8/28/01

MECH 513

Theory of Elasticity

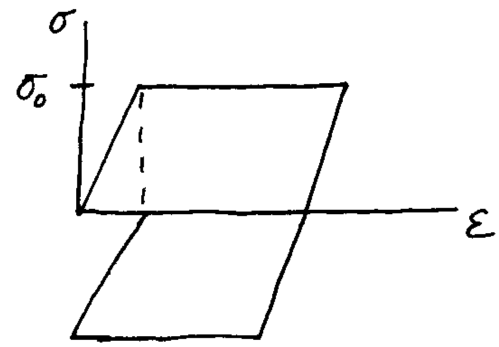
Three components of every solid mechanics problem

1) Equilibrium / Newton's Laws – govern distribution of stresses
2) Compatibility – Geometric relationships relating displacements to strains
3) Constitutive Relationships / Material Law
   Must obey 1st & 2nd Laws of Thermodynamics
   In general these laws provide the relationship between the stress history at a point and the strain history at the point. (local)

Aside: Non-local constitutive laws provide the stress history at a point given the strain history at points in the vicinity of the point.

For elastic materials the stress at a point is only dependent on the current level of strain at that point. In other words the stress is not affected by the history of strain in the material.

Ex)



Elastic-perfectly plastic
versus
Non-linear elastic

- A Green Elastic Material has stress components derived from a strain energy function

- We are concerned with linear elasticity and small strains

$$\therefore \quad \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$$W = \frac{1}{2} C_{ijkl} \, \varepsilon_{ij} \, \varepsilon_{kl}$$

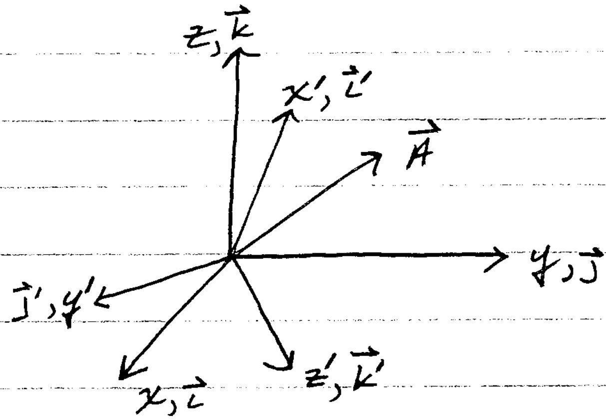
$$\therefore \quad \sigma_{ij} = C_{ijkl} \, \varepsilon_{kl}$$

Cartesian Tensor Transformations

A tensor is a quantity that transforms according to specific rules.

In general the components of a tensor depend on the coordinate system to which they are related.

Ex) A vector is a 1st rank tensor.



$$\vec{A} = (\vec{A}\cdot\vec{i})\vec{i} + (\vec{A}\cdot\vec{j})\vec{j} + (\vec{A}\cdot\vec{k})\vec{k}$$
$$= \sum_{i=1}^{3} A_i \vec{e}_i = A_i \vec{e}_i$$

$$\vec{A} = (\vec{A}\cdot\vec{i}')\vec{i}' + (\vec{A}\cdot\vec{j}')\vec{j}' + (\vec{A}\cdot\vec{k}')\vec{k}'$$
$$= A_i' \vec{e}_i'$$

$$\vec{i}' = (\vec{i}'\cdot\vec{i})\vec{i} + (\vec{i}'\cdot\vec{j})\vec{j} + (\vec{i}'\cdot\vec{k})\vec{k}$$
$$= \underbrace{\cos\theta_{x'x}}_{a_{11}}\vec{i} + \underbrace{\cos\theta_{x'y}}_{a_{12}}\vec{j} + \underbrace{\cos\theta_{x'z}}_{a_{13}}\vec{k}$$

Similar relationships hold for $\vec{j}'$, $\vec{k}'$

$\therefore\ \vec{e}_i' = a_{ij}\vec{e}_j$ where a_{ij} are the direction cosines

$$A_i\vec{e}_i = A_i'\vec{e}_i'$$
$$= A_i' a_{ij}\vec{e}_j$$
$$= A_j' a_{ji}\vec{e}_i$$

$\vec{e}_i = a_{ji}\vec{e}_j'$

$$\therefore\ A_i = a_{ji} A_j'$$

$$(a_{ji})^{-1} = a_{ij} \rightarrow A_i' = a_{ij} A_j$$

Second rank (order) tensor is represented by

$$\underset{\approx}{\sigma} = \sigma_{ij}\,\vec{e}_i \otimes \vec{e}_j = \sigma'_{ij}\,\vec{e}'_i \otimes \vec{e}'_j$$

$$\sigma_{ij}\,a_{ki}\,\vec{e}'_k \otimes a_{lj}\,\vec{e}'_l = \sigma'_{ij}\,\delta_{ik}\,\vec{e}'_k \otimes \delta_{jl}\,\vec{e}'_l$$

$$\therefore \quad \sigma'_{kl} = a_{ki}\,a_{lj}\,\sigma_{ij}$$

General n^{th} rank tensor

$$b_{ijk\ldots n} = a_{ii'}\,a_{jj'}\,a_{kk'}\ldots a_{nn'}\,b_{i'j'k'\ldots n'}$$

Indicial notation: $\sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

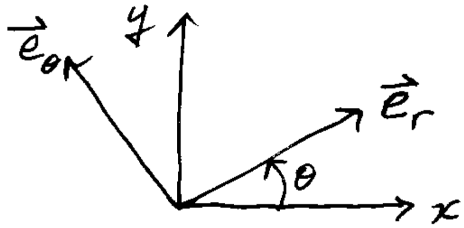
$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231,\ 312 \\ -1 & \text{if } ijk = 321,\ 132,\ 213 \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}$$

$$\sigma_{ij}\,\delta_{ij} = \sigma_{ii}$$

$$\sigma_{ij}\,\delta_{ik} = \sigma_{kj}$$

Ex) Cartesian to Polar 2nd rank tensor transformation



$$a_{xr} = \cos\theta$$
$$a_{yr} = \sin\theta$$
$$a_{x\theta} = -\sin\theta$$
$$a_{y\theta} = \cos\theta$$

$$\sigma_{rr} = a_{xr}\, a_{xr}\, \sigma_{xx} + a_{yr}\, a_{yr}\, \sigma_{yy} + a_{xr}\, a_{yr}\, \sigma_{xy} + a_{yr}\, a_{xr}\, \sigma_{yx}$$

$$\sigma_{rr} = \cos^2\theta\, \sigma_{xx} + \sin^2\theta\, \sigma_{yy} + \sin\theta\cos\theta\, (\sigma_{xy} + \sigma_{yx})$$

$$\sigma_{\theta\theta} = a_{x\theta}\, a_{x\theta}\, \sigma_{xx} + a_{y\theta}\, a_{y\theta}\, \sigma_{yy} + a_{x\theta}\, a_{y\theta}\, \sigma_{xy} + a_{y\theta}\, a_{x\theta}\, \sigma_{yx}$$

$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - (\sigma_{xy} + \sigma_{yx})\sin\theta\cos\theta$$

$$\sigma_{r\theta} = a_{xr}\, a_{x\theta}\, \sigma_{xx} + a_{yr}\, a_{y\theta}\, \sigma_{yy} + a_{xr}\, a_{y\theta}\, \sigma_{xy} + a_{yr}\, a_{x\theta}\, \sigma_{yx}$$

$$\sigma_{r\theta} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \cos^2\theta\, \sigma_{xy} - \sin^2\theta\, \sigma_{yx}$$

$$\sigma_{\theta r} = a_{x\theta}\, a_{xr}\, \sigma_{xx} + a_{y\theta}\, a_{yr}\, \sigma_{yy} + a_{x\theta}\, a_{yr}\, \sigma_{xy} + a_{y\theta}\, a_{xr}\, \sigma_{yx}$$

$$\sigma_{\theta r} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{yx}\cos^2\theta - \sigma_{xy}\sin^2\theta$$

(A)

Index Notation

D_i — 3 components of a vector $\vec{D}$
$i = 1-3$

σ_{ij} — 9 components of a second rank tensor $\underset{\sim}{\sigma}$, $i = 1,3$ $j = 1,3$

d_{kij} — 27 components of a 3rd rank tensor $\underset{\approx}{d}$, $k = 1,3$ $i = 1,3$ $j = 1,3$

etc
⋮

Einstein convention — summation is implied over repeated indices

Ex :
$$S_{ij} S_{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij}$$
$$\sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j$$

Def : <u>Dummy index</u> is an index (i, j, k, etc) that is repeated and therefore summed over.

<u>Free index</u> is not summed over.

Ex: $T_i = \sigma_{ij} n_j$ represents 3 separate equations. One equation for each of the 3 components of the vector $\vec{T}$.

i.e.
$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$T_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$T_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

Ex: $\varepsilon_{ij} = s_{ijkl}\sigma_{kl} + \varepsilon^p_{ij}$

i & j are free indices
k & l are dummy indices $\rightarrow \varepsilon_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} s_{ijkl}\sigma_{kl} + \varepsilon^p_{ij}$

Note that all terms in the equation must have the same free indices, but dummy indices can be changed with no effect.

i.e. $s_{ijkl}\sigma_{kl} = s_{ijmn}\sigma_{mn}$

This notation is used because it allows us to write many equations with many terms in a very compact way.

Aside

Another convention: $(\ )_{,i} \equiv \dfrac{\partial(\ )}{\partial x_i}$ partial diff.

Ex: $D_{i,i} = \dfrac{\partial D_1}{\partial x_1} + \dfrac{\partial D_2}{\partial x_2} + \dfrac{\partial D_3}{\partial x_3} \rightarrow$ summation & partial. diff

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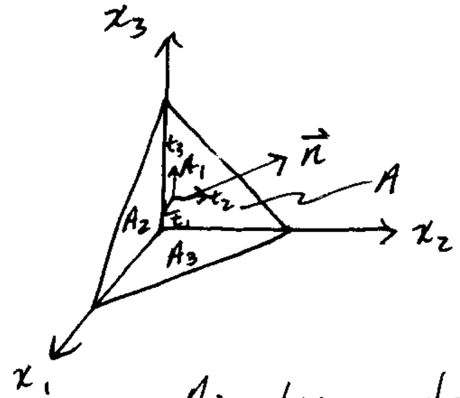
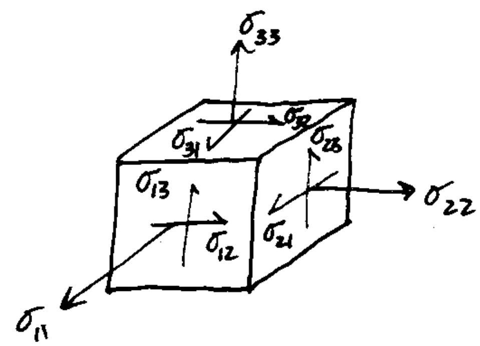
Stress Traction Relationship



A_i = area of the plane normal to the direction x_i.

A_i has stress σ_{ij}



$$\left.\begin{aligned}\sigma_{12} &= \sigma_{21}\\ \sigma_{13} &= \sigma_{31}\\ \sigma_{23} &= \sigma_{32}\end{aligned}\right\}\text{Moment Equilibrium}$$

$$A_i = A n_i$$

Equilibrium (static)

$$x_1:\quad t_1 A \cancel{+} - \sigma_{11} A_1 - \underset{(21)}{\sigma_{21}} A_2 - \underset{(31)}{\sigma_{31}} A_3 = 0$$

$$x_2:\quad t_2 A - \sigma_{12} A_1 - \sigma_{22} A_2 - \sigma_{32} A_3 = 0$$

$$x_3:\quad t_3 A - \sigma_{13} A_1 - \sigma_{23} A_2 - \sigma_{33} A_3 = 0$$

$$x_1:\quad t_1 A - \sigma_{11} n_1 A - \sigma_{21} n_2 A - \sigma_{31} n_3 A = 0$$

$$\therefore\ t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

Similarly: $t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$
$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

or $\quad t_i = \sigma_{ji} n_j$

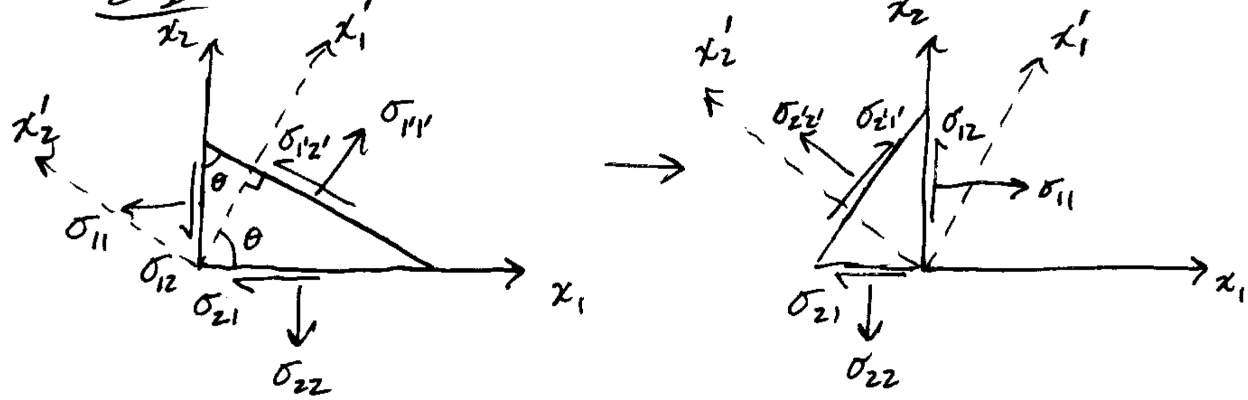
and due to symmetry of σ_{ij}, i.e. $\sigma_{ij} = \sigma_{ji}$

this can also be written as $t_i = \sigma_{ij} n_j$

Normal & shear tractions $\rightarrow$ $T_n = t_i n_i = \sigma_{ij} n_j n_i$, $T_s = t_i s_i = \sigma_{ij} n_j s_i$

Stress components in different coordinate systems

2-D



$$\Sigma F_{1'}: \ \sigma_{1'1'} - \sigma_{22} \sin\theta \sin\theta - \sigma_{11} \cos\theta \cos\theta - \sigma_{21} \sin\theta\cos\theta - \sigma_{12} \sin\theta\cos\theta = 0$$

$$\therefore \ \sigma_{1'1'} = \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + 2\sigma_{12} \sin\theta\cos\theta$$

$$\Sigma F_{2'}: \ \sigma_{1'2'} + \sigma_{11} \sin\theta\cos\theta + \sigma_{21} \sin\theta\sin\theta - \sigma_{12}\cos\theta\cos\theta - \sigma_{22} \sin\theta\cos\theta = 0$$

$$\therefore \ \sigma_{1'2'} = \sigma_{12}(\cos^2\theta - \sin^2\theta) + (\sigma_{22} - \sigma_{11}) \sin\theta\cos\theta$$

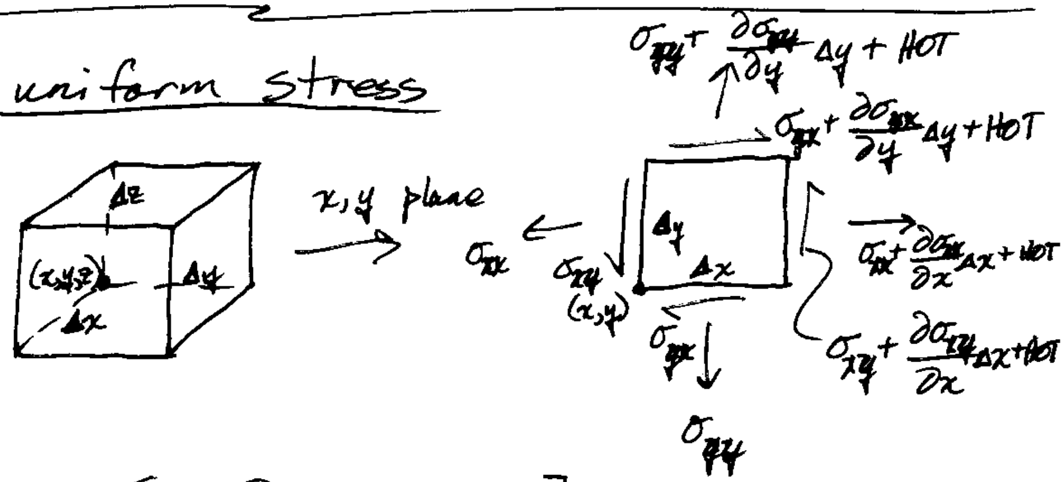
Similarly we can show that

$$\sigma_{2'2'} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\sin\theta\cos\theta$$

$$\sigma_{2'1'} = \sigma_{12}(\cos^2\theta - \sin^2\theta) + (\sigma_{22}-\sigma_{11})\sin\theta\cos\theta$$

Recall from last class how a 2nd order tensor transforms under a coordinate transformation. At least for 2-D we showed that it is exactly the way that stress transforms. This can also be shown in 3-D. Therefore stress is a second rank order tensor.

Non-uniform stress



$$\Sigma F_x = ma_x : \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x + O(\Delta l^2)\right]\Delta y \Delta z - \sigma_{xx}\Delta y\Delta z$$

$$+ \left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\Delta y + O(\Delta l)^2\right]\Delta x\Delta z - \sigma_{yx}\Delta x\Delta z$$

$$+ \left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\Delta z + O(\Delta l)^2\right]\Delta x\Delta y - \sigma_{zx}\Delta x\Delta y$$

$$+ b_x \Delta x\Delta y\Delta z$$

$$= \rho\, \Delta x\Delta y\Delta z\, \ddot{u}_x$$

Divide by $\Delta x \Delta y \Delta z$ & take limit as $\Delta x, \Delta y, \Delta z \to 0$ & we find that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = \rho \ddot{u}_x$$

Similarly we can show

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = \rho \ddot{u}_y$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho \ddot{u}_z$$

or in indicial notation $\boxed{\sigma_{ji,j} + b_i = \rho \ddot{u}_i}$

<u>Moments</u>

$$\Sigma M_z^{CM} = I_{zz}\alpha_z \Rightarrow \left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\Delta y + O(\Delta l^2) + \sigma_{yx}\right]\frac{\Delta y}{2}\Delta x \Delta z$$

$$+ \left[\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}\Delta x + O(\Delta l)^2 + \sigma_{xy}\right]\frac{\Delta x}{2}\Delta y \Delta z$$

$$= \frac{1}{12}\underbrace{\rho \Delta x \Delta y \Delta z}_{m}\left(\Delta x^2 + \Delta y^2\right)$$

* Assumed no distributed body moments.

Again divide by $\Delta x \Delta y \Delta z$ and take limit as $\Delta x, \Delta y, \Delta z \to 0$ and we find that

$$\left.\begin{array}{l} -\sigma_{yx} + \sigma_{xy} = 0 \quad \text{or} \quad \sigma_{xy} = \sigma_{yx} \\ \text{Similarly} \quad \sigma_{xz} = \sigma_{zx}, \ \sigma_{yz} = \sigma_{zy} \end{array}\right\} \boxed{\sigma_{ij} = \sigma_{ji}}$$

Continuum Mechanics Methods

$$\int_S t_i \, dS + \int_V b_i \, dV = \frac{d}{dt}\int_V \rho \dot{u}_i \, dV$$ (Newton's 2nd Law)

$$\int_S \sigma_{ji} n_j \, dS + \int_V b_i \, dV = \frac{d}{dt}\int_V \rho v_i \, dV$$ multiple steps

$$\int_V \sigma_{ji,j} \, dV + \int_V b_i \, dV = \int_V \rho \frac{dv_i}{dt} \, dV$$

choose V to be arbitrary

$$\rightarrow \boxed{\sigma_{ji,j} + b_i = \rho \ddot{u}_i}$$

$$\int_S \underbrace{\epsilon_{rmn} x_m t_n}_{\vec{r} \times \vec{t}} \, dS + \int_V \underbrace{\epsilon_{rmn} x_m b_n}_{\vec{r} \times \vec{b}} \, dV = \frac{d}{dt}\int_V \underbrace{\epsilon_{rmn} x_m \rho v_n}_{\vec{r} \times \rho\vec{v}} \, dV$$

$$\int_V \epsilon_{rmn}\left[(x_m \sigma_{jn})_{,j} + x_m b_n\right] dV = \int_V \epsilon_{rmn} \frac{d}{dt}(x_m v_n) \rho \, dV$$

$$x_{m,j} = \delta_{mj}$$

$$\int_V \epsilon_{rmn}\left[x_m(\sigma_{jn,j} + b_n) + \sigma_{mn}\right] dV = \int_V \epsilon_{rmn}\left[\left(v_m v_n + x_m \frac{dv_n}{dt}\right)\rho\right] dV$$

$$\epsilon_{rmn} v_m v_n = -\epsilon_{rnm} v_m v_n = 0$$
flip

$$\therefore \ \epsilon_{rmn} \sigma_{mn} = 0 \rightarrow \boxed{\sigma_{mn} = \sigma_{nm}}$$

Aside: $\frac{d}{dt}\int_V \rho v_i \, dV$

$$= \int_V \frac{d\rho}{dt} v_i + \rho \frac{dv_i}{dt} + \rho v_i v_{k,k} \, dV$$

$$= \int_V \rho \frac{dv_i}{dt} + v_i \left(\frac{d\rho}{dt} + \rho v_{k,k} \right) dV$$

Conservation of Mass $\rightarrow \frac{d}{dt}\int_V \rho \, dV = 0$

$$\therefore \int_V \frac{d\rho}{dt} + \rho v_{k,k} \, dV = 0$$

arbitrary volume $\rightarrow \frac{d\rho}{dt} + \rho v_{k,k} = 0$

$$\therefore \boxed{\frac{d}{dt}\int_V \rho v_i \, dV = \int_V \rho \frac{dv_i}{dt} \, dV}$$

MECH 516 Supplemental Notes

Eigenvalues and Eigenvectors of a symmetric matrix, an example.

We want to find
$$\begin{bmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{12} & A_{22}-\lambda & A_{23} \\ A_{13} & A_{23} & A_{33}-\lambda \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Procedure :

1) Set [redacted] $\det(A_{ij} - \lambda \delta_{ij}) = 0$
solve for $\lambda_1, \lambda_2, \lambda_3$

2) Plug λ_1 into the equations above and solve for $n_1^{(1)}$ an $n_2^{(1)}$ in terms of $n_3^{(1)}$. Determine n_1, n_2, n_3 by using the constraint $n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$. Repeat for λ_2, λ_3 to determine $n_i^{(2)}, n_i^{(3)}$.

3) Make sure $\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)}$ form a right-handed system. In other words
$$\vec{n}^{(1)} \times \vec{n}^{(2)} = \vec{n}^{(3)}$$
$$\vec{n}^{(2)} \times \vec{n}^{(3)} = \vec{n}^{(1)}$$
$$\vec{n}^{(3)} \times \vec{n}^{(1)} = \vec{n}^{(2)}$$

Note: if $\lambda_1 = \lambda_2 = \lambda_3$ then the tensor is spherically symmetric and any orthogonal triad can be the eigenvectors.
If $\lambda_2 = \lambda_3$ then $\vec{n}^{(1)}$ is unique but $\vec{n}^{(2)}$ and $\vec{n}^{(3)}$ can be any two orthogonal vectors $\perp$ to $\vec{n}^{(1)}$

Ex: $\underset{\approx}{A} = \begin{bmatrix} -20 & -5 & 6 \\ -5 & 5 & 12 \\ 6 & 12 & -8 \end{bmatrix}$ $\underset{\approx}{A} - \lambda \underset{\approx}{I} = \begin{bmatrix} -20-\lambda & -5 & 6 \\ -5 & 5-\lambda & 12 \\ 6 & 12 & -8-\lambda \end{bmatrix}$

$$\text{Det}(\underset{\approx}{A} - \lambda \underset{\approx}{I}) = (-20-\lambda)(5-\lambda)(-8-\lambda) - 5\cdot 12\cdot 6 - 5\cdot 12 \cdot 6$$
$$- 6\cdot 6\cdot(5-\lambda) - 5\cdot 5\cdot(-8-\lambda) - 12\cdot 12\cdot(-20-\lambda)$$

$$= -\lambda^3 - 23\lambda^2 + 185\lambda + 2980 = 0$$

$$\lambda = 12.1974, \ -25.6856, \ -9.51175$$

$$\lambda_1 = 12.1974 \rightarrow -32.1974\, n_1^{(1)} - 5 n_2^{(1)} + 6 n_3^{(1)} = 0$$
$$-5 n_1^{(1)} - 7.1974\, n_2^{(1)} + 12 n_3^{(1)} = 0$$

$$\therefore \ n_1^{(1)} = -0.813395\, n_3^{(1)}$$
$$n_2^{(1)} = 1.72378\, n_3^{(1)}$$

Normalize: $(-0.813395^2 + 1.72378^2 + 1^2)\, {n_3^{(1)}}^2 = 1$

$\therefore \ n_3^{(1)} =$ [redacted] 0.501378

$n_2^{(1)} =$ [redacted] 0.864267

$n_1^{(1)} =$ [redacted] -0.407818

Similar procedures will yield

$\lambda_2 = -25.6856$	$\lambda_3 = -9.51175$
$n_3^{(2)} = 0.493905$	$n_3^{(3)} = 0.710407$
$n_2^{(2)} = -0.324587$	$n_2^{(3)} = -0.384299$
$n_1^{(2)} = -0.806661$	$n_1^{(3)} = 0.589606$

(c)

Now check if $\vec{n}^{(1)}, \vec{n}^{(2)}, \vec{n}^{(3)}$ make a right handed system.

$$\vec{n}^{(1)} \times \vec{n}^{(2)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -0.0407818 & 0.864267 & 0.501378 \\ -0.806661 & -0.324587 & 0.493905 \end{vmatrix} = 0.589607\vec{i} - 0.3843\vec{j} + 0.710408\vec{k} = \vec{n}^{(3)} \checkmark$$

$$\vec{n}^{(2)} \times \vec{n}^{(3)} = -0.0407817\vec{i} + 0.864267\vec{j} + 0.501377\vec{k} = \vec{n}^{(1)} \checkmark$$

$$\vec{n}^{(3)} \times \vec{n}^{(1)} = -0.80666\vec{i} - 0.324587\vec{j} + 0.493905\vec{k} = \vec{n}^{(2)} \checkmark$$

therefore we do have a right handed system.

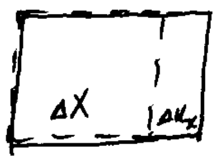
The eigenvalues, i.e. principal values, and eigenvectors, i.e. principal directions of $\underset{\approx}{A}$ are:

$\lambda_1 = 12.1974 \quad \vec{n}^{(1)} = -0.0407818\vec{i} + 0.864267\vec{j} + 0.501378\vec{k}$

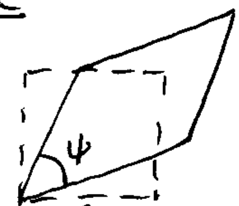
$\lambda_2 = -25.6856 \quad \vec{n}^{(2)} = -0.806661\vec{i} - 0.324587\vec{j} + 0.493905\vec{k}$

$\lambda_3 = -9.51175 \quad \vec{n}^{(3)} = 0.589606\vec{i} - 0.384299\vec{j} + 0.710407\vec{k}$

Small Strain and Rotation

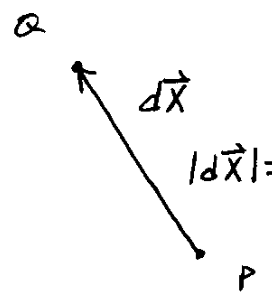


$$\varepsilon_{xx} = \frac{\Delta u_x}{\Delta X}$$



$$\gamma_{xy} = \frac{\pi}{2} - \psi$$

Consider 2 neighboring points P & Q



$\vec{n}$ is a unit vector in the direction of $d\vec{X}$

$$\therefore \; n_i = \frac{dX_i}{dS}$$

we are interested in the normalized relative displacement components $\frac{du_i}{dS}$ because they are strain-like quantities.

$$du_i = u_i^Q - u_i^P$$

$$\frac{du_i}{dS} = \frac{\partial u_i}{\partial X_j}\frac{dX_j}{dS} = \frac{\partial u_i}{\partial X_j} n_j$$

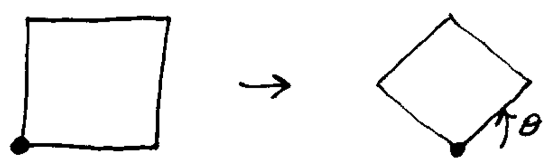
Note that the gradient of a vector is a 2nd order tensor.

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij} \text{ (or } \omega_{ij})}$$

if $\frac{\partial u_i}{\partial x_j} \ll 1$ then Ω_{ij} represents a rigid body rotation and ε_{ij} is a strain tensor

then the stretch of PQ : $\varepsilon = \varepsilon_{ij} n_i n_j$

Check 2-D rigid body rotation



$$u_x = (\cos\theta - 1)X - \sin\theta\, Y$$
$$u_y = \sin\theta\, X + (\cos\theta - 1)Y$$

$$\frac{\partial u_x}{\partial X} = \cos\theta - 1 \qquad \frac{\partial u_x}{\partial Y} = -\sin\theta$$

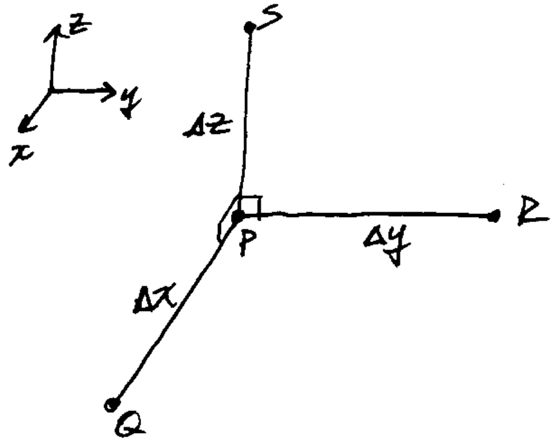
$$\frac{\partial u_y}{\partial X} = \sin\theta \qquad \frac{\partial u_y}{\partial Y} = \cos\theta - 1$$

$$\varepsilon_{ij} = \begin{bmatrix} \cos\theta - 1 & 0 \\ 0 & \cos\theta - 1 \end{bmatrix} \quad \Omega_{ij} = \begin{bmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{bmatrix}$$

$\theta \ll 1 \rightarrow$ to first order

$$\varepsilon_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Omega_{ij} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$$

Another Look at Small Strain & Deformation



We want strains of the line elements PQ, PR, PS and changes in angle of the right angles QPR, QPS, and SPR.

$$u_Q = u_P + \frac{\partial u}{\partial x}\Delta x + \dots \qquad u_R = u_P + \frac{\partial u}{\partial y}\Delta y + \dots$$

$$v_Q = v_P + \frac{\partial v}{\partial x}\Delta x + \dots \qquad v_R = v_P + \frac{\partial v}{\partial y}\Delta y + \dots$$

$$w_Q = w_P + \frac{\partial w}{\partial x} + \Delta x + \dots \qquad w_R = w_P + \frac{\partial w}{\partial y}\Delta y + \dots$$

$$u_S = u_P + \frac{\partial u}{\partial z}\Delta z + \dots$$

$$v_S = v_P + \frac{\partial v}{\partial z}\Delta z + \dots$$

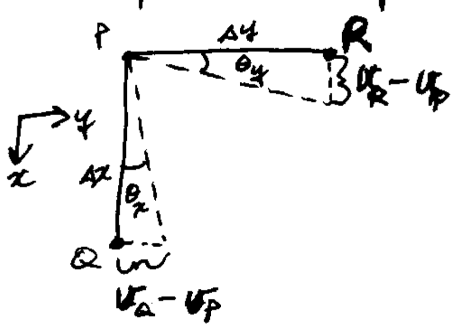
$$w_S = w_P + \frac{\partial w}{\partial z}\Delta z + \dots$$

$$\text{Strain of PQ} = \varepsilon_x = \frac{u_Q - u_P}{\Delta x} = \frac{\partial u}{\partial x}$$

$$\text{Strain of PR} = \varepsilon_y = \frac{v_R - v_P}{\Delta y} = \frac{\partial v}{\partial y}$$

$$\text{Strain of PS} = \varepsilon_z = \frac{w_S - w_P}{\Delta z} = \frac{\partial w}{\partial z}$$

Changes in angle



$$\tan\theta_y = \frac{u_R - u_P}{\Delta y} \approx \theta_y \quad \leftarrow u\text{'s}$$

$$\tan\theta_x = \frac{v_Q - v_P}{\Delta x} \approx \theta_x \quad \leftarrow v\text{'s}$$

$$\therefore \theta_y = \frac{\partial u}{\partial y} \quad \text{and} \quad \theta_x = \frac{\partial v}{\partial x}$$

Recall that $\gamma_{xy} = \theta_x + \theta_y = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}$

Furthermore notice that if $\theta_y = -\theta_x$ then the right angle QPR has undergone a positive (i.e. right-handed) rigid rotation of θ_x.

Hence the z component of the rotation is

$$\omega_z = \frac{1}{2}(\theta_x - \theta_y) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Similarly we can show that

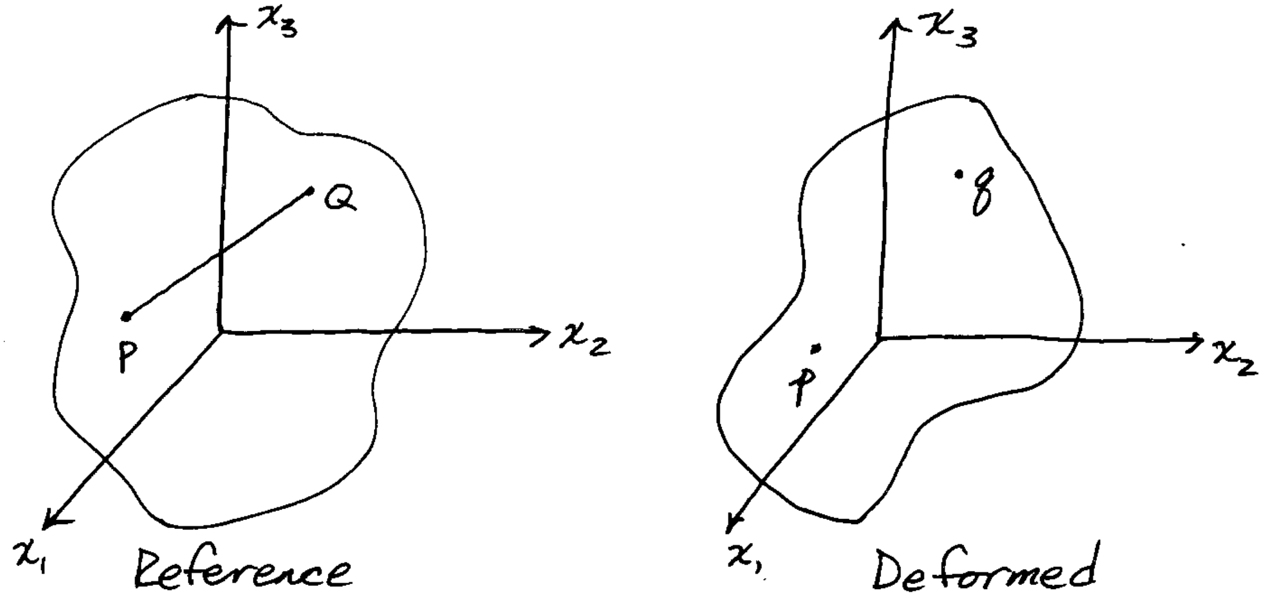
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

Then the small strain and rotation tesors are given as

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}, \quad \underset{\sim}{\Omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Yet another look at Linear Kinematics



Define the strain of line element PQ as

$$\varepsilon^{PQ} = \frac{|pq| - |PQ|}{|PQ|}, \text{ i.e. } \frac{\text{change in length}}{\text{original length}}$$

The coordinates of P and Q are x_i^P and x_i^Q respectively.

Take Q to be in the vicinity of P such that $|PQ|$ is a differential length, then

$$\underbrace{x_i^Q = x_i^P + \Delta x_i}_{} \quad \text{and} \quad |PQ| = \sqrt{\Delta x_i \Delta x_i}$$

Actually, this statement is completely general and does not depend on Q being close to P. However, we will use $\Delta x_i \to 0$ to determine the strain at point P.

After the deformation $P \to p$ and $Q \to q$ such that the positions of p and q are

$$x_i^p = x_i^P + u_i(x_j^P)$$

$$x_i^q = x_i^Q + u_i(x_j^Q)$$

where u_i is the displacement field and $u_i(x_j)$ gives the displacement of a point located at x_j in the reference configuration.

Note that: $u_i(x_j^Q) = u_i(x_j^P + \Delta x_j)$

$$\therefore \; u_i(x_j^Q) = u_i(x_j^P) + \left.\frac{\partial u_i}{\partial x_j}\right|_P \Delta x_j + \frac{1}{2}\left.\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|_P \Delta x_j \Delta x_k$$

$$+ \text{HOT} \ldots$$ (Taylor series expansion)

Assumes smoothness, i.e. no jumps

$$|pq| = \left| x_i^Q + u_i(x_j^Q) - (x_i^P + u_i(x_j^P)) \right|$$ (for lack of better notation)

$$= \left| \Delta x_i + \frac{\partial u_i}{\partial x_j} \Delta x_j + O(\Delta x_j \Delta x_k) \right|$$

Now the fact that Δx_i is differential becomes important

$$|pq| = \sqrt{\Delta x_i \Delta x_i + 2\frac{\partial u_i}{\partial x_j}\Delta x_i \Delta x_j + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\Delta x_j \Delta x_k + O(\Delta x^3)}$$

(C)

Assuming both small deformation <u>and</u> small rotation $\Rightarrow$ $\dfrac{\partial u_i}{\partial x_j} \ll 1$

$$\therefore \quad \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\Delta x_j \Delta x_k \ll 2\frac{\partial u_i}{\partial x_j}\Delta x_i \Delta x_j$$

Finally our small strain, small rotation approximation for $|pq|$ is

$$|pq| = \sqrt{\Delta x_i \Delta x_i + 2\frac{\partial u_i}{\partial x_j}\Delta x_i \Delta x_j}$$

~~now note that with~~

Then :

$$\varepsilon^{PQ} = \sqrt{1 + 2\frac{\partial u_i}{\partial x_j}\frac{\Delta x_i \Delta x_j}{\sqrt{\Delta x_k \Delta x_k}}} - 1$$

take n_i to be the components of a unit vector from P to Q, then $n_i = \dfrac{\Delta x_i}{\sqrt{\Delta x_k \Delta x_k}}$

$$\therefore \quad \varepsilon^{PQ} = \sqrt{1 + 2\frac{\partial u_i}{\partial x_j} n_i n_j} - 1$$

Recall $\dfrac{\partial u_i}{\partial x_j} \ll 1 \quad \therefore \quad \varepsilon^{PQ} \approx 1 + \frac{1}{2}\left(2\frac{\partial u_i}{\partial x_j} n_i n_j\right) - 1$

$$\varepsilon^{PQ} = \frac{\partial u_i}{\partial x_j} n_i n_j \qquad \frac{\partial u_i}{\partial x_j} \equiv \text{displacement gradient}$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_i n_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) n_i n_j$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_i n_j + \frac{1}{2}\frac{\partial u_i}{\partial x_j} n_i n_j - \frac{1}{2}\frac{\partial u_j}{\partial x_i} n_i n_j$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_i n_j + \frac{1}{2}\frac{\partial u_i}{\partial x_j} n_i n_j - \frac{1}{2}\frac{\partial u_i}{\partial x_j} n_j n_i$$

$$\therefore \quad \varepsilon^{PQ} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_i n_j = \varepsilon_{ij} n_i n_j$$

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \quad \text{Small strain tensor}$$

What was the other part of $\frac{\partial u_i}{\partial x_j}$, i.e. $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$?

Hint: It does not cause stretches (strains) of line elements in the vicinity of P.

$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ is the <u>rotation tensor</u>

Note ε_{ij} is the symmetric part of $\frac{\partial u_i}{\partial x_j}$

and ω_{ij} is the anti-symmetric part of $\frac{\partial u_i}{\partial x_j}$

i.e. $\varepsilon_{ij} = \varepsilon_{ji}$ and $\omega_{ij} = -\omega_{ji}$.

(Also called Ω_{ij} on previous pages of notes)

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad , \quad \Omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i})$$

→ 6 compatibility equations

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0$$

$$-\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} - \frac{\partial^2 \varepsilon_{23}}{\partial^2 x_1} + \frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_2} + \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_3} = 0$$

$$-\frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} - \frac{\partial^2 \varepsilon_{13}}{\partial^2 x_2} + \frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_1} + \frac{\partial^2 \varepsilon_{12}}{\partial x_2 \partial x_3} = 0$$

$$-\frac{\partial^2 \varepsilon_{33}}{\partial x_2 \partial x_3} - \frac{\partial^2 \varepsilon_{21}}{\partial^2 x_3} + \frac{\partial^2 \varepsilon_{13}}{\partial x_2 \partial x_3} + \frac{\partial^2 \varepsilon_{23}}{\partial x_1 \partial x_3} = 0$$

Thermoelastic Constitutive Relationships

$$\varepsilon_{ij} = s_{ijkl}\,\sigma_{kl} + \alpha_{ij}\,\Delta T$$

Homogeneous → S_{ijkl}, α_{ij} independent of position

Isotropic → S_{ijkl}, α_{ij} components are invariant after a rotation of coordinate system

Isotropic Material

$$\alpha_{ij} = \alpha \delta_{ij}$$

$$s_{ijkl} = \frac{1+\nu}{E}\delta_{ik}\delta_{jl} - \frac{\nu}{E}\delta_{ij}\delta_{kl}$$

$$c_{ijkl} = (s_{ijkl})^{-1}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{kk} = \frac{1+\nu}{E}\sigma_{kk} - \frac{\nu}{E}\sigma_{mm}\delta_{kk}$$

$$\varepsilon_{kk} = \frac{1+\nu}{E}\sigma_{kk} - \frac{3\nu}{E}\sigma_{kk} = \frac{1-2\nu}{E}\sigma_{kk}$$

$$\therefore \quad \sigma_{kk} = \frac{E}{1-2\nu}\varepsilon_{kk} \quad \rightarrow k = \frac{E}{3(1-2\nu)}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\frac{E}{1-2\nu}\varepsilon_{kk}\delta_{ij}$$

$$\sigma_{ij} = \underbrace{\frac{E}{1+\nu}}_{2\mu}\varepsilon_{ij} + \underbrace{\frac{\nu E}{(1+\nu)(1-2\nu)}}_{\lambda}\varepsilon_{kk}\delta_{ij}$$

Lamé's constants

Another Look at Material Properties

We now have derived the relationships for 2 of the 3 major components of a small strain - small rotation solid mechanics problem. The third component is the constitutive law. As I alluded to previously constitutive frameworks should be derived within the context of thermodynamics.

For elastic materials there is no dissipation, i.e. energy is either stored in the material or results in motion (i.e. kinetic energy)

$$\therefore \quad \text{Work} = KE + PE$$

Actually we will analyze rates of change of these quantities.

$$\underbrace{\int_V b_i v_i \, dV}_{\substack{\text{work rate} \\ \text{of body forces}}} + \underbrace{\int_S T_i v_i \, dS}_{\substack{\text{work rate} \\ \text{of surface} \\ \text{tractions}}} = \frac{d}{dt}\int_V \underbrace{\tfrac{1}{2}\rho v_i v_i}_{\text{KE density}} + \underbrace{W}_{\text{PE density}} \, dV$$

(per unit volume)

* For small deformation - small rotation situations we do not need to make a distinction between reference volume and deformed volume. For large deformation/rotation this distinction is very important and conservation of mass must be used for the analysis.

(B)

Back to our conservation of energy statement.

$$\int_V b_i v_i \, dV + \int_S T_i v_i \, dS = \frac{d}{dt}\int_V \tfrac{1}{2}\rho v_i v_i + W \, dV$$

$$\int_V b_i v_i \, dV + \int_S \sigma_{ji} n_j v_i \, dS = \int_V \rho \dot{v}_i v_i + \dot{W} \, dV$$

Divergence Thm: $\int_V A_{,i} \, dV = \int_S A n_i \, dS$

$$\int_V b_i v_i \, dV + \int_V (\sigma_{ji} v_i)_{,j} \, dV = \int_V \rho \dot{v}_i v_i + \dot{W} \, dV$$

$$\int_V b_i v_i + \sigma_{ji,j} v_i + \sigma_{ji} v_{i,j} \, dV = \int_V \rho \dot{v}_i v_i + \dot{W} \, dV$$

$$\int_V \underbrace{(\sigma_{ji,j} + b_i - \rho \dot{v}_i)}_{=0 \text{ by Equilibrium}} v_i + \sigma_{ji} v_{i,j} \, dV = \int_V \dot{W} \, dV$$

$$\therefore \quad \int_V \sigma_{ji} v_{i,j} - \dot{W} \, dV = 0$$

This must be true for <u>any</u> arbitrary volume including chunks of material that we make arbitrary cuts around.

$$\therefore \quad \sigma_{ji} v_{i,j} - \dot{W} = 0$$

$$\sigma_{ji} \tfrac{1}{2}(v_{i,j} + v_{j,i}) + \sigma_{ji} \tfrac{1}{2}(v_{i,j} - v_{j,i}) = \dot{W}$$

C

For linear kinematics $\frac{1}{2}(v_{i,j}+v_{j,i}) = \frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}) = \dot{\varepsilon}_{ij}$

$$\sigma_{ji}\,\dot{\varepsilon}_{ij} + \frac{1}{2}\sigma_{ji}v_{i,j} - \frac{1}{2}\sigma_{ji}v_{j,i} = \dot{W}$$

but $\sigma_{ji} = \sigma_{ij}$ b/c of moment equilibrium

$$\therefore\quad \sigma_{ij}\,\dot{\varepsilon}_{ij} + \cancel{\frac{1}{2}\sigma_{ij}v_{i,j}} - \cancel{\frac{1}{2}\sigma_{ji}v_{j,i}} = \dot{W}$$

(Note dummy indices can be changed)

$$\therefore\quad \dot{W} = \sigma_{ij}\,\dot{\varepsilon}_{ij}$$

Now take $W = W(\varepsilon_{ij})$

then $\dot{W} = \frac{\partial W}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij}$

$$\therefore\quad \frac{\partial W}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} = \sigma_{ij}\,\dot{\varepsilon}_{ij}$$

Finally, for arbitrary strain rates this must be true, $\therefore$

$$\boxed{\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}}$$ For elastic materials

Green Elastic Material – Linear case

$W = W(\varepsilon_{ij})$ – Strain energy density
Energy per unit volume

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$W = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}$ – Linear elastic case
$\underset{\approx}{C}$ is the 4^{th} rank elastic stiffness tensor

Note: $\dfrac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = C_{ijkl} = \dfrac{\partial^2 W}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = C_{klij}$

$$\sigma_{ij} = \frac{\partial}{\partial \varepsilon_{ij}} \left(\frac{1}{2} C_{pqrs} \varepsilon_{pq} \varepsilon_{rs} \right)$$

$$= \frac{1}{2} C_{pqrs} \delta_{ip} \delta_{jq} \varepsilon_{rs} + \frac{1}{2} C_{pqrs} \varepsilon_{pq} \delta_{ir} \delta_{js}$$

$$= \frac{1}{2} C_{ijrs} \varepsilon_{rs} + \frac{1}{2} C_{pqij} \varepsilon_{pq}$$

$$\boxed{\sigma_{ij} = C_{ijkl} \varepsilon_{kl}}$$

(Using $\dfrac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \dfrac{\partial^2 W}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}}$)
and changing r & p dummy indices to k and s & q to l

Recall that an isotropic material contains spherical symmetry in its material properties, i.e. all directions have identical material response.

The general form of a 4th rank isotropic tensor with $C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{klij}$ is

$$C_{ijkl} = a\,\delta_{ij}\delta_{kl} + b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

<u>Hooke's Law as you have seen it</u>

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} \\
\varepsilon_{22} &= \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{33} \\
\varepsilon_{33} &= \frac{1}{E}\sigma_{33} - \frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} \\
\varepsilon_{12} &= \frac{1}{2}\gamma_{12} = \frac{1}{2}\frac{1}{G}\sigma_{12} \\
\varepsilon_{13} &= \frac{1}{2}\gamma_{13} = \frac{1}{2}\frac{1}{G}\sigma_{13} \qquad \text{with } G = \mu = \frac{E}{2(1+\nu)} \\
\varepsilon_{23} &= \frac{1}{2}\gamma_{23} = \frac{1}{2}\frac{1}{G}\sigma_{23}
\end{aligned}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} = S_{ijkl}\sigma_{kl}$$

$$\therefore\; S_{ijkl} = -\frac{\nu}{E}\delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

$$\underset{\approx}{C} = \underset{\approx}{S}^{-1}$$

$$\therefore\; C_{ijmn}S_{mnkl} = I_{ijkl}$$

where I_{ijkl} is defined such that

$$I_{ijkl}\sigma_{kl} = (\sigma_{ij} + \sigma_{ji})/2 \quad \text{(symmetric 4th rank identity tensor)}$$

$$\therefore\; I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})/2$$

(c)

$$\therefore \left[a\delta_{ij}\delta_{mn} + b(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\right]\left[-\frac{\nu}{E}\delta_{mn}\delta_{kl} + \frac{1+\nu}{2E}(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})\right]$$

$$= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

$$-\frac{\nu}{E}a\,\delta_{ij}\delta_{mm}\delta_{kl} + \frac{1+\nu}{2E}a(\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{lk})$$

$$-\frac{\nu}{E}b\,(\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{kl}) + \frac{1+\nu}{2E}b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

$$= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$$

Note: $\delta_{mm} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

$$\therefore \left(\frac{-3\nu}{E} + \frac{1+\nu}{E}\right)a - \frac{2\nu}{E}b = 0$$

$$\frac{(1+\nu)}{E}b = \frac{1}{2}$$

$$\therefore b = \frac{E}{2(1+\nu)} = \mu$$

$$\frac{1-2\nu}{E}a - \frac{\nu}{1+\nu} = 0$$

$$\therefore a = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$\therefore C_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

and

$$\boxed{\sigma_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}}$$

9/4/01

$$\sigma_{ji,j} + b_i = 0$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

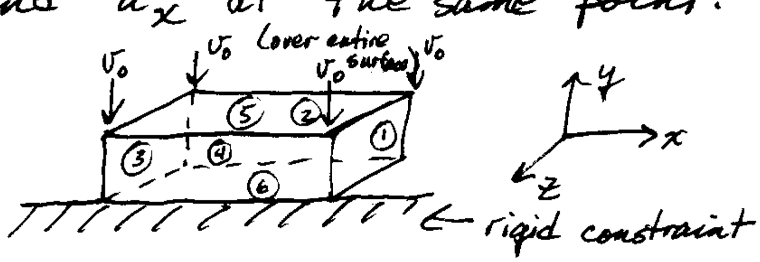
Cylindrical
Spherical

$\left.\begin{array}{l} \text{Cylindrical} \\ \text{Spherical} \end{array}\right\}$ See Handout

Boundary Conditions

At each point on a surface of the body some combination of surface tractions and surface displacements must be specified. Furthermore, like components of surface traction and surface displacement cannot be specified concurrently at the same point. In other words you cannot specify both T_x and u_x at the same point.

Ex)

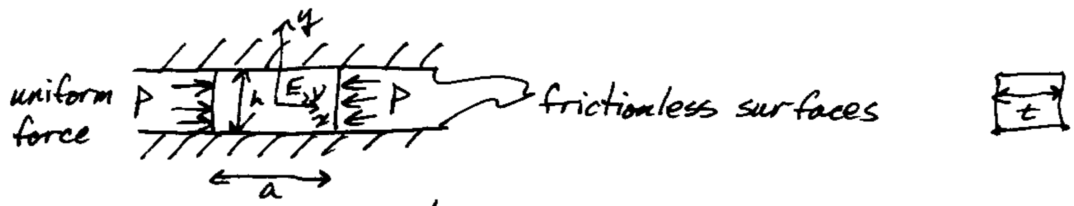


BC's) (1) & (3) Traction free : $\sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0$

(2) & (4) Traction free : $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$

(6) Fixed displacement : $u = v = w = 0$

(5) Mixed : $v = v_0$ [illegible] $= 0$



BC's: $\sigma_{xx} = -P/ht$ at $x = \pm a/2$

$\sigma_{xy} = \sigma_{xz} = 0$ at $x = \pm a/2$

$v = 0$ at $y = \pm h/2$

$\sigma_{xy} = \sigma_{yz} = 0$ at $y = \pm h/2$

$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$ at $z = \pm t/2$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = c$$

assume uniform stress of form $\sigma_{xx} = -P/ht$, $\sigma_{yy} = ?$

$\sigma_{xy} = \sigma_{yz} = \sigma_{zz} = \sigma_{xz} = 0$

$\therefore v = cy + f(x,z)$

$v(x, y = \frac{h}{2}) = 0 \rightarrow f(x,z) = -c\frac{h}{2}$

$v(x, y = -\frac{h}{2}) = 0 \rightarrow -c\frac{h}{2} - c\frac{h}{2} = 0 \rightarrow c = 0$

$\therefore$ $v = 0$ everywhere

$$\varepsilon_{yy} = 0 = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}(-P/ht)$$

$$\therefore \sigma_{yy} = -\nu P/ht$$

$$\varepsilon_{xx} = \frac{1}{E}(-P/ht) - \frac{\nu}{E}(-\nu P/ht) = \frac{1-\nu^2}{E}\frac{P}{ht} = \frac{\partial u}{\partial x}$$

$$\therefore u = \frac{1-\nu^2}{E} P/ht \; x + f(y,z)$$

~~choose u=0 at (x,y,z)=(0,0,arbitrary)~~

$$\varepsilon_{xy} = 0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \rightarrow \frac{\partial u}{\partial y} = 0$$

$$\varepsilon_{zz} = \frac{-\nu}{E}\left(\frac{-P}{ht} - \frac{\nu P}{ht}\right) = -\frac{\nu(1+\nu)}{E}\frac{P}{ht} = \frac{\partial w}{\partial z}$$

$$\therefore \quad w = -\frac{\nu(1+\nu)}{E} P/ht \, z + g(x,y)$$

$$\varepsilon_{yz} = 0 = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \rightarrow \frac{\partial w}{\partial y} = 0$$

$$\therefore \quad u = \frac{1-\nu^2}{E} P/ht \, x + f(z)$$

$$w = \frac{-\nu(1+\nu)}{E} P/ht \, z + g(x)$$

$$v = 0$$

$$\varepsilon_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(\frac{df}{dz} + \frac{dg}{dx}\right)$$

$$\therefore \quad \frac{df}{dz} = -\frac{dg}{dx} = c$$

$$\therefore \quad u = \frac{1-\nu^2}{E}\frac{P}{ht} x + cz + c_1$$

$$v = 0$$

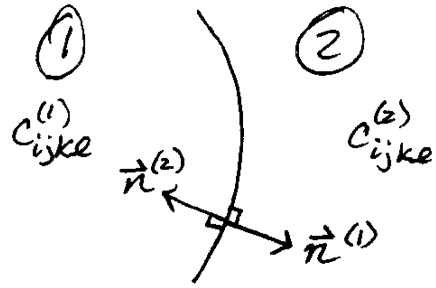
$$w = \frac{-\nu(1+\nu)}{E}\frac{P}{ht} z - cx + c_2$$

choose $(u,v,w) = (0,0,0)$ at $(x,y,z) = (0,0,0)$ (arbitrary), $\therefore c_2 = c_1 = 0$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(c - -c) = c = 0$$

$$\therefore \quad \boxed{\begin{aligned} u &= \frac{1-\nu^2}{E}\frac{P}{ht} x \\ v &= 0 \\ w &= \frac{-\nu(1+\nu)}{E}\frac{P}{ht} z \end{aligned}}$$

Material Interfaces



Assume a well bonded interface. How are components of stress, strain and displacement related across the interface?

(1) Displacements are continuous. In other words $u^{(1)} = u^{(2)}$, $v^{(1)} = v^{(2)}$, $w^{(1)} = w^{(2)}$ on the interface. Or in indicial notation:

$$\| u_i \| = 0$$

where $\| \; \|$ denotes a jump in the included quantity.

~~Eq~~ (1) implies certain relationships for the strain components across the interface. At a point on the surface let's take an orthogonal coordinate system such that $\vec{e}_3 = \vec{n}^{(1)}$, $\vec{e}_2 \cdot \vec{e}_3 = 0$, $\vec{e}_1 \cdot \vec{e}_3 = 0$, $\vec{e}_1 \cdot \vec{e}_2 = 0$, $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$ and $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$.

$$\therefore \; \| \varepsilon_{ij} \| = \left\| \tfrac{1}{2} (u_{i,j} + u_{j,i}) \right\|$$

let's expand u_i into a Taylor series about the point in question

$$u_1^{(1)} = u_1^{(1)}(x_1^o, x_2^o, x_3^o) + \frac{\partial u_1}{\partial x_1}\Big|_o^{(1)} \Delta x_1 + \frac{\partial u_1}{\partial x_2}\Big|_o^{(1)} \Delta x_2 + \frac{\partial u_1}{\partial x_3}\Big|_o^{(1)} \Delta x_3 + \ldots$$

$$u_2^{(1)} = u_2^{o(1)} + \frac{\partial u_2}{\partial x_1}^{(1)} \Delta x_1 + \frac{\partial u_2}{\partial x_2}^{o(1)} \Delta x_2 + \frac{\partial u_2}{\partial x_3}^{(1)} \Delta x_3 + \ldots$$

$$u_3^{(1)} = u_3^{o(1)} + \frac{\partial u_3}{\partial x_1}^{(1)} \Delta x_1 + \frac{\partial u_3}{\partial x_2}^{(1)} \Delta x_2 + \frac{\partial u_3}{\partial x_3}^{(1)} \Delta x_3 + \ldots$$

$$u_1^{(2)} = u_1^{o(2)} + \frac{\partial u_1}{\partial x_1}^{(2)} \Delta x_1 + \frac{\partial u_1}{\partial x_2}^{(2)} \Delta x_2 + \frac{\partial u_1}{\partial x_3}^{(2)} \Delta x_3 + \ldots$$
$$u_2^{(2)} = u_2^{o(2)} + \frac{\partial u_2}{\partial x_1}^{(2)} \Delta x_1 + \frac{\partial u_2}{\partial x_2}^{(2)} \Delta x_2 + \frac{\partial u_2}{\partial x_3}^{(2)} \Delta x_3 + \ldots$$
$$u_3^{(2)} = u_3^{o(2)} + \frac{\partial u_3}{\partial x_1}^{(2)} \Delta x_1 + \frac{\partial u_3}{\partial x_2}^{(2)} \Delta x_2 + \frac{\partial u_3}{\partial x_3}^{(2)} \Delta x_3 + \ldots$$

To 1st order for a smooth surface

→ All points on the interface in the vicinity of (x_1^o, x_2^o, x_3^o) can be given by $(x_1^o+\Delta x_1, x_2^o+\Delta x_2, x_3^o)$

and on the interface $u_i^{(1)} = u_i^{(2)}$

$$\therefore\ u_1^{o(1)} + u_{1,1}^{(1)}\Delta x_1 + u_{1,2}^{(1)}\Delta x_2 = u_1^{o(2)} + u_{1,1}^{(2)}\Delta x_1 + u_{1,2}^{(2)}\Delta x_2$$
$$u_2^{o(1)} + u_{2,1}^{(1)}\Delta x_1 + u_{2,2}^{(1)}\Delta x_2 = u_2^{o(2)} + u_{2,1}^{(2)}\Delta x_1 + u_{2,2}^{(2)}\Delta x_2$$
$$u_3^{o(1)} + u_{3,1}^{(1)}\Delta x_1 + u_{3,2}^{(1)}\Delta x_2 = u_3^{o(2)} + u_{3,1}^{(2)}\Delta x_1 + u_{3,2}^{(2)}\Delta x_2$$

note that Δx_1 & Δx_2 are arbitrary (but small)

$$\therefore\ u_1^{o(1)} = u_1^{o(2)},\quad u_2^{o(1)} = u_2^{o(2)},\quad u_3^{o(1)} = u_3^{o(2)}$$

$$\|u_{1,1}\| = \|u_{1,2}\| = \|u_{2,1}\| = \|u_{2,2}\| = \|u_{3,1}\| = \|u_{3,2}\| = 0$$

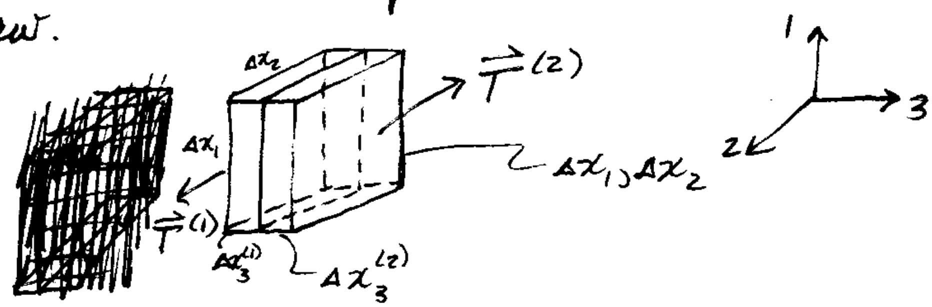
$$\therefore\ \|\varepsilon_{11}\| = \|\varepsilon_{12}\| = \|\varepsilon_{22}\| = 0$$

however $\|\varepsilon_{33}\|$, $\|\varepsilon_{13}\|$, $\|\varepsilon_{23}\|$ not necessarily $\emptyset$

→ Note that we have taken the 3-direction to be ⊥ to the interface

(2) Tractions are continuous. Result of Newton's 3rd Law, or a limiting process with Newton's 2nd Law.

Ex:



$$\Sigma F_1 = T_1^{(2)} \Delta x_1 \Delta x_2 - T_1^{(1)} \Delta x_1 \Delta x_2 = \underbrace{(\rho^{(1)} \Delta x_1 \Delta x_2 \Delta x_3^{(1)} + \rho^{(2)} \Delta x_1 \Delta x_2 \Delta x_3^{(2)}) \ddot{u}_1}_{\text{Inertia term}}$$

$$\therefore \boxed{T_i^{(1)} = T_i^{(2)} \text{ as } \Delta x_3^{(1)} \,\&\, \Delta x_3^{(2)} \to 0}$$

$$\therefore \sigma_{ij}^{(1)} n_j^{(1)} = - \sigma_{ij}^{(2)} n_j^{(2)}$$

$$n_j^{(1)} = -n_j^{(2)} \rightarrow \sigma_{ij}^{(1)} n_j^{(1)} = \sigma_{ij}^{(2)} n_j^{(1)}$$

for our case we have $n_j^{(1)} = \delta_{j3}$

$$\therefore \delta_{j3} \sigma_{ij}^{(1)} = \sigma_{ij}^{(2)} \delta_{j3}$$

$$\therefore \boxed{\sigma_{i3}^{(1)} = \sigma_{i3}^{(2)}}$$

Again with the normal to the interface in the 3-direction.