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Spherical void in Infinite Medium Loaded by Hydrostatic Stress.

Spherical Symmetry $\Rightarrow$ all field quantities are functions of r only, i.e. no θ or ϕ dependence.

Furthermore, since the stress field at a given point is invariant under any rotation of the coordinate axes it can be argued that $\sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\theta\phi} = 0$ and $\sigma_{\theta\theta} = \sigma_{\phi\phi}$ by rotating the coordinate system by $90°$ about the r-direction.

$\therefore$ Equilibrium: $\dfrac{d\sigma_{rr}}{dr} + \dfrac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$ (1)

Compatibility: $\varepsilon_{rr} = \dfrac{du_r}{dr}$, $\varepsilon_{\theta\theta} = \dfrac{u_r}{r}$

Note: $\dfrac{d}{dr}\left(\dfrac{u_r}{r}\right) = \dfrac{1}{r}\dfrac{du_r}{dr} - \dfrac{1}{r}\dfrac{u_r}{r}$

$\dfrac{d}{dr}(\varepsilon_{\theta\theta}) = \dfrac{1}{r}\varepsilon_{rr} - \dfrac{1}{r}\varepsilon_{\theta\theta}$ (2)

Hooke's Law: $\varepsilon_{rr} = \dfrac{1}{E}\sigma_{rr} - \dfrac{2\nu}{E}\sigma_{\theta\theta}$ (3)

$\varepsilon_{\theta\theta} = \dfrac{1-\nu}{E}\sigma_{\theta\theta} - \dfrac{\nu}{E}\sigma_{rr}$ (4)

4 Equations for 4 unknowns $(\sigma_{rr}, \sigma_{\theta\theta}, \varepsilon_{rr}, \varepsilon_{\theta\theta})$

(1) $\rightarrow$ $\sigma_{\theta\theta} = \frac{r}{2}\frac{d\sigma_{rr}}{dr} + \sigma_{rr}$

(3) $\rightarrow$ $\varepsilon_{rr} = \frac{1}{E}\sigma_{rr} - \frac{2\nu}{E}\sigma_{rr} - \frac{\nu}{E} r \frac{d\sigma_{rr}}{dr}$

$\varepsilon_{rr} = \frac{1-2\nu}{E}\sigma_{rr} - \frac{\nu}{E} r \frac{d\sigma_{rr}}{dr}$

(4) $\rightarrow$ $\varepsilon_{\theta\theta} = \frac{1-\nu}{E}\sigma_{rr} + \frac{1-\nu}{E}\frac{r}{2}\frac{d\sigma_{rr}}{dr} - \frac{\nu}{E}\sigma_{rr}$

$\varepsilon_{\theta\theta} = \frac{1-2\nu}{E}\sigma_{rr} + \frac{1}{2}\frac{1-\nu}{E} r \frac{d\sigma_{rr}}{dr}$

(2) $\rightarrow$ $\frac{d}{dr}\left[\frac{1-2\nu}{E}\sigma_{rr} + \frac{1}{2}\frac{1-\nu}{E} r \frac{d\sigma_{rr}}{dr}\right] = \frac{1}{\cancel{r}}\left[\frac{-\nu}{E}\cancel{r}\frac{d\sigma_{rr}}{dr} - \frac{1}{2}\frac{1-\nu}{E}\cancel{r}\frac{d\sigma_{rr}}{dr}\right]$

$\frac{2(1-2\nu)}{\cancel{2E}}\frac{d\sigma_{rr}}{dr} + \frac{1}{\cancel{2}}\frac{1-\nu}{\cancel{E}}\frac{d\sigma_{rr}}{dr} + \frac{1}{\cancel{2}}\frac{1-\nu}{\cancel{E}} r \frac{d^2\sigma_{rr}}{dr^2} = -\frac{1+\nu}{\cancel{2E}}\frac{d\sigma_{rr}}{dr}$

$$(1-\nu) r \frac{d^2\sigma_{rr}}{dr^2} + \left[2(1-2\nu) + (1-\nu) + (1+\nu)\right]\frac{d\sigma_{rr}}{dr} = 0$$

$2-4\nu + 1-\nu + 1+\nu$

$4 - 4\nu = 4(1-\nu)$

$\therefore$ $\boxed{\frac{d^2\sigma_{rr}}{dr^2} + \frac{4}{r}\frac{d\sigma_{rr}}{dr} = 0}$ Governing <u>equidimensional</u> ordinary differential equation for $\sigma_{rr}(r)$

<u>Boundary Conditions</u>: Traction-free void of radius a

$\therefore$ $\sigma_{rr}(r=a) = 0$

Hydrostatic stress as $r\to\infty$ $\therefore$ $\sigma_{rr}(r\to\infty) = +\sigma_0$

<u>Solution Procedure</u>: Equidimensional $\rightarrow$ $\sigma_{rr} = A r^p$

$\sigma_{rr}' = A p r^{p-1}$

$\sigma_{rr}'' = A p(p-1) r^{p-2}$

$\therefore \cancel{A} p(p-1)\cancel{r^{p-2}} + 4\cancel{A}p\,\cancel{r^{p-2}} = 0$

$p(p-1) + 4p = 0$ ← characteristic equation

$p^2 + 3p = 0$

$p = 0, -3$ (These are the eigenvalues of the characteristic equation)

$\therefore \sigma_{rr} = A + B/r^3 \rightarrow \sigma_{\theta\theta} = \frac{r}{2}\frac{-3B}{r^4} + A + \frac{B}{r^3}$

$\therefore \sigma_{\theta\theta} = A - \frac{1}{2} B/r^3$

$\sigma_{rr}(r \to \infty) = +\sigma_o = A \quad \therefore A = +\sigma_o$

$\sigma_{rr}(r \to a) = 0 = +\sigma_o + B/a^3 \rightarrow B = -a^3\sigma_o$

$$\therefore \boxed{\begin{aligned} \sigma_{rr} &= +\sigma_o\left[1 - \left(\frac{a}{r}\right)^3\right] \\ \sigma_{\theta\theta} &= \sigma_o\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right] \end{aligned}}$$

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2-D Elasticity

Plane Stress $\rightarrow$ $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, $\therefore \varepsilon_{xz} = \varepsilon_{yz} = 0$

$$\therefore \left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx} \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \end{aligned}\right\} \begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{\nu E}{1-\nu^2}\varepsilon_{yy} \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\varepsilon_{yy} + \frac{\nu E}{1-\nu^2}\varepsilon_{xx} \\ \sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \end{aligned}$$

Plane Strain $\rightarrow$ $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$, $\therefore \sigma_{xz} = \sigma_{yz} = 0$

$$\varepsilon_{zz} = \frac{1}{E}\sigma_{zz} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = 0, \therefore \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\nu(\sigma_{xx} + \sigma_{yy})$$

$$\therefore \begin{aligned} \varepsilon_{xx} &= \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy} \\ \varepsilon_{yy} &= \frac{1-\nu^2}{E}\sigma_{yy} - \frac{\nu(1+\nu)}{E}\sigma_{xx} \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \end{aligned}$$

$$\begin{aligned} \sigma_{xx} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{yy} \\ \sigma_{yy} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{yy} - \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{xx} \\ \sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \end{aligned}$$

If we define $E^*, \nu^* = \begin{cases} E, \nu & \rightarrow \text{Plane stress} \\ \frac{E}{1-\nu^2}, \frac{\nu}{1-\nu} & \rightarrow \text{Plane Strain} \end{cases}$

then

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E^*}\sigma_{xx} - \frac{\nu^*}{E^*}\sigma_{yy} \\ \varepsilon_{yy} &= \frac{1}{E^*}\sigma_{yy} - \frac{\nu^*}{E^*}\sigma_{xx} \\ \varepsilon_{xy} &= \frac{1+\nu^*}{E^*}\sigma_{xy} \end{aligned}\right\}$$

$$\sigma_{xx} = \frac{E^*}{1-\nu^{*2}}\,\varepsilon_{xx} + \frac{\nu^* E^*}{1-\nu^{*2}}\,\varepsilon_{yy}$$
$$\sigma_{yy} = \frac{E^*}{1-\nu^{*2}}\,\varepsilon_{yy} + \frac{\nu^* E^*}{1-\nu^{*2}}\,\varepsilon_{xx}$$
$$\sigma_{xy} = \frac{E^*}{1+\nu^*}\,\varepsilon_{xy}$$

check plane strain: $\dfrac{E^*}{1+\nu^*} = \dfrac{E}{(1-\nu^2)\left(\frac{1-\nu}{1-\nu} + \frac{\nu}{1-\nu}\right)} \stackrel{\checkmark}{=} \dfrac{E\cancel{(1-\nu)}}{\cancel{(1-\nu)}(1+\nu)\cancel{(1)}}$

Equilibrium: $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Satisfied if
$$\left.\begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} \\ \sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned}\right\} \phi \equiv \text{Airy Stress Function}$$

Compatibility: $$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\therefore \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0$$

$$\varepsilon_{xx} = \frac{1}{E^*}\sigma_{xx} - \frac{\nu^*}{E^*}\sigma_{yy} = \frac{1}{E^*}\frac{\partial^2 \phi}{\partial y^2} - \frac{\nu^*}{E^*}\frac{\partial^2 \phi}{\partial x^2}$$
$$\varepsilon_{yy} = \frac{1}{E^*}\sigma_{yy} - \frac{\nu^*}{E^*}\sigma_{xx} = \frac{1}{E^*}\frac{\partial^2 \phi}{\partial x^2} - \frac{\nu^*}{E^*}\frac{\partial^2 \phi}{\partial y^2}$$
$$\varepsilon_{xy} = \frac{1+\nu^*}{E^*}\sigma_{xy} = -\left(\frac{1+\nu^*}{E^*}\right)\frac{\partial^2 \phi}{\partial x \partial y}$$

$$\therefore \frac{1}{\cancel{E^*}}\frac{\partial^4 \phi}{\partial y^4} - \frac{\cancel{\nu^*}}{\cancel{E^*}}\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{1}{\cancel{E^*}}\frac{\partial^4 \phi}{\partial x^4} - \frac{\cancel{\nu^*}}{\cancel{E^*}}\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{2+\cancel{2\nu^*}}{\cancel{E^*}}\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$$

$$\therefore \boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial y^2 \partial x^2} + \frac{\partial^4 \phi}{\partial y^4} = 0}$$

$$\nabla^4 \phi = 0 \quad \leftarrow \text{Biharmonic Equation}$$

$$\nabla^2(\nabla^2 \phi) = 0 \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{(Cartesian)}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2} \quad \text{(Polar)}$$

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r}\frac{\partial^3 \phi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^3}\frac{\partial \phi}{\partial r} + \frac{2}{r^2}\frac{\partial^4 \phi}{\partial r^2 \partial \theta^2}$$

$$- \frac{2}{r^3}\frac{\partial^3 \phi}{\partial r \partial \theta^2} + \frac{4}{r^4}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 \phi}{\partial \theta^4} = 0$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

We will return to polar coordinates later.
Now let's look at some solutions in Cartesian coordinates.

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Solution by polynomials

$$\phi = \frac{A}{2}x^2 + {}^{-}Bxy + \frac{C}{2}y^2$$

$$\rightarrow \sigma_{xx} = C, \quad \sigma_{yy} = A, \quad \sigma_{xy} = B$$

Cantilever Beams

Take a hint from beam theory: $\sigma_{xx} = Ay$, $\sigma_{yy} = \sigma_{xy} = 0$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = Ay \rightarrow \phi = \frac{1}{6}Ay^3$$

$$M = b\int_{-h/2}^{h/2} \sigma_{xx}\, y\, dy = b\int_{-h/2}^{h/2} Ay^2\, dy$$

$$= b\,A\,\frac{1}{3}y^3\Big]_{-h/2}^{h/2}$$

$$= A\,\frac{bh^3}{12}$$

$$\therefore\ A = \frac{12}{bh^3}M$$

$$\sigma_{xx} = \frac{12}{bh^3}My \qquad \left(\sigma_{xx} = \frac{My}{I}\right)$$

$$\sigma_{yy} = \sigma_{xy} = 0$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E}\frac{12}{bh^3}My \rightarrow u = \frac{M}{EI}yx + f(y)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{-\nu}{E}\frac{12}{bh^3}My \rightarrow v = -\frac{1\nu M}{2EI}y^2 + g(x)$$

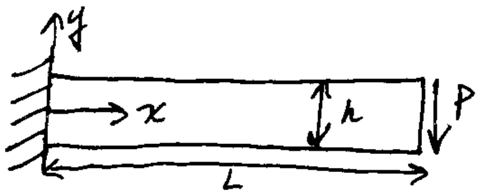
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 = \frac{1}{2}\left[\frac{M}{EI}x + f'(y) + g'(x)\right]$$

$$u(x=0, y) = 0 \rightarrow f(y) = 0,\ f'(y) = 0$$

$$\therefore\ g'(x) = -\frac{M}{EI}x \rightarrow g(x) = -\frac{1}{2}\frac{M}{EI}x^2 + c$$

$$\therefore\ v = -\frac{1}{2}\frac{M}{EI}\left(\nu y^2 + x^2\right) + \cancel{c}^{\,0} \text{ b/c } v(0,0) = 0$$

$$v(x, y=0) = -\frac{1}{2}\frac{M}{EI}x^2 \rightarrow \text{Agrees exactly w/ beam theory}$$



Beam theory $\rightarrow \sigma_{xx} = \frac{My}{I}$, $M = P(L-x)$

$$\sigma_{xx} = Ay + Bxy = \frac{\partial^2 \phi}{\partial y^2}$$

$$\therefore \phi = \frac{1}{6} A y^3 + \frac{1}{6} B x y^3 + Cxy$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0 \quad , \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{1}{2} B y^2 - C$$

BCs: $\sigma_{xy} = 0$ at $y = \pm \frac{h}{2} \rightarrow -\frac{1}{2} B \frac{h^2}{4} - C = 0$

$$\therefore C = \frac{-Bh^2}{8}$$

$$+b \int_{-\frac{h}{2}}^{h/2} \sigma_{xy}(x=L)\, dy = -P$$

$$b \int_{-h/2}^{h/2} \frac{1}{2} B \left(\frac{h^2}{4} - y^2\right) dy = -P$$

$$\frac{b}{2} B \left(\frac{h^3}{4} - \frac{h^3}{12}\right) = -P$$

$$B \frac{bh^3}{12} = -P$$

$$B = \frac{-12P}{bh^3} = \frac{-P}{I}$$

$$\sigma_{xx}(x=L, y) = 0 = Ay + \frac{-P}{I} L y \rightarrow A = \frac{PL}{I}$$

$$\therefore \sigma_{xx} = \frac{PL}{I} y - \frac{P}{I} xy = \frac{P(L-x)}{I} y$$

$$\sigma_{yy} = 0$$

$$\sigma_{xy} = \frac{1}{2} \frac{P}{I} \left(y^2 - \frac{h^2}{4}\right)$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E}\frac{P(L-x)}{I}y \rightarrow u = \frac{PL}{EI}yx - \frac{1}{2}\frac{P}{EI}yx^2 + f(y)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{-\nu}{E}\left(\frac{PL}{I} - \frac{Px}{I}\right)y \rightarrow v = -\frac{1}{2}\nu\frac{PL}{EI}y^2 - \frac{1}{2}\nu\frac{P}{EI}xy^2 + g(x)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left[\frac{PL}{EI}x - \frac{1}{2}\frac{P}{EI}x^2 + f'(y) - \frac{1}{2}\nu\frac{P}{EI}y^2 + g'(x)\right]$$
$$= \frac{1+\nu}{E}\frac{1}{2}\frac{P}{I}\left(y^2 - \frac{h^2}{4}\right)$$

$$\therefore \left.\begin{aligned} g'(x) &= \frac{1}{2}\frac{P}{EI}x^2 - \frac{PL}{EI}x + c_1 \\ f'(y) &= (1+\nu)\frac{P}{EI}y^2 + \frac{1}{2}\nu\frac{P}{EI}y^2 + c_2 \end{aligned}\right\} c_1 + c_2 = -\frac{1+\nu}{E}\frac{P}{I}\frac{h^2}{4}$$

$$\therefore g(x) = \frac{1}{6}\frac{P}{EI}x^3 - \frac{1}{2}\frac{PL}{EI}x^2 + c_1 x + \cancel{c_3}^{\,0} \text{ b/c } v(0,0)=0$$

$$v(x, y=0) = \frac{1}{6}\frac{P}{EI}x^3 - \frac{1}{2}\frac{PL}{EI}x^2 + c_1 x$$

$$\frac{\partial v}{\partial x}(x=0, y=0) = 0 \rightarrow c_1 = 0 \rightarrow c_2 = -\frac{1+\nu}{E}\frac{P}{I}\frac{h^2}{4}$$

$$\therefore v(x, y=0) = \frac{1}{6}\frac{P}{EI}x^3 - \frac{1}{2}\frac{PL}{EI}x^2$$ (Exactly the same as beam theory)

Note we could not enforce $u = v = 0$ over the entire built in section of the beam. Furthermore we could not enforce an arbitrary distribution of the applied loads.

St. Venant's Principle

The stresses and strains produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part.

Fourier Series Solutions

$$\phi = X(x)\sin ny$$

$$\nabla^4\phi = X''''\cancel{\sin ny} - 2n^2 X''\cancel{\sin ny} + n^4 X\cancel{\sin ny} = 0$$

$$X = Ae^{\alpha x}, \quad X'' = A\alpha^2 e^{\alpha x}, \quad X'''' = A\alpha^4 e^{\alpha x}$$

$$\therefore \quad \alpha^4 - 2n^2\alpha^2 + n^4 = 0$$

$$(\alpha^2 - n^2)(\alpha^2 - n^2) = 0$$

Two double roots $\alpha = \pm n$

$$\therefore \ X = A_1 e^{nx} + A_2 e^{-nx} + A_3 x e^{-nx} + A_4 x e^{nx}$$

$$Y = B_1 \sin ny + B_2 \cos ny$$

Supplement to Lecture on 9/11/01

(A)

Plane Stress is an Approximation?

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} \\ \sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned}$$

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{aligned}\right\} \quad \begin{aligned} &\varepsilon_{xx,yy} + \varepsilon_{yy,xx} - 2\varepsilon_{xy,xy} = 0 & (1) \\ &\varepsilon_{xx,zz} + \varepsilon_{zz,xx} - 2\varepsilon_{xz,xz} = 0 & (2) \\ &\varepsilon_{yy,zz} + \varepsilon_{zz,yy} - 2\varepsilon_{yz,yz} = 0 & (3) \\ &-\varepsilon_{xx,yz} - \varepsilon_{yz,xx} + \varepsilon_{xz,xy} + \varepsilon_{xy,xz} = 0 & (4) \\ &-\varepsilon_{yy,xz} - \varepsilon_{xz,yy} + \varepsilon_{yz,xy} + \varepsilon_{xy,yz} = 0 & (5) \\ &-\varepsilon_{zz,xy} - \varepsilon_{xy,zz} + \varepsilon_{xz,yz} + \varepsilon_{yz,xz} = 0 & (6) \end{aligned}$$

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} = \frac{1}{E}\frac{\partial^2 \phi}{\partial y^2} - \frac{\nu}{E}\frac{\partial^2 \phi}{\partial x^2} \\ \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx} = \frac{1}{E}\frac{\partial^2 \phi}{\partial x^2} - \frac{\nu}{E}\frac{\partial^2 \phi}{\partial y^2} \\ \varepsilon_{zz} &= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{\nu}{E}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} = -\frac{1+\nu}{E}\frac{\partial^2 \phi}{\partial x \partial y} \\ \varepsilon_{xz} &= 0 \\ \varepsilon_{yz} &= 0 \end{aligned}$$

$$(1) \rightarrow \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0 \rightarrow \nabla^4_{xy}\phi = 0$$

$$(2) \rightarrow -\frac{\nu}{E}\left(\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^2 \partial x^2}\right) + \frac{1}{E}\frac{\partial^4 \phi}{\partial y^2 \partial z^2} - \frac{\nu}{E}\frac{\partial^4 \phi}{\partial x^2 \partial z^2} = 0$$

$$(3) \rightarrow -\frac{\nu}{E}\left(\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^2}\right) + \frac{1}{E}\frac{\partial^4 \phi}{\partial x^2 \partial z^2} - \frac{\nu}{E}\frac{\partial^4 \phi}{\partial y^2 \partial z^2} = 0$$

$$(2)+(3) \rightarrow -\frac{\nu}{E}\left[\nabla^4_{xy}\phi\right] + \frac{1-\nu}{E}\frac{\partial^2}{\partial z^2}\left[\nabla^2_{xy}\phi\right] = 0$$

$\therefore$ from 1 & 2 + 3 we have $\frac{\partial^2}{\partial z^2}\left[\nabla^2_{xy}\phi\right] = 0$

$$4 \rightarrow \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left(\nabla^2_{xy}\phi\right)\right] = 0$$

$$5 \rightarrow \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\left(\nabla^2_{xy}\phi\right)\right] = 0$$

$$\therefore \quad \nabla^2_{xy}\phi = cz + \theta_0(x,y)$$

$$\text{and } 1 \rightarrow \nabla^2_{xy}\theta_0 = 0$$

$$6 \rightarrow -\nu\frac{\partial^2}{\partial x\partial y}\left(\nabla^2_{xy}\phi\right) = (1+\nu)\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi}{\partial x\partial y}\right) = (1+\nu)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\phi}{\partial z^2}\right)$$

$$-\nu\frac{\partial^2}{\partial x\partial y}\left[cz + \theta_0(x,y)\right] = (1+\nu)\frac{\partial^2}{\partial x\partial y}\frac{\partial^2\phi}{\partial z^2}$$

$$-\nu\left[cz + \theta_0(x,y)\right] = (1+\nu)\frac{\partial^2\phi}{\partial z^2} + a(z)x + b(z)y + d(z)$$

$$\therefore \quad \phi = \frac{-\nu}{1+\nu}\left[\frac{1}{3}cz^3 + \frac{1}{2}z^2\theta_0(x,y)\right] + A(z)x + B(z)y + D(z)$$
$$+ \phi_1(x,y)z + \phi_0(x,y)$$

Note that stresses are unaffected if $c = A(z) = B(z) = D(z) = 0$

Furthermore, if stresses are symmetric about the midplane, i.e. $\sigma_{ij}(z) = \sigma_{ij}(-z)$, then $\phi_1(x,y) = 0$

$$\therefore \quad \phi = \phi_0(x,y) - \frac{1}{2}\frac{\nu}{1+\nu}z^2\theta_0(x,y)$$

C

but $\nabla^2_{xy}\phi = \theta_0(x,y)$

$$\nabla^2_{xy}\phi = \nabla^2_{xy}\phi_0 - \frac{1}{2}\frac{\nu}{1+\nu} z^2 \nabla^2_{xy}\theta_0$$

but $\nabla^2_{xy}\theta_0 = 0 \rightarrow \nabla^2_{xy}\phi = \nabla^2_{xy}\phi_0$

and $\theta_0(x,y) = \nabla^2_{xy}\phi_0$

$$\therefore \boxed{\phi = \phi_0(x,y) - \frac{1}{2}\frac{\nu}{1+\nu} z^2 \nabla^2_{xy}\phi_0}$$

and $\nabla^4_{xy}\phi = 0$

$$\rightarrow \nabla^4_{xy}\phi_0 - \frac{1}{2}\frac{\nu}{1+\nu} z^2 \nabla^2_{xy}\left(\cancelto{0}{\nabla^2_{xy}\theta_0}\right) = 0$$

$$\therefore \boxed{\nabla^4_{xy}\phi_0 = 0}$$

Hence, stresses have a parabolic distribution in the z direction. But, if the thickness of the plate is very small then the contribution to the stresses is very small due to the z-varying term.

9/13/01

Back to Polar Coordinates

Solutions of the form $\phi = R(r)\cos n\theta$

$$R''''\cos n\theta + \frac{2}{r}R'''\cos n\theta - \frac{1}{r^2}R''\cos n\theta + \frac{1}{r^3}R'\cos n\theta$$
$$- \frac{1}{r^2}2n^2R''\cos n\theta + \frac{1}{r^3}2n^2R'\cos n\theta - \frac{1}{r^4}4n^2R\cos n\theta + \frac{1}{r^4}n^4R\cos n\theta = 0$$

$$R'''' + \frac{2}{r}R''' - \frac{1}{r^2}(2n^2+1)R'' + \frac{1}{r^3}(2n^2+1)R' + \frac{1}{r^4}n^2(n^2-4)R = 0$$

Equidimensional $\rightarrow$ $R = Ar^p$, $R' = Apr^{p-1}$, $R'' = Ap(p-1)r^{p-2}$

$R''' = Ap(p-1)(p-2)r^{p-3}$, $R'''' = Ap(p-1)(p-2)(p-3)r^{p-4}$

$$\therefore p(p-1)(p-2)(p-3) + 2p(p-1)(p-2) - (2n^2+1)p(p-1) + (2n^2+1)p + n^2(n^2-4) = 0$$

$$(p-n)(p+n)[p-(n+2)][p+(n-2)] = 0$$

$$\therefore p = \pm n, \quad p = n+2, \quad p = 2-n$$

Double roots when $n=0$ $\rightarrow$ $R = A\ln r$
and $R = Ar^2\ln r$
are solutions

$n = \pm 1$ $\rightarrow$ $R = Ar\ln r$
is a solution

J. Dundurs

POLAR COMPONENTS OF STRESS AND DISPLACEMENT

$\varkappa = 3 - 4\nu$ for plane strain,

$\varkappa = (3 - \nu)/(1 + \nu)$ for plane stress

Rigid body displacements:

$u_r = C_1\cos\theta + C_2\sin\theta$

$u_\theta = -C_1\sin\theta + C_2\cos\theta + C_3 r$

$U = \phi$	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$	$2Gu_r$	$2Gu_\theta$
r^2	2	0	2	$(\varkappa-1)r$	0
$\log r$	$1/r^2$	0	$-1/r^2$	$-1/r$	0
θ	0	$1/r^2$	0	0	$-1/r$
$r^2\log r$	$2\log r + 1$	0	$2\log r + 3$	$(\varkappa-1)r\log r - r$	$(\varkappa + 1)r\theta$
$r^2\theta$	2θ	-1	2θ	$(\varkappa-1)r\theta$	$-(\varkappa + 1)r\log r$
$r^3\cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$	$(\varkappa-2)r^2\cos\theta$	$(\varkappa+2)r^2\sin\theta$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$	$(\varkappa-2)r^2\sin\theta$	$-(\varkappa+2)r^2\cos\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0	$\frac{1}{2}[(\varkappa-1)\theta\sin\theta+(\varkappa+1)\log r\cos\theta-\cos\theta]$	$\frac{1}{2}[(\varkappa-1)\theta\cos\theta-(\varkappa+1)\log r\sin\theta-\sin\theta]$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0	$\frac{1}{2}[(\varkappa-1)\theta\cos\theta-(\varkappa+1)\log r\sin\theta+\sin\theta]$	$\frac{1}{2}[-(\varkappa-1)\theta\sin\theta-(\varkappa+1)\log r\cos\theta-\cos\theta]$
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$	$\frac{1}{2}[(\varkappa+1)\theta\sin\theta+(\varkappa-1)\log r\cos\theta-\cos\theta]$	$\frac{1}{2}[(\varkappa+1)\theta\cos\theta-(\varkappa-1)\log r\sin\theta-\sin\theta]$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$	$\frac{1}{2}[-(\varkappa+1)\theta\cos\theta+(\varkappa-1)\log r\sin\theta-\sin\theta]$	$\frac{1}{2}[(\varkappa+1)\theta\sin\theta+(\varkappa-1)\log r\cos\theta+\cos\theta]$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$\cos\theta/r^2$	$\sin\theta/r^2$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$	$\sin\theta/r^2$	$-\cos\theta/r^2$

$U = \Phi$	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$	$2Gu_r$	$2Gu_\theta$
$r^2\cos2\theta$	$-2\cos2\theta$	$2\sin2\theta$	$2\cos2\theta$	$-2r\cos2\theta$	$2r\sin2\theta$
$r^2\sin2\theta$	$-2\sin2\theta$	$-2\cos2\theta$	$2\sin2\theta$	$-2r\sin2\theta$	$-2r\cos2\theta$
$r^4\cos2\theta$	0	$6r^2\sin2\theta$	$12r^2\cos2\theta$	$-(3-\kappa)r^3\cos2\theta$	$(3+\kappa)r^3\sin2\theta$
$r^4\sin2\theta$	0	$-6r^2\cos2\theta$	$12r^2\sin2\theta$	$-(3-\kappa)r^3\sin2\theta$	$-(3+\kappa)r^3\cos2\theta$
$\cos2\theta/r^2$	$-6\cos2\theta/r^4$	$-6\sin2\theta/r^4$	$6\cos2\theta/r^4$	$2\cos2\theta/r^3$	$2\sin2\theta/r^3$
$\sin2\theta/r^2$	$-6\sin2\theta/r^4$	$6\cos2\theta/r^4$	$6\sin2\theta/r^4$	$2\sin2\theta/r^3$	$-2\cos2\theta/r^3$
$\cos2\theta$	$-4\cos2\theta/r^2$	$-2\sin2\theta/r^2$	0	$(\kappa+1)\cos2\theta/r$	$-(\kappa-1)\sin2\theta/r$
$\sin2\theta$	$-4\sin2\theta/r^2$	$2\cos2\theta/r^2$	0	$(\kappa+1)\sin2\theta/r$	$(\kappa-1)\cos2\theta/r$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$(n+1)nr^n\sin n\theta$	$(n+2)(n+1)r^n\cos n\theta$	$-(n+1-\kappa)r^{n+1}\cos n\theta$	$(n+1+\kappa)r^{n+1}\sin n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-(n+1)nr^n\cos n\theta$	$(n+2)(n+1)r^n\sin n\theta$	$-(n+1-\kappa)r^{n+1}\sin n\theta$	$-(n+1+\kappa)r^{n+1}\cos n\theta$
$\cos n\theta/r^n$	$-(n+1)n\cos n\theta/r^{n+2}$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$	$n\cos n\theta/r^{n+1}$	$n\sin n\theta/r^{n+1}$
$\sin n\theta/r^n$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$	$(n+1)n\sin n\theta/r^{n+2}$	$n\sin n\theta/r^{n+1}$	$-n\cos n\theta/r^{n+1}$
$\cos n\theta/r^{n-2}$	$-(n+2)(n-1)\cos n\theta/r^n$	$-n(n-1)\sin n\theta/r^n$	$(n-1)(n-2)\cos n\theta/r^n$	$(n-1+\kappa)\cos n\theta/r^{n-1}$	$(n-1-\kappa)\sin n\theta/r^{n-1}$
$\sin n\theta/r^{n-2}$	$-(n+2)(n-1)\sin n\theta/r^n$	$n(n-1)\cos n\theta/r^n$	$(n-1)(n-2)\sin n\theta/r^n$	$(n-1+\kappa)\sin n\theta/r^{n-1}$	$-(n-1-\kappa)\cos n\theta/r^{n-1}$

Radially Symmetric Solutions

No θ dependence in ϕ $\rightarrow$ $n = 0$ for $\phi = R(r)\cos n\theta$

governing equation: $R'''' + \frac{2}{r}R''' - \frac{1}{r^2}R'' + \frac{1}{r^3}R' = 0$

$$R = A r^p$$

$$p = 0, \; p = 2 \quad \text{(double roots)}$$

$$\therefore \; R(r) = \phi = \cancel{A} + B\ln r + Cr^2 + Dr^2 \ln r$$

does not affect stresses or strains

$$\sigma_{rr} = B/r^2 + 2C + 2D\ln r + D$$

$$\sigma_{\theta\theta} = -B/r^2 + 2C + 2D\ln r + 3D$$

$$\sigma_{r\theta} = 0$$ (as we would expect for radial symmetry)

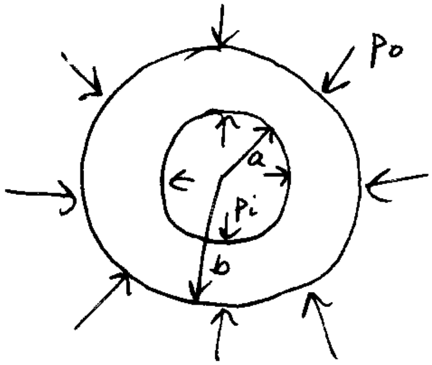
$$2\mu u_r = -B/r + C(\chi - 1)r + D(\chi - 1)r\ln r - Dr$$

$$2\mu u_\theta = D(\chi + 1)r\theta$$ $\Leftarrow$ Note D leads to θ dependence in u_θ

$$\chi = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

$D \neq 0$ for pure bending of a curved bar

Pressure Vessel (no θ dependence in u_θ)



$\sigma_{rr}(r=a) = -p_i$

$\sigma_{rr}(r=b) = -p_o$

$$\sigma_{rr}(r=a) = B/a^2 + 2C = -p_i$$

$$\sigma_{rr}(r=b) = B/b^2 + 2C = -p_o$$

$$\frac{B(b^2-a^2)}{a^2b^2} = p_o - p_i$$

$$B = \frac{(p_o-p_i)a^2b^2}{b^2-a^2}$$

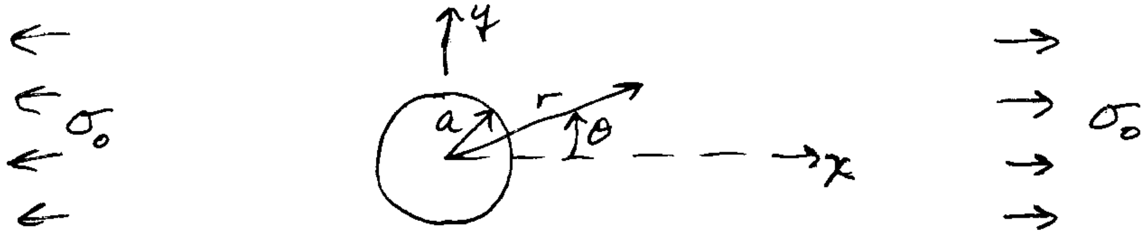
$$\therefore \frac{(p_o-p_i)b^2}{b^2-a^2} + 2C = -p_i = \frac{-p_i b^2 + p_i a^2}{b^2-a^2}$$

$$\cancel{\tfrac{1}{2}}\cancel{2}C = \frac{1}{2}\frac{p_i a^2 - p_o b^2}{b^2-a^2}$$

$$\therefore \sigma_{rr} = \frac{p_o - p_i}{1-(\frac{a}{b})^2}\left(\frac{a}{r}\right)^2 + \frac{p_i(\frac{a}{b})^2 - p_o}{1-(\frac{a}{b})^2}$$

$$\sigma_{\theta\theta} = \frac{p_i - p_o}{1-(\frac{a}{b})^2}\left(\frac{a}{r}\right)^2 + \frac{p_i(\frac{a}{b})^2 - p_o}{1-(\frac{a}{b})^2}$$

Hole in an Infinite Plate Subject to Simple Tension

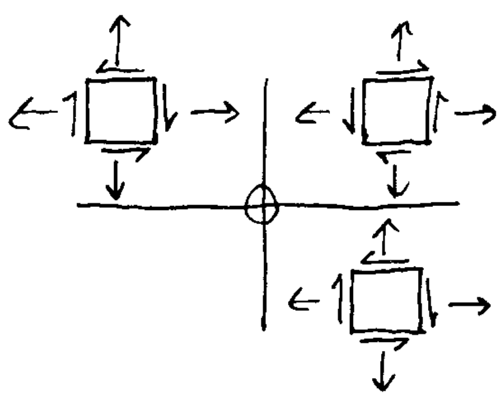


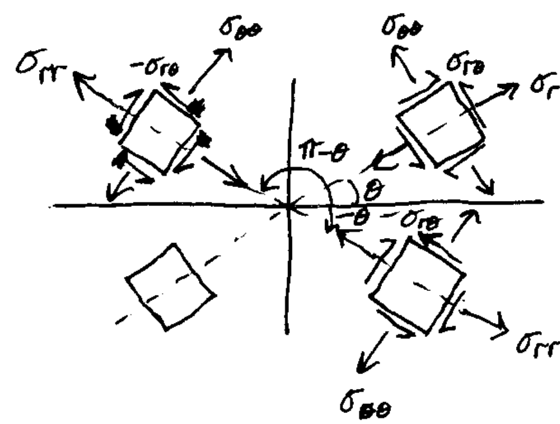
BCs: $\sigma_{rr} = \sigma_{r\theta} = 0$ at $r = a$, all θ

$\sigma_{xx} = \sigma_0$, $\sigma_{yy} = \sigma_{xy} = 0$ as $r \to \infty$

Symmetry:

$$\sigma_{xx}(x,y) = \sigma_{xx}(x,-y)$$
$$\sigma_{yy}(x,y) = \sigma_{yy}(x,-y)$$
$$\sigma_{xy}(x,y) = -\sigma_{xy}(x,-y)$$
$$\sigma_{rr}(x,y) = \sigma_{rr}(x,-y), \quad \sigma_{rr}(r,\theta) = \sigma_{rr}(r,\pi-\theta)$$
$$\sigma_{\theta\theta}(x,y) = \sigma_{\theta\theta}(x,-y), \quad \sigma_{\theta\theta}(r,\theta) = \sigma_{\theta\theta}(r,\pi-\theta)$$
$$\sigma_{r\theta}(x,y) = -\sigma_{r\theta}(x,-y), \quad \sigma_{r\theta}(r,\theta) = -\sigma_{r\theta}(r,\pi-\theta)$$





Note that for $\phi = R_{cn}(r)\cos n\theta + R_{sn}(r)\sin n\theta$

$$\sigma_{rr} = \sigma_{rr}^{cn}(r)\cos n\theta + \sigma_{rr}^{sn}(r)\sin n\theta$$
$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{cn}(r)\cos n\theta + \sigma_{\theta\theta}^{sn}(r)\sin n\theta$$
$$\sigma_{r\theta} = \sigma_{r\theta}^{cn}(r)\sin n\theta + \sigma_{r\theta}^{sn}(r)\cos n\theta$$

$$\sigma_{rr}(r,\theta) = \sigma_{rr}(r,-\theta),\ \sigma_{\theta\theta}(r,\theta) = \sigma_{\theta\theta}(r,-\theta),\ \sigma_{r\theta}(r,\theta) = -\sigma_{r\theta}(r,-\theta)$$

$\longrightarrow$ we want $\phi = R_c(r)\cos n\theta$ terms only

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -n R_{cn}''(r)\cos n\theta$$

$$\sigma_{\theta\theta}(r,\theta) = \sigma_{\theta\theta}(r,\pi-\theta) \rightarrow \cancel{-n R_{cn}''(r)}\cos n\theta = \cancel{-n R_{cn}''(r)}\cos n(\pi-\theta)$$

$$\cos n\theta = \cos n(\pi-\theta) = \cos(n\pi - n\theta)$$
$$\cos n\theta = \cos n\pi \cos n\theta + \sin n\pi \sin n\theta$$

$\therefore$ we want $n\pi = 0, 2\pi, 4\pi, 6\pi \ldots$

$\therefore\ n = 0, 2, 4, 6 \ldots$

$\therefore$ The most general solution for this type of symmetry has the form

$$\phi = \sum_{m=0}^{\infty} R_{2m}(r)\cos 2m\theta$$

1st term: $R_0(r) = B\ln r + Cr^2 + Dr^2\ln r$

$u_\theta(r,\pi) = u_\theta(r,-\pi) \rightarrow D = 0$

2nd term: $R_2(r) = Er^2 + F/r^2 + Gr^4 + H$

$$\phi = B\ln r + Cr^2 + Er^2\cos 2\theta + F\frac{1}{r^2}\cos 2\theta + Gr^4\cos 2\theta + H\cos 2\theta$$

$$\sigma_{rr} = \frac{B}{r^2} + 2C - 2E\cos 2\theta - 6F\frac{1}{r^4}\cos 2\theta - 4H\frac{1}{r^2}\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{-B}{r^2} + 2C + 2E\cos 2\theta + 6F\frac{1}{r^4}\cos 2\theta + \overset{12G}{12G}r^2\cos 2\theta$$

$$\sigma_{r\theta} = 2E\sin 2\theta - 6F\frac{1}{r^4}\sin 2\theta + 6Gr^2\sin 2\theta - 2H\frac{1}{r^2}\sin 2\theta$$

at ∞ :

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta = \tfrac{1}{2}\sigma_{xx}(1+\cos 2\theta) = \frac{\sigma_0}{2}(1+\cos 2\theta)$$
$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta = \frac{\sigma_0}{2}(1-\cos 2\theta)$$
$$\sigma_{r\theta} = -\sigma_{xx}\sin\theta\cos\theta = -\frac{\sigma_0}{2}\sin 2\theta$$

all stresses finite as $r\to\infty \rightarrow G = 0$

$\therefore$ for $r\to\infty$

$$2C - 2E\cos 2\theta = \frac{\sigma_0}{2} + \frac{\sigma_0}{2}\cos 2\theta$$
$$2C + 2E\cos 2\theta = \frac{\sigma_0}{2} - \frac{\sigma_0}{2}\cos 2\theta$$
$$2E\sin 2\theta = -\frac{\sigma_0}{2}\sin 2\theta$$

$$\therefore\quad C = \frac{\sigma_0}{4}\,,\quad E = \frac{-\sigma_0}{4}$$

at $r = a$:

$$\sigma_{rr} = \frac{B}{a^2} + \frac{\sigma_0}{2} + \left(\frac{\sigma_0}{2} - \frac{6F}{a^4} - \frac{4H}{a^2}\right)\cos 2\theta = 0$$

$$\sigma_{r\theta} = \left(-\frac{\sigma_0}{2} - \frac{6F}{a^4} - \frac{2H}{a^2}\right)\sin 2\theta = 0$$

Must be true for all $\theta \rightarrow B = \dfrac{-\sigma_0 a^2}{2}$

and $\dfrac{\sigma_0}{2} - \dfrac{6F}{a^4} - \dfrac{4H}{a^2} = 0\,,\quad \dfrac{-\sigma_0}{2} - \dfrac{6F}{a^4} - \dfrac{2H}{a^2} = 0$

$$\therefore \quad \sigma_o - \frac{2H}{a^2} = 0 \rightarrow H = \frac{\sigma_o a^2}{2}$$

$$\frac{\sigma_o}{2} - \frac{6F}{a^4} - \frac{4\sigma_o}{2} = 0 \rightarrow F = \frac{-\sigma_o a^4}{4}$$

Then: $$\sigma_{rr} = \frac{\sigma_o}{2}\left[1-\left(\frac{a}{r}\right)^2\right] + \frac{\sigma_o}{2}\left[1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_o}{2}\left[1+\left(\frac{a}{r}\right)^2\right] + \frac{\sigma_o}{2}\left[-1-3\left(\frac{a}{r}\right)^4\right]\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma_o}{2}\left[-1-2\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$

Stress concentrations

$$\sigma_{\theta\theta}\left(r=a,\ \theta=\pm\frac{\pi}{2}\right) = \sigma_o - 2\sigma_o\cos\pi = 3\sigma_o$$

$$\sigma_{\theta\theta}\left(r=a,\ \theta=0,\pi\right) = \sigma_o - 2\sigma_o\cos 2\pi = -\sigma_o$$

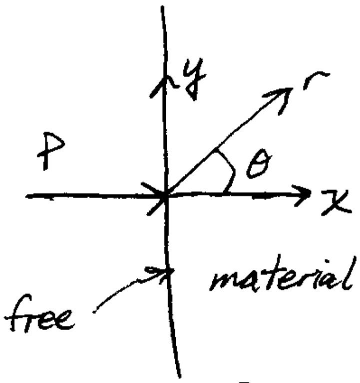
Let's say a glass material has a tensile strength of σ_c.

Then a glass plate with a hole in it will break under tension when $\sigma_o = \sigma_c/3$ and under compression when the magnitude of the applied compressive stress reaches σ_c.

9/18/01

Point Loads

Point Load on a half space



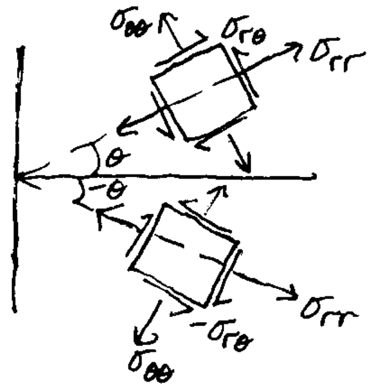
Dimensional Analysis : $P = \frac{F}{L}$

$$\sigma_{ij} = \frac{F}{L^2} \rightarrow \sigma_{ij} \sim \frac{P}{r}$$

Look for ϕ that give stresses like $\frac{1}{r}$

<u>Symmetry</u>

$$\begin{cases} \sigma_{rr}(r,\theta) = \sigma_{rr}(r,-\theta) \\ \sigma_{\theta\theta}(r,\theta) = \sigma_{\theta\theta}(r,-\theta) \\ \sigma_{r\theta}(r,\theta) = -\sigma_{r\theta}(r,-\theta) \end{cases}$$



→ Look for even functions for $\sigma_{rr}, \sigma_{\theta\theta}$, odd for $\sigma_{r\theta}$

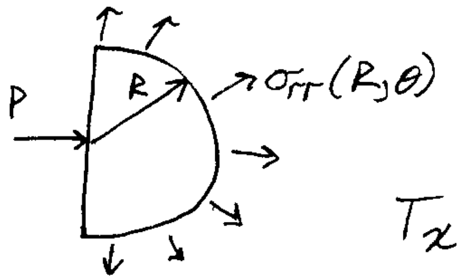
BC's :
$$\sigma_{\theta\theta}(r \neq 0, \theta = \pm\tfrac{\pi}{2}) = 0$$
$$\sigma_{r\theta}(r \neq 0, \theta = \pm\tfrac{\pi}{2}) = 0$$
$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \quad \text{as } r \rightarrow \infty$$

$$\phi = A r\theta \sin\theta + B r \ln r \cos\theta$$

$$\sigma_{rr} = 2A\frac{1}{r}\cos\theta + B\frac{1}{r}\cos\theta$$
$$\sigma_{\theta\theta} = B\frac{1}{r}\cos\theta$$
$$\sigma_{r\theta} = B\frac{1}{r}\sin\theta$$

$$\sigma_{r\theta}\left(r \neq 0, \theta = \pm\frac{\pi}{2}\right) = \pm\frac{B}{r} = 0 \rightarrow B = 0$$

$$\therefore \quad \sigma_{rr} = 2A\frac{1}{r}\cos\theta \ , \ \sigma_{\theta\theta} = \sigma_{r\theta} = 0$$



$$P + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} T_x \, R \, d\theta = 0$$

$$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y$$

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta \qquad n_x = \cos\theta$$

$$\sigma_{xy} = \sigma_{rr}\sin\theta\cos\theta \qquad n_y = \sin\theta$$

$$\therefore \quad T_x = \left(\sigma_{rr}\cos^2\theta + \sigma_{rr}\sin^2\theta\right)\cos\theta = \sigma_{rr}\cos\theta$$

$$\therefore \quad P + \int_{-\pi/2}^{\pi/2} 2A \frac{1}{\cancel{R}}\cos^2\theta \, \cancel{R} \, d\theta = 0$$

$$P + 2A\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\, d\theta = 0$$

$$P + A\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\pi/2} = 0$$

$$P + A\pi = 0 \rightarrow A = \frac{-P}{\pi}$$

$$\therefore \quad \sigma_{rr} = -\frac{2P}{\pi}\frac{1}{r}\cos\theta \ , \ \sigma_{r\theta} = \sigma_{\theta\theta} = 0$$

$$u_r = \frac{-P}{4\pi\mu}\left[(\kappa-1)\theta\sin\theta + (\kappa+1)\ln r\cos\theta - \cos\theta\right] + C_1\cos\theta$$

$$u_\theta = \frac{-P}{4\pi\mu}\left[(\kappa-1)\theta\cos\theta - (\kappa+1)\ln r\sin\theta - \sin\theta\right] - C_1\sin\theta$$

$$u_\theta(r, \theta = \pm\pi/2) = \frac{\pm P}{4\pi\mu}\left[(\chi+1)\ln r + 1\right] \pm C_1$$

Let's pick $u_\theta = 0$ at some distance d from the origin.

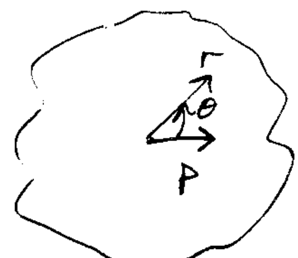
Then $\pm\frac{P}{4\pi\mu}[(\chi+1)\ln d + 1] \pm C_1 = 0$

$$C_1 = [-(\chi+1)\ln d - 1]\frac{P}{4\pi\mu}$$

$$\therefore\ u_\theta = \frac{-P}{4\pi\mu}\left[(\chi-1)\theta\cos\theta + (\chi+1)\ln\left(\frac{d}{r}\right)\sin\theta\right]$$

$$u_r = \frac{-P}{4\pi\mu}\left[(\chi-1)\theta\sin\theta - (\chi+1)\ln\left(\frac{d}{r}\right)\cos\theta\right]$$

<u>Point Load In Infinite Space</u>



Can we use the same solution as the half space but with $\sigma_{rr} = \frac{-P}{\pi}\frac{1}{r}\cos\theta$?

Only BCs are that $\sigma_{ij} \to 0$ as $r \to \infty$

and $P + \int_{-\pi}^{\pi} T_x\, R\, d\theta = 0 \rightarrow A = \frac{-P}{2\pi}$

However we must also have $u_r(r,\pi) = u_r(r,-\pi)$

$$u_\theta(r,\pi) = u_\theta(r,-\pi)$$

<u>Check</u>

$u_r(r,\pm\pi) = \frac{-P}{2\pi\mu}\left[(\chi+1)\ln\left(\frac{d}{r}\right)\right]$ ✓ continuous

$u_\theta(r,\pm\pi) = \frac{-P}{2\pi\mu}\left[(\chi-1)(\mp\pi)\right]$ ✗ not continuous

Return to general solution : $\phi = A r\theta \sin\theta + B r \ln r \cos\theta$

$$\sigma_{rr} = (2A+B)\frac{1}{r}\cos\theta$$
$$\sigma_{\theta\theta} = B\frac{1}{r}\cos\theta$$
$$\sigma_{r\theta} = B\frac{1}{r}\sin\theta$$

Again $\quad T_x = \sigma_{xx} n_x + \sigma_{xy} n_y \;,\quad n_x = \cos\theta \;,\; n_y = \sin\theta$

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - 2\sigma_{r\theta}\sin\theta\cos\theta$$
$$\sigma_{xy} = \sigma_{rr}\sin\theta\cos\theta - \sigma_{\theta\theta}\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta)$$

$$T_x = \sigma_{rr}(\cos^3\theta + \sin^2\theta\cos\theta) + \cancel{\sigma_{\theta\theta}(\sin^2\theta\cos\theta - \sin^2\theta\cos\theta)} + \sigma_{r\theta}(\cancel{\cos^2\theta\sin\theta} - \sin^3\theta - \cancel{2}\sin\theta\cos^2\theta)$$

$$T_x = \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta$$

$$\therefore \; P + \int_{-\pi}^{\pi}\left[(2A+B)\cancel{\tfrac{1}{r}}\cos^2\theta - B\cancel{\tfrac{1}{r}}\sin^2\theta\right]\cancel{r}\,d\theta = 0$$

$$P + (2A + \cancel{B})\pi - \cancel{B\pi} = 0 \rightarrow A = \frac{-P}{2\pi}$$

Displacement Continuity at $\theta = \pm\pi$

$$2\mu u_r = \frac{A}{2}\left[(\chi-1)\theta\sin\theta + (\chi+1)\ln r\cos\theta - \cos\theta\right] + \frac{B}{2}\left[(\chi+1)\theta\sin\theta + (\chi-1)\ln r\cos\theta - \cos\theta\right] + RBD$$

$$2\mu u_\theta = \frac{A}{2}\left[(\chi-1)\theta\cos\theta - (\chi+1)\ln r\sin\theta - \sin\theta\right] + \frac{B}{2}\left[(\chi+1)\theta\cos\theta - (\chi-1)\ln r\sin\theta - \sin\theta\right] + RBD$$

$$2\mu u_r(r,\pm\pi) = \frac{A}{2}\left[-(\chi+1)\ln r+1\right] + \frac{B}{2}\left[-(\chi-1)\ln r+1\right] + RBD$$

$$2\mu u_\theta(r,\pm\pi) = \mp\frac{A}{2}(\chi-1)\pi \mp \frac{B}{2}(\chi+1)\pi + RBD$$

$$\therefore\ A(\chi-1) + B(\chi+1) = 0$$

$$B = -A\,\frac{\chi-1}{\chi+1} = \frac{P}{2\pi}\,\frac{\chi-1}{\chi+1}$$

$$\therefore\ \sigma_{rr} = -\frac{P}{2\pi}\,\frac{\chi+3}{\chi+1}\,\frac{1}{r}\cos\theta$$

$$\sigma_{\theta\theta} = \frac{P}{2\pi}\,\frac{\chi-1}{\chi+1}\,\frac{1}{r}\cos\theta$$

$$\sigma_{r\theta} = \frac{P}{2\pi}\,\frac{\chi-1}{\chi+1}\,\frac{1}{r}\sin\theta$$

Cartesian Components

Point Load on Half-Space (2-D)

$$\sigma_{xx} = -\frac{2P}{\pi}\,\frac{1}{r}\cos^3\theta\ , \quad \sigma_{xy} = \frac{-2P}{\pi}\,\frac{1}{r}\sin\theta\cos^2\theta$$

$$\sigma_{yy} = \frac{-2P}{\pi}\,\frac{1}{r}\sin^2\theta\cos\theta$$

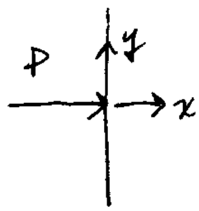
Point Load in Infinit Space (2-D)

$$\sigma_{xx} = \frac{-P}{2\pi}\,\frac{1}{r}\cos\theta\,\frac{\left[(\chi+3)\cos^2\theta + (\chi-1)\sin^2\theta\right]}{\chi+1}$$

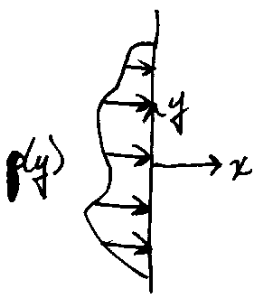
$$\sigma_{yy} = \frac{P}{2\pi}\,\frac{1}{r}\cos\theta\,\frac{\left[(\chi-1)\cos^2\theta + (\chi-5)\sin^2\theta\right]}{\chi+1}$$

$$\sigma_{xy} = \frac{P}{2\pi}\,\frac{1}{r}\sin\theta\,\frac{\left[-(\chi-1)\sin^2\theta - (\chi+3)\cos^2\theta\right]}{\chi+1}$$

Using Point Forces to Determine Stress fields for distributed loads



$$\left.\begin{aligned}\sigma_{xx} &= -\frac{2P}{\pi}\,\frac{x^3}{r^4}\\ \sigma_{yy} &= -\frac{2P}{\pi}\,\frac{xy^2}{r^4}\\ \sigma_{xy} &= -\frac{2P}{\pi}\,\frac{x^2y}{r^4}\end{aligned}\right\}\ r^4 = (x^2+y^2)^2$$



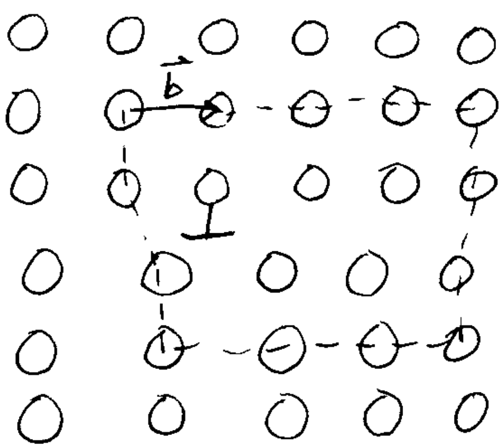
$$\sigma_{xx} = -\frac{2}{\pi}\int \frac{x^3}{[x^2+(y-y')^2]^2}\,p(y')\,dy'$$

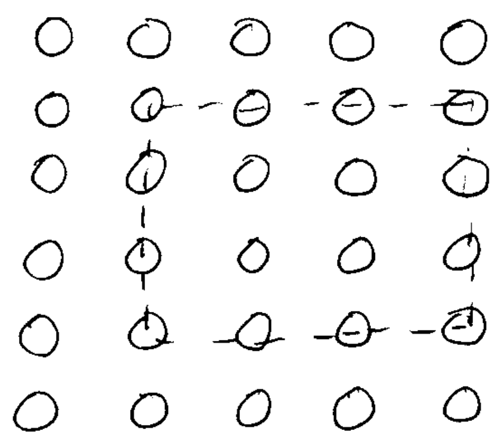
$$\sigma_{yy} = -\frac{2}{\pi}\int \frac{x(y-y')^2}{[x^2+(y-y')^2]^2}\,p(y')\,dy'$$

$$\sigma_{xy} = -\frac{2}{\pi}\int \frac{x^2(y-y')}{[x^2+(y-y')^2]^2}\,p(y')\,dy'$$

Superposition of many small point loads.

9/20/01 **Dislocation (Edge)**





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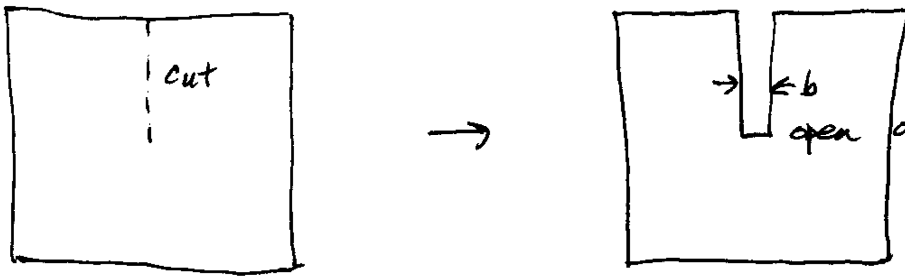
2 ways to create an elastic dislocation



$u_\theta(\frac{\pi}{2}^-) - u_\theta(\frac{\pi}{2}^+) = b$

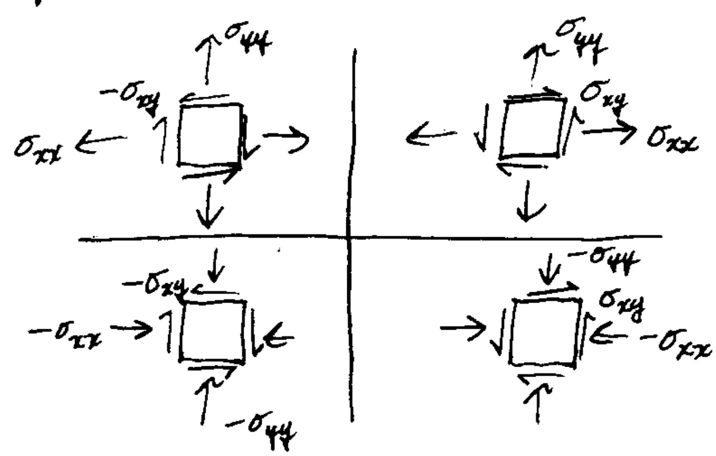
or

$u_\theta(-\frac{3\pi}{2}) - u_\theta(\frac{\pi}{2}) = b$

$u_r(-\pi) - u_r(\pi) = b$

Stresses in either case are the same but displacement fields differ.

Symmetry:



We want σ_{rr} with odd functions of θ and a displacement jump at $\theta = \theta_0$ and $\theta = \theta_0 + 2\pi$.

$$\phi = A r \ln r \sin\theta + B r \theta \cos\theta$$

$$\therefore\ 2\mu u_r = \frac{A}{2}\left[-(\chi+1)\theta\cos\theta + (\chi-1)\ln r\sin\theta - \sin\theta\right] + \frac{B}{2}\left[(\chi-1)\theta\cos\theta - (\chi+1)\ln r\sin\theta + \sin\theta\right] + RBD$$

$$2\mu u_\theta = \frac{A}{2}\left[(\chi+1)\theta\sin\theta + (\chi-1)\ln r\cos\theta + \cos\theta\right] + \frac{B}{2}\left[-(\chi-1)\theta\sin\theta - (\chi+1)\ln r\cos\theta - \cos\theta\right] + RBD$$

$$2\mu u_r(-\pi) - 2\mu u_r(\pi) = \frac{A}{2}\left[-2\pi(\chi+1)\right] + \frac{B}{2}\left[2\pi(\chi-1)\right]$$

$$2\mu u_\theta\left(-\tfrac{3\pi}{2}\right) - 2\mu u_\theta\left(\tfrac{\pi}{2}\right) = \frac{A}{2}\left[-2\pi(\chi+1)\right] + \frac{B}{2}\left[2\pi(\chi-1)\right]$$

$$\therefore\ \pi\chi(B-A) - \pi(A+B) = 2\mu b$$

$$\sigma_{r\theta}(\pi) = +\frac{A}{r}, \quad \sigma_{\theta\theta}\left(\tfrac{\pi}{2}\right) = \frac{A}{r}$$

No net force within any arc $\rightarrow$

$$\int_{-\pi}^{\pi} (A+2B)\cos^2\theta - A\sin^2\theta \, d\theta = 0$$

$$\therefore\ 2\pi B = 0 \rightarrow B = 0$$

$$\therefore\ \therefore\ -\pi A(\chi+1) = b^{*}2\mu \rightarrow A = \frac{-b\,2\mu}{\pi(\chi+1)}$$

$$\therefore\ \sigma_{rr} = \frac{-b\mu}{2\pi(1-\nu)}\frac{\sin\theta}{r}, \quad \sigma_{xx} = -\frac{\mu b}{2\pi(1-\nu)}\frac{y(3x^2+y^2)}{(x^2+y^2)^2}$$

$$\sigma_{\theta\theta} = \frac{-b\mu}{2\pi(1-\nu)}\frac{\sin\theta}{r}, \quad \sigma_{yy} = \frac{\mu b}{2\pi(1-\nu)}\frac{y(x^2-y^2)}{(x^2+y^2)^2}$$

$$\sigma_{r\theta} = \frac{b\mu}{2\pi(1-\nu)}\frac{\cos\theta}{r}, \quad \sigma_{xy} = \frac{\mu b}{2\pi(1-\nu)}\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

Energy of the Dislocation

$$u = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$$

$$\sigma_{rr} = \sigma_{\theta\theta} = \frac{-\mu b}{2\pi(1-\nu)}\frac{\sin\theta}{r} \qquad \rightarrow \varepsilon_{rr} = \varepsilon_{\theta\theta} = \frac{(1+\nu)(1-2\nu)}{E}\sigma_{rr}$$

$$\sigma_{zz} = \nu(\sigma_{rr}+\sigma_{\theta\theta}) \qquad \varepsilon_{rr} = \varepsilon_{\theta\theta} = \frac{1-2\nu}{2\mu}\sigma_{rr}$$

$$\therefore u = \frac{1}{2}\sigma_{rr}\varepsilon_{rr} + \frac{1}{2}\sigma_{\theta\theta}\varepsilon_{\theta\theta} + \frac{1}{2}\sigma_{r\theta}\varepsilon_{r\theta} + \frac{1}{2}\sigma_{\theta r}\varepsilon_{\theta r} = \sigma_{rr}\varepsilon_{rr} + \sigma_{r\theta}\varepsilon_{r\theta}$$

$$\therefore u = \frac{\mu b^2(1-2\nu)}{8\pi^2(1-\nu)^2}\frac{\sin^2\theta}{r^2} + \frac{b^2\mu}{8\pi^2(1-\nu)^2}\frac{\cos^2\theta}{r^2}$$

$$U = \int_0^\infty \int_{-\pi}^{\pi} \frac{\mu b^2}{8\pi^2(1-\nu)^2}\left(\frac{1}{r^2} - \frac{2\nu\sin^2\theta}{r^2}\right)\cancel{r}\,dr\,d\theta$$

$$= \frac{\mu b^2}{8\pi^2(1-\nu)^2}\left[2\pi \ln r - 2\nu\pi \ln r\right]_0^\infty$$

$$U = \frac{\mu b^2}{4\pi(1-\nu)}\ln\left(\frac{R}{r_0}\right)$$

Another Approach

$$U = \frac{-1}{2}\int_0^\infty \sigma_{\theta\theta}\left(\theta=\frac{\pi}{2}\right)b\,dr = -\frac{1}{2}\int_{0}^{\infty}\sigma_{r\theta}(\theta=\pi)\,b\,dr$$

$$= \frac{\mu b^2}{4\pi(1-\nu)}\int_0^\infty \frac{1}{r}\,dr = \frac{\mu b^2}{4\pi(1-\nu)}\int_0^\infty \frac{1}{r}\,dr$$

$$U = \frac{\mu b^2}{4\pi(1-\nu)}\ln\left(\frac{R}{r_0}\right) = \frac{\mu b^2}{4\pi(1-\nu)}\ln\left(\frac{R}{r_0}\right)$$

Force on a dislocation

$$F_i = -\frac{\partial W}{\partial x_i} \quad \text{where} \quad W = \text{strain energy} + \text{potential energy of loads}$$

W is an energy per length & F is a force per length. For straight dislocations

$$F_i = \epsilon_{ijk}\, b_p\, \sigma_{pj}\, s_k$$

$\vec{b}$ is the Burgers vector
$\vec{s}$ is a vector in the direction of the dislocation
$\underset{\approx}{\sigma}$ is the stress that would exist at the core if the $\perp$ were not present

For the edge dislocation $b_p = b\,\delta_{p1}$, $s_k = \delta_{k3}$

$$F_i = b\,\epsilon_{ijk}\,\delta_{p1}\,\sigma_{pj}\,\delta_{k3} = b\,\epsilon_{ij3}\,\sigma_{1j}$$

$\therefore$

$F_1 = b\,\sigma_{12}$ $\leftarrow$ glide force
$F_2 = -b\,\sigma_{11}$ $\leftarrow$ climb force
$F_3 = 0$