}{2}\right)^{n+1}\frac{C_n}{(n+1)(n+2)}\left(2^{n+2}-2\right) = 0$$

$$\frac{\partial\pi}{\partial u_2} = \frac{2EA}{L}(u_2-u_1) - \left(\frac{L}{2}\right)^{n+1}\frac{1}{(n+1)(n+2)}C_n\left(1+n2^{n+1}\right) = 0$$

Impose the boundary condition that $u_0 = 0$ then $\frac{\partial \pi}{\partial u_0}$ yields the reaction force at the wall.

$$\rightarrow \sigma_0 A = \left(\frac{L}{2}\right)^{n+1} \frac{C_n}{(n+1)(n+2)} + \frac{2EA}{L} u_1$$

$$\frac{\partial \pi}{\partial u_1} = 0 \rightarrow \frac{4EA}{L} u_1 - \frac{2EA}{L} u_2 = \left(\frac{L}{2}\right)^{n+1} \left(2^{n+2} - 2\right) \frac{C_n}{(n+1)(n+2)}$$

$$\frac{\partial \pi}{\partial u_2} = 0 \rightarrow -\frac{2EA}{L} u_1 + \frac{2EA}{L} u_2 = \left(\frac{L}{2}\right)^{n+1} \left(1 + n 2^{n+1}\right) \frac{C_n}{(n+1)(n+2)}$$

$$\therefore u_2 = u_1 + \frac{L}{2EA} \left(\frac{L}{2}\right)^{n+1} \left(1 + n 2^{n+1}\right) \frac{C_n}{(n+1)(n+2)}$$

$$\therefore \frac{2EA}{L} u_1 = \left(\frac{L}{2}\right)^{n+1} \frac{C_n}{(n+1)(n+2)} \left[2^{n+1}(n+2) - 1\right]$$

<u>Approximate</u>

$$u_1 = \frac{L}{2EA} \left(\frac{L}{2}\right)^{n+1} \frac{C_n}{(n+1)(n+2)} \left[(n+2) 2^{n+1} - 1\right]$$

$$u_2 = \frac{L}{2EA} \left(\frac{L}{2}\right)^{n+1} \frac{C_n}{(n+1)(n+2)} (2n+2) 2^{n+1}$$

<u>Exact</u>

$$u_1 = \frac{L}{2EA} \left(\frac{L}{2}\right)^{n+1} \frac{C_n}{(n+1)(n+2)} \left[(n+2) 2^{n+1} - 1\right]$$

$$u_2 = \frac{L}{2EA} \left(\frac{L}{2}\right)^{n+1} \frac{C_n}{(n+1)(n+2)} (2n+2) 2^{n+1}$$

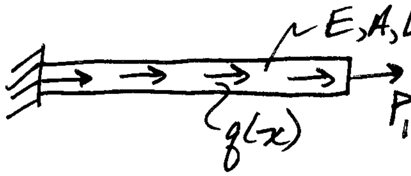
Exact and Approximate solutions are <u>identical</u>! (at nodes)

Note that this does not usually happen.

However, stresses and strains are not identical. For $n=0$ the σ and ε are correct at the midpoint of each element, and for $n>0$ the σ and ε are correct to the right of the midpoint of each element.

(A)

Rayleigh - Ritz



$\rightarrow$ Exact : $\dfrac{d^2u}{dx^2} + \dfrac{1}{EA} q(x) = 0$

Taylor series for $q(x) = \displaystyle\sum_{n=0}^{\infty} \left.\frac{d^n q}{dx^n}\right|_{x=0} x^n \frac{1}{n!}$

$$\therefore \quad \frac{du}{dx} = -\frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{n+1} \left.\frac{d^n q}{dx^n}\right|_{x=0} x^{n+1} \frac{1}{n!} + a$$

$$u = -\frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \left.\frac{d^n q}{dx^n}\right|_{x=0} x^{n+2} \frac{1}{n!} + ax + b$$

Approximate with linear interpolation

$$u = \frac{u_1 - u_0}{L} x + u_0$$

$$\pi = \int_0^L \frac{1}{2} EA \left(\frac{u_1 - u_0}{L}\right)^2 dx - \int_0^L \sum_{n=0}^{\infty} \left.\frac{d^n q}{dx^n}\right|_{x=0} \frac{x^n}{n!} \left(\frac{u_1 - u_0}{L} x + u_0\right) dx - P_0 u_0 - P_1 u_1$$

$$= \frac{1}{2} \frac{EA}{L} (u_1 - u_0)^2 - \int_0^L \frac{u_1 - u_0}{L} \sum_{n=0}^{\infty} \left.\frac{d^n q}{dx^n}\right|_{x=0} \frac{x^{n+1}}{n!} dx$$

$$- \int_0^L u_0 \sum_{n=0}^{\infty} \left.\frac{d^n q}{dx^n}\right|_{x=0} \frac{x^n}{n!} dx$$

$$- P_0 u_0 - P_1 u_1$$

$$= \frac{1}{2} \frac{EA}{L} (u_1 - u_0)^2 - \frac{u_1 - u_0}{L} \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d^n q}{dx^n} L^{n+2} \frac{1}{n!}$$

$$- u_0 \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n q}{dx^n} L^{n+1} \frac{1}{n!}$$

$$- P_0 u_0 - P_1 u_1$$

(B)

then $\frac{\partial\pi}{\partial u_0} = 0 = \frac{EA}{L}(u_0 - u_1) + \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+1}}{n!} - P_0$

$$\frac{\partial\pi}{\partial u_1} = 0 = \frac{EA}{L}(u_1 - u_0) - \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+1}}{n!} - P_1$$

Take $u_0 = 0$ (we must fix our structure in the x-direction somewhere)

then $\boxed{u_1^{Approx} = \frac{P_1 L}{EA} + \frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+2}}{n!}}$

Exact: $u_0 = 0 \rightarrow b = 0$

$$P_1 = EA \frac{du}{dx}\Big|_{x=L} \rightarrow P_1 = -\sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+1}}{n!} + EAa$$

$$\therefore\ a = \frac{P_1}{EA} + \frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+1}}{n!}$$

$$\rightarrow u^{exact} = -\frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{x^{n+2}}{n!} + \frac{P_1 x}{AE} + \frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+1} x}{n!}$$

$$u^{exact}(x=L) = -\frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+2}}{n!} + \frac{P_1 L}{AE} + \frac{L}{EA} \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+2}}{n!}$$

$$= \frac{P_1 L}{EA} + \frac{1}{EA} \sum_{n=0}^{\infty} \frac{d^n g}{dx^n}\Big|_{x=0} L^{n+2} \left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} \right) \frac{1}{n!}$$

$$= \frac{P_1 L}{EA} + \frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d^n g}{dx^n}\Big|_{x=0} \frac{L^{n+2}}{n!} = u_1^{Approx}$$

(C)

Now let's try a "bad" interpolation.

$$u = \frac{u_1 - u_0}{L^2} x^2 + u_0$$

$$\varepsilon = \frac{2(u_1 - u_0)}{L^2} x$$

$$\pi = \int_0^L \frac{1}{2} EA \frac{4(u_1-u_0)^2}{L^4} x^2 \, dx - \int_0^L \sum_{n=0}^{\infty} \frac{d^n q}{dx^n}\bigg|_{x=0} \frac{x^n}{n!} \left(\frac{u_1-u_0}{L^2} x^2 + u_0 \right) dx$$

$$- P_0 u_0 - P_1 u_1$$

$$= \frac{1}{2} EA \frac{4}{3} \frac{1}{L} (u_1 - u_0)^2 - \frac{u_1-u_0}{L^2} \sum_{n=0}^{\infty} \frac{1}{n+3} \frac{d^n q}{dx^n}\bigg|_{x=0} L^{n+3} \frac{1}{n!}$$

$$- u_0 \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n q}{dx^n}\bigg|_{x=0} L^{n+1} \frac{1}{n!}$$

$$\frac{\partial \pi}{\partial u_0} = 0 = \frac{4}{3} \frac{EA}{L} (u_0 - u_1) + \sum_{n=0}^{\infty} \frac{1}{n+3} \frac{d^n q}{dx^n}\bigg|_{x=0} \frac{L^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{1}{n+1} \frac{d^n q}{dx^n}\bigg|_{x=0} \frac{L^{n+1}}{n!} - P_0$$

$$\frac{\partial \pi}{\partial u_1} = 0 = \frac{4}{3} \frac{EA}{L} (u_1 - u_0) - \sum_{n=0}^{\infty} \frac{1}{n+3} \frac{d^n q}{dx^n}\bigg|_{x=0} \frac{L^{n+1}}{n!} - P_1$$

$$u_0 = 0$$

$$\rightarrow u_1^{Approx} = \frac{3}{4} \frac{P_1 L}{EA} + \frac{3}{4} \frac{1}{EA} \sum_{n=0}^{\infty} \frac{1}{n+3} \frac{d^n q}{dx^n}\bigg|_{x=0} L^{n+2} \frac{1}{n!}$$

$$\neq u^{exact}(x=L)$$

Notice that even if $q=0$, this approximate method does not give the exact result if a load is applied at the end.

In general any interopolation or approximate displacement field can be used in the Rayleigh-Ritz method. However, if polynomials are used then it is usually safest to used a polynomial of 1 order less than the number of degrees of freedom.

Also, note that 2 approximate solutions can always be compared against one another by computing the potential energy of each solution. The solution with the lower PE is usually better.

In general, for the 1-D problem we are considering a general interpolation can be written as

$$u = \sum_{i=0}^{N} a_i x^i \rightarrow \varepsilon = \sum_{i=0}^{N} i a_i x^{i-1}$$

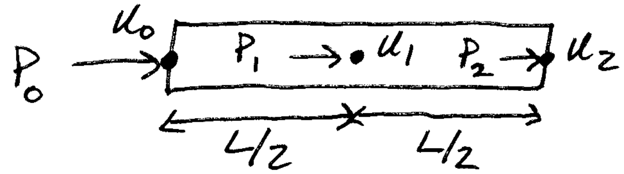
Also let me define $c_n = \frac{1}{n!} \frac{d^n g}{dx^n}$

$$\pi = \int_0^L \frac{1}{2} EA \left(\sum_{i=0}^{N} i a_i x^{i-1} \right)^2 dx$$

$$- \int_0^L \left(\sum_{n=0}^{\infty} c_n x^n \right) \left(\sum_{i=0}^{N} a_i x^i \right) dx$$

$$- P_0 a_0 - P_L \sum_{i=0}^{N} a_i L^i$$

Lastly, I would like to consider a higher order interpolation and find out if the exact solution is recovered at internal nodes as well as external nodes on each element.



(External nodes are shared by more than 1 element or are on a bondary)

We can write the interpolation in 2 ways.

$$u = a_0 + a_1 x + a_2 x^2$$

or

$$u = \underbrace{\underbrace{2\left(\tfrac{1}{2} - \tfrac{x}{L}\right)\left(1 - \tfrac{x}{L}\right)}_{N_0} u_0 + \underbrace{4\tfrac{x}{L}\left(1 - \tfrac{x}{L}\right)}_{N_1} u_1 + \underbrace{2\tfrac{x}{L}\left(\tfrac{x}{L} - \tfrac{1}{2}\right)}_{N_2}}_{\text{These are called shape functions in the finite element method.}} u_2$$

Note : $N_0 = 1$ at node 0 & $N_0 = 0$ at 1 & 2
$N_1 = 1$ at node 1 & $N_1 = 0$ at 0 & 2
$N_2 = 1$ at node 2 & $N_2 = 0$ at 0 & 1

$$u = \begin{bmatrix} N_0 & N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} \rightarrow \varepsilon = \begin{bmatrix} \frac{dN_0}{dx} & \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix}$$

(F)

Again we write the potential energy as

$$\pi = \int_0^L \frac{1}{2} EA\,\varepsilon^2\,dx - \int_0^L q(x)\,u(x)\,dx - P_0u_0 - P_1u_1 - P_2u_2$$

Using Mathematica this give the following

$$\frac{\partial \pi}{\partial u_0} = \frac{EA}{3L}(7u_0 - 8u_1 + u_2) - P_0 + \sum_{n=0}^{\infty} c_n \frac{(n-1)L^{n+1}}{(n+1)(n+2)(n+3)} = 0$$

$$\frac{\partial \pi}{\partial u_1} = \frac{EA}{3L}(-8u_0 + 16u_1 - 8u_2) - P_1 - \sum_{n=0}^{\infty} c_n \frac{4L^{n+1}}{(n+2)(n+3)} = 0$$

$$\frac{\partial \pi}{\partial u_2} = \frac{EA}{3L}(u_0 - 8u_1 + 7u_2) - P_2 - \sum_{n=0}^{\infty} c_n \frac{(n+1)L^{n+1}}{(n+2)(n+3)}$$

Then with $u_0 = 0$, $P_1 = 0$, $P_2 = P$
we get

$$u_1 = \frac{7}{4EA}\sum_{n=0}^{\infty} \frac{c_n L^{n+2}}{(n+2)(n+3)} + \frac{1}{2EA}\sum_{n=0}^{\infty} \frac{c_n (n+1) L^{n+2}}{(n+2)(n+3)} + \frac{PL}{2EA}$$

$$u_2 = \frac{1}{EA}\sum_{n=0}^{\infty} \frac{c_n L^{n+2}}{n+2} + \frac{PL}{EA}$$

Note $u_2 = u^{exact}(x=L)$

but $u_1 \neq u^{exact}(x=\frac{L}{2}) = \frac{PL}{2EA} + \frac{1}{EA}\sum_{n=0}^{\infty} c_n L^{n+2}\left[\frac{1}{2(n+1)} - \frac{1}{2^{n+2}(n+1)\cdot(n+2)}\right]$

∴ The displacement at nodes internal to an element are <u>not</u> exact. However, for this simple 1-D problem, displacements at nodes on the edges/ends of elements are exact if the interpolation is built up appropriately.

Elastodynamics - Waves

$$\sigma_{ji,j} = \rho \ddot{u}_i \quad , \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\sigma_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

$$\sigma_{ij} = \frac{E}{2(1+\nu)}(u_{i,j} + u_{j,i}) + \frac{E\nu}{(1+\nu)(1-2\nu)}u_{k,k}\delta_{ij}$$

$$\sigma_{ij,j} = \frac{E}{2(1+\nu)}(u_{i,jj} + u_{j,ij}) + \frac{E\nu}{(1+\nu)(1-2\nu)}u_{k,ki}$$

$$\sigma_{ij,j} = \frac{E}{2(1+\nu)}u_{i,jj} + \frac{E}{2(1+\nu)(1-2\nu)}u_{j,ji}$$

$$\frac{E}{2(1+\nu)}\left[u_{i,jj} + \frac{1}{1-2\nu}u_{j,ji}\right] = \rho\ddot{u}_i$$

Helmholtz decomposition $\rightarrow \vec{u} = \vec{\nabla}\phi + \vec{\nabla}\times\vec{\Psi}$

$$u_i = \phi_{,i} + \epsilon_{ijk}\psi_{k,j} = u_i^G + u_i^c$$

$$\epsilon_{ijk}u^G_{k,j} = \epsilon_{ijk}\phi_{,kj} = -\epsilon_{ikj}\phi_{,kj} = -\epsilon_{ijk}\phi_{,jk}$$

$$= -\epsilon_{ijk}\phi_{,kj} \rightarrow \epsilon_{ijk}\phi_{,kj} = 0$$

$\therefore u_i^G$ is irrotational

$$u^c_{i,i} = \epsilon_{ijk}\psi_{k,ji} = -\epsilon_{jik}\psi_{k,ji} = -\epsilon_{ijk}\psi_{k,ij} = -\epsilon_{ijk}\psi_{k,ji}$$

$$\therefore \epsilon_{ijk}\psi_{k,ji} = 0$$

$\therefore u_i^c$ is equi-volumnal (no volume change)

Substitute $u_i = \phi_{,i} + \epsilon_{ijk}\psi_{k,j}$ into Navier's Eqs.

$$\frac{E}{2(1+\nu)}\left[\phi_{,ijj} + \epsilon_{ijk}\psi_{k,jll} + \frac{1}{1-2\nu}\phi_{,ijj} + \frac{1}{1-2\nu}\underbrace{\epsilon_{jkl}\psi_{l,kji}}_{0}\right]$$
$$= \rho\ddot{\phi}_{,i} + \rho\,\epsilon_{ijk}\ddot{\psi}_{k,j}$$

$$\frac{2(1-\nu)}{1-2\nu}\phi_{,jji} + \epsilon_{ijk}\psi_{k,jll} = \frac{\rho}{\mu}\ddot{\phi}_{,i} + \frac{\rho}{\mu}\epsilon_{ijk}\ddot{\psi}_{k,j}$$

$$\left[\frac{2(1-\nu)}{1-2\nu}\phi_{,jj} - \frac{\rho}{\mu}\ddot{\phi}\right]_{,i} + \epsilon_{ijk}\left[\psi_{k,ll} - \frac{\rho}{\mu}\ddot{\psi}_k\right]_{,j} = 0$$

<u>Special Solutions</u>

$$\psi_k = 0 \;\rightarrow\; \phi_{,jj} = \frac{\rho(1-2\nu)(1+\nu)}{E(1-\nu)}\ddot{\phi}$$

$$c_L = \sqrt{\frac{E(1-\nu)}{\rho(1-2\nu)(1+\nu)}} = \text{Longitudinal or irrotational wave speed}$$

$$\rightarrow \nabla^2\phi = \frac{1}{c_L^2}\ddot{\phi}$$

$$\phi = 0 \;\rightarrow\; \psi_{k,ll} = \frac{\rho}{\mu}\ddot{\psi}_k$$

$$c_s = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2(1+\nu)\rho}} = \text{shear wave speed}$$

$$\rightarrow \nabla^2\psi_k = \frac{1}{c_s^2}\ddot{\psi}_k$$

Note: $\dfrac{c_L^2}{c_s^2} = \dfrac{2(1-\nu)}{1-2\nu} = 1 + \dfrac{1}{1-2\nu} > 1 \;,\; -1<\nu<0.5$

$\therefore$ Longitudinal waves are faster than shear waves

<u>Longitudinal wave example</u>: $\phi = f(x_1 - c_L t)$

$$\phi_{,2} = \phi_{,3} = 0 \ , \quad \phi_{,1} = f' \qquad \dot{\phi} = -c_L f'$$
$$\phi_{,11} = f'' \qquad \ddot{\phi} = +c_L^2 f''$$

$$\rightarrow \nabla^2 \phi = \frac{1}{c_L^2}\ddot{\phi} \rightarrow f'' = \frac{c_L^2}{c_L^2} f'' \checkmark$$

$\therefore$ The wave equation is satisfied for <u>any</u> f.

$$u_i = \phi_{,i} \rightarrow u_2 = u_3 = 0 \ , \quad u_1 = f'(x_1 - c_L t)$$

$$\rightarrow \varepsilon_{11} = u_{1,1} = f''(x_1 - c_L t) \text{ all other } \varepsilon_{ij} = 0$$

$$\sigma_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{11} \ , \quad \sigma_{22} = \sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{11}$$

This is a wave of "shape" $f(x_1)$ moving in the x_1 direction with speed c_L. The displacement is in the x_1 direction as well, i.e. in the direction of the wave propagation.

Note that solutions of the form $\phi = f(x_1 + c_L t)$ propagate ~~to~~ in the $-x_1$ direction.

There is another 1-D result with $c = \sqrt{\frac{E}{\rho}}$ which is an approximate solution for $\sigma_{22} = \sigma_{33} = 0$ that is a good approximation for thin wires.

<u>Shear Wave Example</u> : $\psi_1 = f(x_2 - c_s t)$
$\psi_2 = \psi_3 = 0$

$\psi_{1,1} = \psi_{1,3} = 0, \quad \psi_{1,2} = f' \qquad \dot{\psi}_1 = -c_s f'$
$\psi_{1,22} = f'' \qquad \ddot{\psi}_1 = c_s^2 f''$

$\rightarrow \nabla^2 \psi_k = \frac{1}{c_s^2}\ddot{\psi}_k \rightarrow f'' = \frac{c_s^2}{c_s^2} f''$ ✓

$u_1 = 0, \ u_2 = 0, \ u_3 = \epsilon_{321}\psi_{1,2} = -f'(x_2 - c_s t)$

→ Displacement is perpendicular to the propagation direction.

$\varepsilon_{23} = -\frac{1}{2} f'' \rightarrow \sigma_{23} = -\mu f''$ all other $\varepsilon_{ij} = 0$
all other $\sigma_{ij} = 0$

This is a shear wave propagating in the x_2 direction.

<u>General Longitudinal Plane Wave</u>
Let $\pm n_i$ be the propagation direction.
Take $\psi_i = 0 \qquad \phi = f(n_i x_i \pm c_L t)$

$u_i = \phi_{,i} = n_i f'(n_i x_i \pm c_L t)$

$\phi_{,ik} = n_i n_k f''(n_i x_i \pm c_L t) \rightarrow \phi_{,kk} = \underbrace{n_k n_k}_{1} f''(n_i x_i \pm c_L t)$

$\ddot{\phi} = c_L^2 f''(n_i x_i \pm c_L t) \rightarrow \phi_{,kk} = \frac{1}{c_L^2}\ddot{\phi}$ ✓

$$\therefore \quad \dot{u}_i = \pm c_L n_i f''(n_i x_i \pm c_L t)$$

Hence both u_i and $\dot{u}_i$ are parallel to the direction of propagation, i.e. // to n_i. The wave speed is c_L and strains are uniaxial in the $\vec{n}$ direction.

<u>General Shear Plane Wave</u>

$$\phi = 0 \quad , \quad \psi_i = A_i f(n_i x_i \pm c_s t)$$ A_i is a constant vector

$$\psi_{i,j} = A_i n_j f'(n_i x_i \pm c_s t)$$
$$\psi_{i,jj} = A_i \underbrace{n_j n_j}_{1} f''(n_i x_i \pm c_s t)$$

$$\ddot{\psi}_i = A_i c_s^2 f''(n_i x_i \pm c_s t)$$

$$\therefore \quad \psi_{i,jj} = \frac{1}{c_s^2} \ddot{\psi}_i \quad \checkmark$$

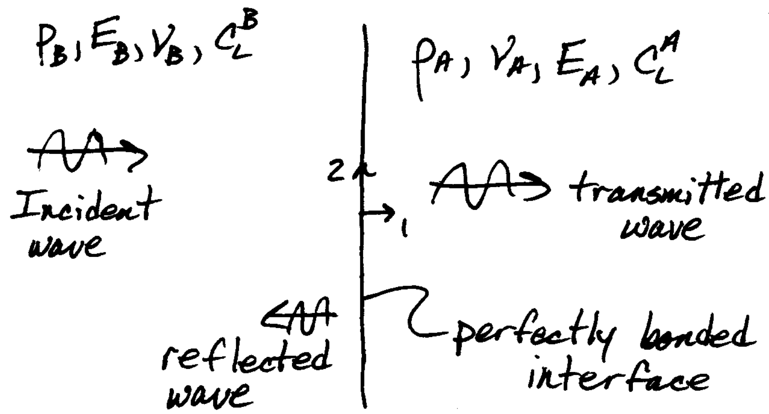
$$u_i = \epsilon_{ijk} \psi_{k,j} = \epsilon_{ijk} A_k n_j f'(n_m x_m \pm c_s t)$$

$$\dot{u}_i = \epsilon_{ijk} \dot{\psi}_{k,j} = \epsilon_{ijk} A_k n_j (\pm c_s) f''(n_m x_m \pm c_s t)$$

i.e. $\vec{u} = \vec{n} \times \vec{A} f'$, $\dot{\vec{u}} = \pm c_s \vec{n} \times \vec{A} f''$

$\therefore$ Both displacement and velocity are perpendicular to the direction of wave propagation, $\vec{n}$, and the wave speed is c_s.

<u>Reflection & Transmission</u> of a plane wave normally incident to an interface between dissimilar materials.



Look for solution with $u_2 = u_3 = 0$

Incident wave $\quad u_1^I = u_0 \sin\left[\dfrac{x - c_L^B t}{\lambda_B}\right]$

u_0 and λ_B are specified, wavelength $= \lambda_B / 2\pi$
frequency $= \omega_B = c_L^B / \lambda_B$

Look for reflected and transmitted waves of the form

$$u_1^B = u_1^I + u_1^R \quad , \quad u_1^A = u_1^T$$

where $\quad u_1^R = R\, u_0 \sin\left[\dfrac{x + c_L^B t}{\lambda_B}\right]$

$$u_1^T = T\, u_0 \sin\left[\dfrac{x - c_L^A t}{\lambda_A}\right]$$

Continuity of u_1 across $x_1 = 0$

$$\rightarrow \; (1-R)\sin\frac{c_L^B t}{\lambda_B} = T \sin\frac{c_L^A t}{\lambda_A}$$

$$\therefore \quad \boxed{T = 1 - R} \quad \text{and} \quad \frac{c_L^B}{\lambda_B} = \frac{c_L^A}{\lambda_A}$$

$$\rightarrow \boxed{\lambda_A = \frac{c_L^A}{c_L^B}\lambda_B} \qquad \omega_A = \frac{c_L^A}{\lambda_A} = \frac{c_L^B}{\lambda_B} = \omega_B$$

Continuity of σ_{11} (traction) at $x = 0$

$$\therefore \quad \frac{E_B(1-\nu_B)}{(1+\nu_B)(1-2\nu_B)} u^B_{1,1} = \frac{E_A(1-\nu_A)}{(1+\nu_A)(1-2\nu_A)} u^A_{1,1}$$

$$\rho_B c_L^{B\,2} u^B_{1,1} = \rho_A c_L^{A\,2} u^A_{1,1} \quad \text{at } x = 0$$

$$\rho_B c_L^{B\,2} u_0 \frac{1}{\lambda_B}(1+R)\cos\frac{c_L^B t}{\lambda_B} = \rho_A c_L^{A\,2} u_0 \frac{1}{\lambda_A} T \cos\frac{c_L^A t}{\lambda_A}$$

$$\therefore \quad \rho_B c_L^{B\,2} \frac{1}{\lambda_B}(1+R) = \rho_A c_L^{A\,2} \frac{1}{\lambda_A} T$$

$$(1+R) = \frac{\rho_A}{\rho_B}\left(\frac{c_L^A}{c_L^B}\right)^2 \frac{\lambda_B}{\lambda_A} T = \underbrace{\frac{\rho_A}{\rho_B}\frac{c_L^A}{c_L^B}}_{\alpha} T$$

$$\therefore \quad 1 + R = \alpha T \quad , \quad 1 - R = T$$

$$\boxed{T = \frac{2}{1+\alpha} \quad , \quad R = \frac{\alpha - 1}{\alpha + 1} \quad , \quad \frac{\lambda_A}{\lambda_B} = \frac{c_L^A}{c_L^B}}$$

$\alpha = 1 \rightarrow T = 1$ and $R = 0$

i.e. the entire wave is transmitted.

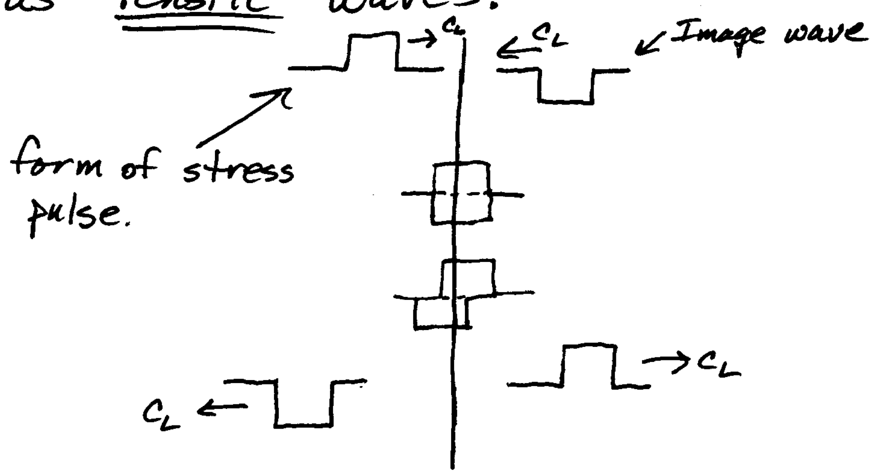
- traction free interface $\rightarrow \alpha = 0$

$\rightarrow \quad T = 2 \quad , \quad R = -1$

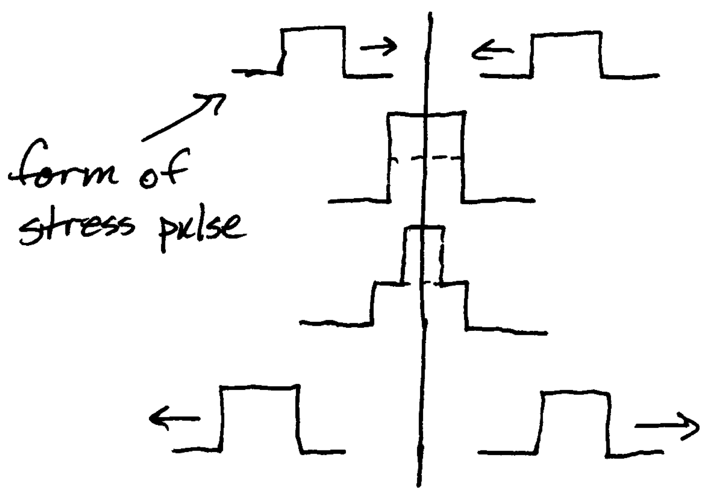
But we are only interested in the reflected wave.

$R = -1 \rightarrow u_1^R = -u_1^I$ the reflected wave is the opposite of the incident wave.

* Incident compressive waves are reflected as <u>tensile</u> waves.



Rigid interface: either very stiff or massive
$\rightarrow \alpha \rightarrow \infty \qquad T = 0, \quad R = 1$

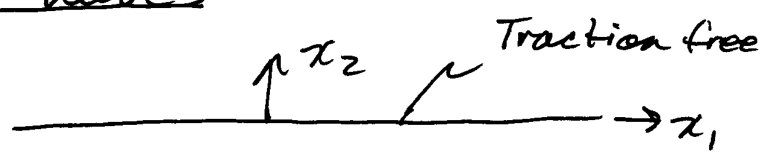


* Incident compressive waves are reflected as compressive waves.

If the incident wave approaches at an angle to the interface both shear and longitudinal waves can be reflected and transmitted.

Rayleigh Surface Waves

Rayleigh showed that waves propagating // to a free surface that decay exponentially into the bulk exist.



Look for plane strain solutions: $u_3 = 0$, $u_\alpha(x_1, x_2)$ $\alpha = 1, 2$

$$u_\alpha = \phi_{,\alpha} + \epsilon_{\alpha\beta 3}\psi_{3,\beta} \quad , \quad \psi_1 = \psi_2 = 0$$

$$\phi = f(x_2) \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$
$$\psi_3 = g(x_2) \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$

c is the propagation velocity, x_1 the direction

$$\phi_{,1} = \frac{i}{\lambda} f \exp\left[i\frac{x_1 - ct}{\lambda}\right] \qquad \dot{\phi} = \frac{-ic}{\lambda} f \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$
$$\phi_{,11} = -\frac{1}{\lambda^2} f \exp\left[i\frac{x_1 - ct}{\lambda}\right] \qquad \ddot{\phi} = \frac{-c^2}{\lambda^2} f \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$
$$\phi_{,2} = f' \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$
$$\phi_{,22} = f'' \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$

$$\psi_{3,11} = -\frac{1}{\lambda^2} g \exp\left[i\frac{x_1 - ct}{\lambda}\right] \qquad \ddot{\psi}_3 = \frac{-c^2}{\lambda^2} g \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$
$$\psi_{3,22} = g'' \exp\left[i\frac{x_1 - ct}{\lambda}\right]$$

Navier's Eqs $\rightarrow \left[\left\{\frac{2(1-\nu)}{1-2\nu}\left(f'' - \frac{1}{\lambda^2}f\right) + \left(\frac{\rho}{\mu}\frac{c^2}{\lambda^2}f\right)\right\}\exp\left[i\,\frac{x_1-ct}{\lambda}\right]\right]_{,i}$

$$+\,\epsilon_{ij3}\left[\left(g'' - \frac{1}{\lambda^2}g + \frac{\rho}{\mu}\frac{c^2}{\lambda^2}g\right)\exp\left[i\,\frac{x_1-ct}{\lambda}\right]\right]_{,j} = 0$$

$\therefore$ if $\quad f'' - \frac{1}{\lambda^2}\left(1 - \frac{\rho(1-2\nu)}{2\mu(1-\nu)}c^2\right)f = 0$

and $\quad g'' - \frac{1}{\lambda^2}\left(1 - \frac{\rho}{\mu}c^2\right)g = 0$

we can satisfy the pde's

$$f'' - \frac{1}{\lambda^2}\left(1 - \frac{c^2}{c_L^2}\right)f = 0 \rightarrow f = A\exp[\alpha_L x_2/\lambda]$$
$$g'' - \frac{1}{\lambda^2}\left(1 - \frac{c^2}{c_s^2}\right)g = 0 \rightarrow g = B\exp[\alpha_s x_2/\lambda]$$

$$\alpha_L^2 = 1 - \frac{c^2}{c_L^2}\,, \quad \alpha_s^2 = 1 - \frac{c^2}{c_s^2} \quad \left(\text{assume } \frac{c}{c_s} < 1\right)$$

Traction free condition: $x_2 = 0 \rightarrow \sigma_{12} = \sigma_{22} = 0$

on $x_2 = 0$

$$\varepsilon_{11} = u_{1,1} = \frac{1}{\lambda^2}\left(-A + i\alpha_s B\right)\exp\left[i\,\frac{x_1-ct}{\lambda}\right]$$
$$\varepsilon_{22} = u_{2,2} = \frac{1}{\lambda^2}\left(\alpha_L^2 A - i\alpha_s B\right)\exp\left[i\,\frac{x_1-ct}{\lambda}\right]$$
$$2\varepsilon_{12} = u_{1,2} + u_{2,1} = \frac{1}{\lambda^2}\left(+2i\alpha_L A + (1+\alpha_s^2)B\right)\exp\left[i\,\frac{x_1-ct}{\lambda}\right]$$

$$\sigma_{12} = 0 \rightarrow \varepsilon_{12} = 0 \rightarrow \boxed{+2i\alpha_L A + (1+\alpha_s^2)B = 0}$$

$$\sigma_{22} = 0 \rightarrow (1-\nu)\varepsilon_{22} + \nu\varepsilon_{11} = 0$$

$$\rightarrow \boxed{\left[(1-\nu)\alpha_L^2 - \nu\right]A - i(1-2\nu)\alpha_s B = 0}$$

Rearrange: $\quad -2\alpha_L A + (1+\alpha_s^2)(iB) = 0$

$$\left[(1-\nu)\alpha_L^2 - \nu\right]A - (1-2\nu)\alpha_s(iB) = 0$$

*Note f and g out of phase.

$$\rightarrow \quad iB = \frac{(1-\nu)\alpha_L^2 - \nu}{(1-2\nu)\alpha_s} A$$

$$\therefore \quad -2\alpha_L \cancel{A} + (1+\alpha_s^2)\frac{(1-\nu)\alpha_L^2 - \nu}{(1-2\nu)\alpha_s}\cancel{A} = 0$$

$$(1+\alpha_s^2)\left[(1-\nu)\alpha_L^2 - \nu\right] = 2(1-2\nu)\alpha_L\alpha_s$$

$$\alpha_s^2 = 1 - \left(\frac{c}{c_s}\right)^2 \qquad \left(\frac{c_s}{c_L}\right)^2 = \frac{1-2\nu}{2(1-\nu)}$$

$$\alpha_L^2 = 1 - \left(\frac{c}{c_L}\right)^2 = 1 - \left(\frac{c}{c_s}\right)^2\left(\frac{c_s}{c_L}\right)^2 = 1 - \frac{1-2\nu}{2(1-\nu)}\left(\frac{c}{c_s}\right)^2$$

$$2\cdot\left[2 - \left(\frac{c}{c_s}\right)^2\right]\left[\cancel{(1-2\nu)}^{\,1} - \frac{1}{2}\cancel{(1-2\nu)}\left(\frac{c}{c_s}\right)^2\right] = 2\cdot 2\cancel{(1-2\nu)}\,\alpha_L\alpha_s$$

$$\boxed{\left[2 - \left(\frac{c}{c_s}\right)^2\right]^2 = 4\alpha_L\alpha_s = 4\sqrt{\left[1-\left(\frac{c}{c_s}\right)^2\right]\left[1 - \frac{1-2\nu}{2(1-\nu)}\left(\frac{c}{c_s}\right)^2\right]}}$$

Solve for $\frac{c}{c_s}$ as a function of ν

ν	$\frac{c}{c_s}$
1/2	0.955
1/4	0.919
0	0.874

<u>Some real values</u>

Al : $E = 70\ GPa$ $\quad c_s = 5{,}082\ m/s$
$\rho = 2710\ kg/m^3$ $\quad c_L = 10{,}090\ m/s$
$\nu = 0.33$

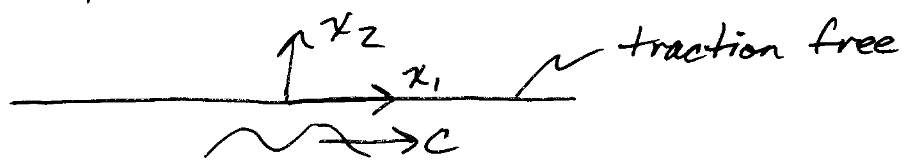
Steel: $E = 200\ GPa$ $\quad c_s = 5{,}044\ m/s$
$\rho = 7860\ kg/m^3$ $\quad c_L = 9{,}275\ m/s$
$\nu = 0.29$

A Note on using $\exp[i \frac{x - c_s t}{\lambda}]$ to solve problems.

Applying this method is OK if all we want to do is determine C_s for a certain type of wave.

However if we want to solve a physical problem we must use sines and cosines. Otherwise there will always be an imaginary part in the solution which is not physically possible. This is why the $\exp[i \frac{x - c_L t}{\lambda}]$ approach could not be used to solve the reflection and transmission problem.

Rayleigh Surface Waves



Look for plane strain solutions : $u_3 = 0$
$u_1 = u_1(x_1, x_2)$
$u_2 = u_2(x_1, x_2)$

$$u_1 = \phi_{,1} + \psi_{3,2} \quad , \quad u_2 = \phi_{,2} - \psi_{3,1} \quad , \quad \psi_1 = \psi_2 = 0$$

Try: $$\phi = f(x_2) \sin\left(\frac{x_1 - ct}{\lambda}\right)$$

$$\psi_3 = g(x_2) \cos\left(\frac{x_1 - ct}{\lambda}\right)$$

Governing PDE's for ϕ, ψ_3

$$x_1: \frac{E}{2(1+\nu)}\Big[(\phi_{,11} + \phi_{,22})_{,1} + (\psi_{3,11} + \psi_{3,22})_{,2}$$
$$+ \frac{1}{1-2\nu}(\phi_{,11} + \phi_{,22})_{,1}\Big] = \rho\ddot{\phi}_{,1} + \rho\ddot{\psi}_{3,2}$$

$$x_2: \frac{E}{2(1+\nu)}\Big[\frac{2(1-\nu)}{1-2\nu}(\phi_{,11} + \phi_{,22})_{,2} - (\psi_{3,11} + \psi_{3,22})_{,1}\Big]$$
$$= \rho\ddot{\phi}_{,2} - \rho\ddot{\psi}_{3,1}$$

$$\rightarrow c_L^2 \nabla^2\phi_{,1} + c_S^2 \nabla^2\psi_{3,2} = \ddot{\phi}_{,1} + \ddot{\psi}_{3,2}$$

$$c_L^2 \nabla^2\phi_{,2} - c_S^2 \nabla^2\psi_{3,1} = \ddot{\phi}_{,2} - \ddot{\psi}_{3,1}$$

or $\left[c_L^2 \nabla^2\phi - \ddot{\phi}\right]_{,i} + \epsilon_{ij3}\left[c_s^2 \nabla^2\psi_3 - \ddot{\psi}_3\right]_{,j} = 0$

Substitute in our proposed solution.

$$\phi_{,1} = +\frac{f}{\lambda}\cos\left(\frac{x_1-ct}{\lambda}\right) \qquad \phi_{,11} = -\frac{f}{\lambda^2}\sin\left(\frac{x_1-ct}{\lambda}\right)$$

$$\phi_{,2} = f'\sin\left(\frac{x_1-ct}{\lambda}\right) \qquad \phi_{,22} = f''\sin\left(\frac{x_1-ct}{\lambda}\right)$$

$$\dot{\phi} = -\frac{c}{\lambda}f\cos\left(\frac{x_1-ct}{\lambda}\right) \qquad \ddot{\phi} = -\frac{c^2}{\lambda^2}f\sin\left(\frac{x_1-ct}{\lambda}\right)$$

$$\psi_{3,1} = -\frac{g}{\lambda}\sin\left(\frac{x_1-ct}{\lambda}\right) \qquad \psi_{3,11} = -\frac{g}{\lambda^2}\cos\left(\frac{x_1-ct}{\lambda}\right)$$

$$\psi_{3,2} = g'\cos\left(\frac{x_1-ct}{\lambda}\right) \qquad \psi_{3,22} = g''\cos\left(\frac{x_1-ct}{\lambda}\right)$$

$$\dot{\psi}_3 = \frac{c}{\lambda}g\sin\left(\frac{x_1-ct}{\lambda}\right) \qquad \ddot{\psi}_3 = -g\frac{c^2}{\lambda^2}\cos\left(\frac{x_1-ct}{\lambda}\right)$$

$$\therefore \left[c_L^2\left(f'' - \frac{1}{\lambda^2}f + \frac{c^2}{c_L^2\lambda^2}f\right)\sin\left(\frac{x_1-ct}{\lambda}\right)\right]_{,i}$$

$$+ \epsilon_{ij3}\left[c_s^2\left(g'' - \frac{1}{\lambda^2}g + \frac{c^2}{c_s^2\lambda^2}g\right)\cos\left(\frac{x_1-ct}{\lambda}\right)\right]_{,j} = 0$$

Satisfied if: $f'' - \frac{1}{\lambda^2}\left(1-\frac{c^2}{c_L^2}\right)f = 0$

and

$$g'' - \frac{1}{\lambda^2}\left(1-\frac{c^2}{c_s^2}\right)g = 0$$

$$\rightarrow \quad f = A\exp\left[\frac{\alpha_L x_2}{\lambda}\right], \quad \alpha_L = \sqrt{1-\frac{c^2}{c_L^2}}$$

$$g = B\exp\left[\frac{\alpha_s x_2}{\lambda}\right], \quad \alpha_s = \sqrt{1-\frac{c^2}{c_s^2}}$$

C

Next, we need to apply BCs on $x_2 = 0$.

on $x_2 = 0$ $\quad \sigma_{12} = \sigma_{22} = 0$, Traction free.

$\varepsilon_{11} = u_{1,1} = \phi_{,11} + \psi_{3,21}$ [redacted]

$\varepsilon_{22} = u_{2,2} = \phi_{,22} - \psi_{3,12}$

$2\varepsilon_{12} = u_{1,2} + u_{2,1} = 2\phi_{,12} + \psi_{3,22} - \psi_{3,11}$

$\psi_{3,21} = -\frac{g'}{\lambda} \sin\left(\frac{x_1 - ct}{\lambda}\right) = -B \frac{\alpha_s}{\lambda^2} \exp\left[\frac{\alpha_s x_2}{\lambda}\right] \sin\left(\frac{x_1 - ct}{\lambda}\right)$

$\phi_{,12} = \frac{f'}{\lambda} \cos\left(\frac{x_1 - ct}{\lambda}\right) = A \frac{\alpha_L}{\lambda^2} \exp\left[\frac{\alpha_L x_2}{\lambda}\right] \cos\left(\frac{x_1 - ct}{\lambda}\right)$

$$\therefore \sigma_{22}(x_2 = 0) = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left(\phi_{,22} - \psi_{3,21}\right) + \frac{E\nu}{(1-\nu)(1-2\nu)} \left(\phi_{,11} + \psi_{3,21}\right)$$

$$= \cancel{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}} \left(A \frac{\alpha_L^2}{\cancel{\lambda^2}} + B \frac{\alpha_s}{\cancel{\lambda^2}}\right) \cancel{\sin\left(\frac{x_1 - ct}{\lambda}\right)}$$

$$+ \frac{E\nu}{\cancel{(1+\nu)(1-2\nu)}} \left(-\frac{A}{\cancel{\lambda^2}} - B \frac{\alpha_s}{\cancel{\lambda^2}}\right) \cancel{\sin\left(\frac{x_1 - ct}{\lambda}\right)} = 0$$

$$\rightarrow \boxed{\left[(1-\nu)\alpha_L^2 - \nu\right] A + (1-2\nu)\alpha_s B = 0}$$

$$\sigma_{12}(x_2 = 0) = \frac{E}{2(1+\nu)} \left(2\phi_{,12} + \psi_{3,22} - \psi_{3,11}\right)$$

$$= \cancel{\frac{E}{2(1-\nu)}} \left(2A \frac{\alpha_L}{\cancel{\lambda^2}} + B \frac{\alpha_s^2}{\cancel{\lambda^2}} + B \frac{1}{\cancel{\lambda^2}}\right) \cancel{\cos\left(\frac{x_1 - ct}{\lambda}\right)} = 0$$

$$\boxed{2\alpha_L A + (1 + \alpha_s^2) B = 0}$$

D

$$\therefore \quad B = -\frac{(1-\nu)\alpha_L^2 - \nu}{(1-2\nu)\alpha_S} A = -\frac{2\alpha_L}{1+\alpha_S^2} A$$

$$\therefore \quad (1+\alpha_S^2)\left[(1-\nu)\alpha_L^2 - \nu\right] = 2\alpha_L \alpha_S (1-2\nu)$$

$$\alpha_S^2 = 1 - \left(\frac{c}{c_S}\right)^2 \qquad \left(\frac{c_S}{c_L}\right)^2 = \frac{1-2\nu}{2(1-\nu)}$$

$$\alpha_L^2 = 1 - \left(\frac{c}{c_L}\right)^2 = 1 - \left(\frac{c}{c_S}\right)^2\left(\frac{c_S}{c_L}\right)^2 = 1 - \frac{1-2\nu}{2(1+\nu)}\left(\frac{c}{c_S}\right)^2$$

$$\rightarrow 2\left[2 - \left(\frac{c}{c_S}\right)^2\right]\left[\overset{1}{\cancel{(1-2\nu)}} - \frac{1}{2}\cancel{(1-2\nu)}\left(\frac{c}{c_S}\right)\right] = 2\cdot 2\,\cancel{(1-2\nu)}\,\alpha_L \alpha_S$$

$$\therefore \quad \boxed{\left[2 - \left(\frac{c}{c_S}\right)^2\right]\left[2 - \left(\frac{c}{c_S}\right)^2\right] = 4\sqrt{\left[1 - \left(\frac{c}{c_S}\right)^2\right]\left[1 - \frac{1-2\nu}{2(1+\nu)}\left(\frac{c}{c_S}\right)^2\right]}}$$

Non-linear eqn for $\frac{c}{c_S}$ as a function of ν

ν	$\frac{c}{c_S}$
1/2	0.955
1/4	0.919
0	0.874

Aluminum: $E = 70$ GPa, $\rho = 2710$ kg/m³, $\nu = 0.33$ $\qquad c_S = 5082$ m/s, $c_L = 10{,}090$ m/s, $c_R = 4807$ m/s

Steel: $E = 210$ GPa, $\rho = 7860$ kg/m³, $\nu = 0.29$ $\qquad c_S = 5044$ m/s, $c_L = 9275$ m/s, $c_R = 4754$ m/s