9/25/01

Cracks

- Asymptotic Analysis of the crack tip.
- Obviously, stresses & strains are singular as the crack tip is approached. Therefore, we want the dominant, i.e. "most" singular, terms only.

$$\sigma_{ij} = r^p f_{ij}(\theta) \leftarrow \text{Look for solutions of this form.}$$

$$\phi = r^{p+2}\left[A\cos p\theta + B\cos(p+2)\theta + C\sin p\theta + D\sin(p+2)\theta\right]$$

$$\therefore \sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = (p+2)r^p\left[A\cos p\theta + B\cos(p+2)\theta + C\sin p\theta + D\sin(p+2)\theta\right]$$
$$+ r^p\left[-Ap^2\cos p\theta - B(p+2)^2\cos(p+2)\theta - Cp^2\sin p\theta - D(p+2)^2\sin(p+2)\theta\right]$$

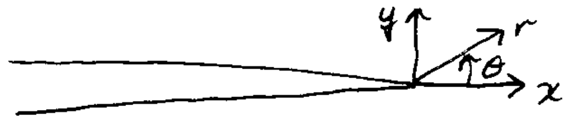
$$\sigma_{rr} = r^p\Big[\overset{(-p+2)(p+1)}{(-p^2+p+2)}A\cos p\theta + \overset{-(p+2)(p+1)}{(-p^2-3p-2)}B\cos(p+2)\theta$$
$$+(-p^2+p+2)C\sin p\theta + (-p^2-3p-2)D\sin(p+2)\theta\Big]$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = (p+2)(p+1)r^p\left[A\cos p\theta + B\cos(p+2)\theta + C\sin p\theta + D\sin(p+2)\theta\right]$$

$$\sigma_{\theta\theta} = r^p\Big[(p+2)(p+1)A\cos p\theta + (p+2)(p+1)B\cos(p+2)\theta$$
$$+(p+2)(p+1)C\sin p\theta + (p+2)(p+1)D\sin(p+2)\theta\Big]$$

$$\sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} = -(p+2)r^p\left[-Ap\sin p\theta - B(p+2)\sin(p+2)\theta + Cp\cos p\theta + D(p+2)\cos(p+2)\theta\right]$$
$$+ r^p\left[-Ap\sin p\theta - B(p+2)\sin(p+2)\theta + Cp\cos p\theta + D(p+2)\cos(p+2)\theta\right]$$

$$\sigma_{r\theta} = r^p\Big[p(p+1)A\sin p\theta + (p+2)(p+1)B\sin(p+2)\theta$$
$$-p(p+1)C\cos p\theta - (p+2)(p+1)D\cos(p+2)\theta\Big]$$



BCs: $\sigma_{\theta\theta}(r, \theta = \pm\pi) = 0$ ①
$\sigma_{r\theta}(r, \theta = \pm\pi) = 0$ ②

① $\rightarrow$ $A\cos p\pi + B\cos(p+2)\pi + C\sin p\pi + D\sin(p+2)\pi = 0$

② $\rightarrow$ $p\cancel{(p+1)}A\sin p\pi + \cancel{(p+1)}(p+2)B\sin(p+2)\pi - p\cancel{(p+1)}C\cos p\pi - \cancel{(p+1)}(p+2)D\cos(p+2)\pi = 0$

$\therefore$ $p = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ and $C + D = 0$, $pA + (p+2)B = 0$

or

$p = 0, \pm 1, \pm 2, \pm 3, \ldots$ and $A + B = 0$, $pC + (p+2)D = 0$

We are looking for singular terms for stresses
$\therefore$ $p < 0 \rightarrow p = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$

This also implies u_i remain bounded

* Now let us further restrict our search to solutions that have a finite amount of energy stored in a region near the crack tip.

$$U = \frac{1}{2}\int_A \sigma_{ij}\varepsilon_{ij}\,dA \quad , \quad \sigma_{ij} \propto r^p f(\theta) \, , \, \varepsilon_{ij} \propto r^p g(\theta)$$

$$\therefore U \propto \frac{1}{2}\int_0^R \int_{-\pi}^{\pi} r^{2p} h(\theta)\, r\, d\theta\, dr$$

$$U \propto c\int_0^R r^{2p+1}\,dr = c\,\frac{1}{2p+2}\,r^{2p+2}\Big]_0^R$$

$$U < \infty \rightarrow 2p+2 > 0 \rightarrow \underline{p > -1}$$

$\therefore$ The only singular solution that will satisfy this condition is $p = -\frac{1}{2}$

$\therefore$ $p = -\frac{1}{2}$ and $C + D = 0$, $-\frac{1}{2}A + \frac{3}{2}B = 0$

$C = -D$, $A = 3B$ or $B = \frac{1}{3}A$

$$\therefore \sigma_{rr} = A r^{-1/2}\left[\tfrac{5}{4}\cos\tfrac{\theta}{2} - \tfrac{1}{4}\cos\tfrac{3\theta}{2}\right] + C r^{-1/2}\left[-\tfrac{5}{4}\sin\tfrac{\theta}{2} + \tfrac{3}{4}\sin\tfrac{3\theta}{2}\right]$$

$$\sigma_{\theta\theta} = A r^{-1/2}\left[\tfrac{3}{4}\cos\tfrac{\theta}{2} + \tfrac{1}{4}\cos\tfrac{3\theta}{2}\right] + C r^{-1/2}\left[-\tfrac{3}{4}\sin\tfrac{\theta}{2} - \tfrac{3}{4}\sin\tfrac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = A r^{-1/2}\left[\tfrac{1}{4}\sin\tfrac{\theta}{2} + \tfrac{1}{4}\sin\tfrac{3\theta}{2}\right] + C r^{-1/2}\left[\tfrac{1}{4}\cos\tfrac{\theta}{2} + \tfrac{3}{4}\cos\tfrac{3\theta}{2}\right]$$

A convention, due to Irwin, calls for $\sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}}$ and $\sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi x}}$ on the plane ahead of the crack. We will show that this convention leads to $\mathcal{G} = K_I^2/E' + K_{II}^2/E'$.

$$\sigma_{\theta\theta}(r, \theta=0) = \sigma_{yy}(x, y=0) = A r^{-1/2} = A x^{-1/2} = \frac{K_I}{\sqrt{2\pi}} x^{-1/2}$$

$$\therefore \underline{\underline{A = K_I/\sqrt{2\pi}}}$$

$$\sigma_{r\theta}(r, \theta=0) = \sigma_{xy}(x, y=0) = C r^{-1/2} = C x^{-1/2} = \frac{K_{II}}{\sqrt{2\pi}} x^{-1/2}$$

$$\therefore C = \frac{K_{II}}{\sqrt{2\pi}}$$

$$\therefore\ \sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]$$

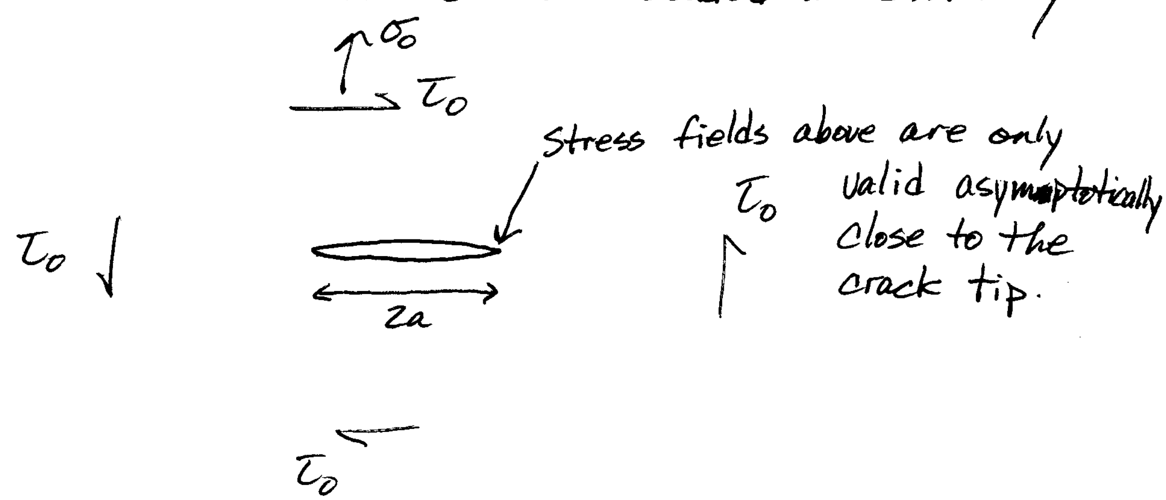
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right]$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right]$$

K_I, K_{II} have dimensions of $\sigma\sqrt{L}$ and are called the Mode I and Mode II stress intesity factors. There is also a Mode III stress intensity factor that we will get to when we deal with anti-plane strain.

The specific form of K_I, K_{II} depends on the geometry and loading for the problem.

Ex) Crack in Infinite Plate Loaded at Infinity



$$K_I = \sigma_0\sqrt{\pi a}, \quad K_{II} = \tau_0\sqrt{\pi a}$$

* Shear & tensile SIF's are usually not "equal".

Displacement Field:

$$u_r = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[(2\chi-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$

$$+ \frac{K_{II}}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi-1)\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\right]$$

$$u_\theta = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]$$

$$+ \frac{K_{II}}{2E}\sqrt{\frac{r}{2\pi}}\,(1+\nu)\left[-(2\chi+1)\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right]$$

Crack opening displacement $= COD = -u_\theta(r,\pi) + u_\theta(r,-\pi)$

Crack sliding displacement $= CSD = -u_r(r,\pi) + u_r(r,-\pi)$

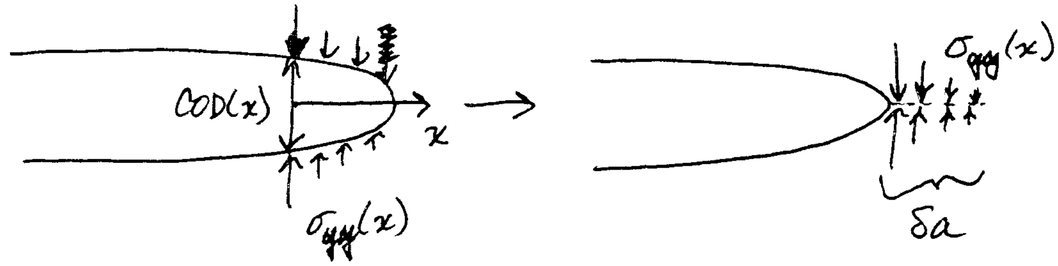
$$COD = \frac{K_I}{E}\sqrt{\frac{2r}{\pi}}\,(1+\nu)(\chi+1)$$

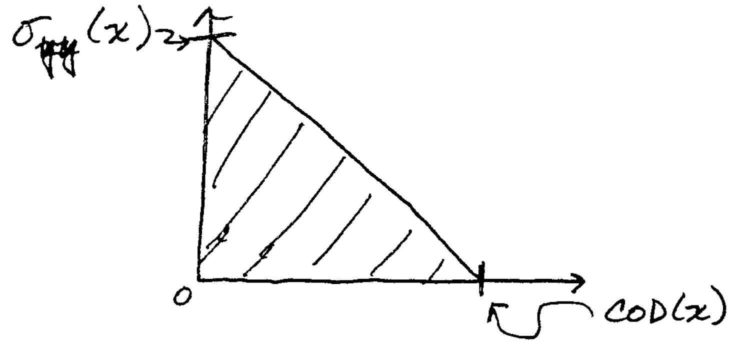
$$CSD = \frac{K_{II}}{E}\sqrt{\frac{2r}{\pi}}\,(1+\nu)(\chi+1)$$

How much energy is released if the crack is able to propagate by some small distance δa?

We can answer this question by figuring out how much work must be done to close the crack by an amount δa.

9/27/01





Work done by some small part of $\sigma_{yy}(x)$

$$dW = \frac{1}{2}\sigma_{yy}(x)\,COD(x)\,dx$$

$$W = \frac{1}{2}\int_0^{\delta a}\sigma_{yy}(x)\,COD(x)\,dx$$

$$\sigma_{yy}(x) = \frac{K_I}{\sqrt{2\pi x}}\ , \quad COD(x) = \frac{K_I}{E}\sqrt{\frac{2(\delta a - x)}{\pi}}\,(1+\nu)(\kappa+1)$$

$$W = \frac{K_I^2}{2E}(1+\nu)(\kappa+1)\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\int_0^{\delta a}\sqrt{\frac{\delta a - x}{x}}\,dx$$

transformation: $x = \delta a \sin^2\theta$

$dx = 2\delta a \sin\theta\cos\theta\, d\theta$

$$\therefore W = \frac{K_I^2}{2E}(1+\nu)(\kappa+1)\frac{1}{\pi}\int_0^{\pi/2}\sqrt{\frac{\cancel{\delta a} - \cancel{\delta a}\sin^2\theta}{\cancel{\delta a}\,\cancel{\sin^2\theta}}}\;2\delta a\,\cancel{\sin\theta}\cos\theta\,d\theta$$

$$= \frac{K_I^2}{2E}(1+\nu)(\kappa+1)\frac{1}{\pi}\int_0^{\pi/2} 2\delta a\cos^2\theta\,d\theta$$

$$W = \frac{K_I^2}{2E}(1+\nu)(\kappa+1)\frac{1}{\pi}\,2\delta a\,\frac{\pi}{4}$$

$$W = \mathcal{G}\,\delta a = \frac{1}{2}\frac{K_I^2}{2E}(1+\nu)(\chi+1)\,\delta a$$

$$\frac{(1+\nu)(\chi+1)}{2\cdot 2E} = \begin{cases} \dfrac{(1+\nu)(4-4\nu)}{2\cdot 2E} = \dfrac{(1+\nu)(1-\nu)}{E} = \dfrac{1-\nu^2}{E} & \text{plane strain} \\ \dfrac{(1+\nu)\,4}{4(1+\nu)E} = \dfrac{1}{E} & \text{plane stress} \end{cases}$$

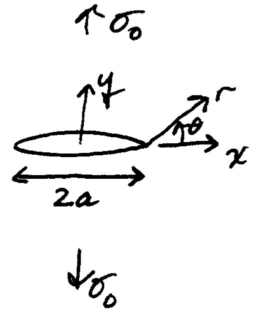
$$\therefore\ \mathcal{G}\,\delta a = \frac{K_I^2}{E'}\,\delta a \rightarrow \mathcal{G} = \frac{K_I^2}{E'}$$

Note that we could use a similar procedure on the CSD & shear stresses ahead of the crack. Furthermore, we could first close the COD then close the CSD & the energies add together.

$$\therefore\ \mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'}$$

Note that K_I, K_{II} are linear functions of the loading but $\mathcal{G}$ is not. Hence K's can be added ~~for~~ superposed problems, but $\mathcal{G}$ cannot.

Determining K_I, K_{II} from a BVP solution.



Solution

$$\sigma_{yy}(|x| \geq a, y=0) = \frac{\sigma x}{\sqrt{x^2-a^2}}$$

$$\lim_{r\to 0} \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

on $y=0$, $x = a+r$

$$\therefore \lim_{r\to 0} \sigma_{yy} = \lim_{r\to 0} \frac{\sigma(a+r)}{\sqrt{\cancel{a^2}+2ar+r^2-\cancel{a^2}}}$$

$$= \lim_{r\to 0} \frac{\sigma(a+r)}{\sqrt{r^2+2ar}} = \frac{\sigma a}{\sqrt{2ar}} = \frac{K_I}{\sqrt{2\pi r}}$$

$$\therefore K_I = \frac{\sigma a}{\sqrt{2a\cancel{r}}}\sqrt{2\pi \cancel{r}} = \sigma\sqrt{\pi a}$$

Similar procedures can be used if you know any of the stress components or any of the displacement components.

For brittle materials fracture occurs when $K_I = K_{Ic}$. You should interpret K_I as a loading parameter that is similar but not identical to applied stress. Then K_{Ic} is a material property that is similar to the materials yield strength.

Antiplane / Longitudinal Shear

$$u = v = 0 \;,\quad w = w(x,y) \rightarrow \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0$$

$$\therefore \quad \sigma_{zz} = \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial w}{\partial x} \;,\quad \gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial w}{\partial y}$$

$$\sigma_{xz} = \mu \gamma_{xz} \;,\quad \sigma_{yz} = \mu \gamma_{yz}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \rightarrow \text{Stress potential } \psi$$

$$\sigma_{yz} = \frac{\partial \psi}{\partial x} \;,\quad \sigma_{xz} = -\frac{\partial \psi}{\partial y}$$

$$\mu \frac{\partial \gamma_{xz}}{\partial x} + \mu \frac{\partial \gamma_{yz}}{\partial y} = 0 \;,\qquad -\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} = 0$$

$$\mu \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0$$

$$\therefore \quad \boxed{\nabla^2 w = 0} \qquad \boxed{\nabla^2 \psi = 0}$$

Separation of variables Solutions

Cartesian

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \;,\quad w = X(x)\,Y(y)$$

$$\therefore \quad \frac{X''Y}{XY} + \frac{XY''}{XY} = 0 \rightarrow \underbrace{\frac{X''}{X}}_{f(x)} + \underbrace{\frac{Y''}{Y}}_{g(y)} = 0$$

$$\therefore \quad \frac{X''}{X} = -\frac{Y''}{Y} = -c^2$$

$$\therefore \quad X'' + c^2 X = 0 \quad , \quad Y'' - c^2 Y = 0$$

$$X = A\cos cx + B\sin cx \quad , \quad Y = Ce^{cy} + De^{-cy}$$

$$\left.\begin{aligned} \therefore \quad w = {} & \exp[cy](A\cos cx + B\sin cx) \\ & + \exp[-cy](C\cos cx + D\sin cx) \\ \text{Double root at } c = 0 \rightarrow \quad & w = Ay, Bx \end{aligned}\right\}$$

Note that solutions with x and y interchanged are of course also valid.

<u>Polar</u>

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0 \quad , \quad w = R(r)\Theta(\theta)$$

$$\therefore \quad \frac{R''\Theta}{R\Theta} + \frac{\frac{1}{r}R'\Theta}{R\Theta} + \frac{\frac{1}{r^2}R\Theta''}{R\Theta} = 0$$

$$\underbrace{r^2\frac{R''}{R} + \frac{r^2}{r}\frac{R'}{R}}_{f(r)} + \underbrace{\frac{r^2}{r^2}\frac{\Theta''}{\Theta}}_{g(\theta)} = 0$$

$$\therefore \quad r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = c^2$$

$$\therefore \quad r^2 R'' + rR' - c^2 R = 0 \quad , \quad \Theta'' + c^2\Theta = 0$$

$$R = Ar^p \; , \; R' = Apr^{p-1} , \; R'' = Ap(p-1)r^{p-2} , \; \Theta = A\sin c\theta + B\cos c\theta$$

$$p(p-1) + p - c^2 = 0$$

$$p^2 - c^2 = 0$$

$$p = \pm c$$

$$R = Cr^c + Dr^{-c}$$

or $\quad R = Cr^p + Dr^{-p} \; , \; \Theta = A\sin p\theta + B\cos p\theta$

$$\therefore \quad \omega = r^{p}(A\sin p\theta + B\cos p\theta) + r^{-p}(C\sin p\theta + D\cos p\theta)$$ All solutions

Double root at $p=0 \rightarrow \omega = A\ln r, B\theta$

10/2/01 Complex Variables

$$i = \sqrt{-1}$$

$$z = x+iy = re^{i\theta} = r\cos\theta + ir\sin\theta$$

$$\bar{z} = x-iy = re^{-i\theta} = r\cos\theta - ir\sin\theta$$

$$x = \frac{1}{2}(z+\bar{z}) = \mathrm{Re}(z)$$

$$y = \frac{1}{2i}(z-\bar{z}) = \mathrm{Im}(z)$$

A complex function $f(x,y)+ig(x,y)$ is analytic iff $\underbrace{\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \text{ and } \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}}_{\text{Cauchy-Riemann conditions.}}$.

- A function of a complex variable is analytic iff $\frac{\partial f}{\partial \bar{z}} = 0$.

- Proof

$$f(z,\bar{z}) = P(x,y) + iQ(x,y)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial P}{\partial x}\frac{1}{2} + i\frac{\partial Q}{\partial x}\frac{1}{2} + \frac{\partial P}{\partial y}\frac{-1}{2i} + i\frac{\partial Q}{\partial y}\frac{-1}{2i}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{iff} \quad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \quad \text{and} \quad \frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$$ **QED**

- If $f(z)$ is analytic then $\nabla^2[Re(f(z))] = 0$ and $\nabla^2\{Im[f(z)]\} = 0$

-Proof- $f(z) = P(x,y) + i\,Q(x,y)$

analytic $\rightarrow \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y}, \quad \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x}$

$$\left.\begin{aligned} \therefore \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} &= \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial y} = 0 = \nabla^2 P \\ \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} &= \frac{-\partial^2 P}{\partial x \partial y} + \frac{\partial^2 P}{\partial x \partial y} = 0 = \nabla^2 Q \end{aligned}\right\} \textbf{QED}$$

$\therefore$ Both P & Q are harmonic functions.

Back to Anti-plane Shear

The function $\omega(z) = \psi(x,y) + i\mu w(x,y)$ is analytic.

$$\frac{d\omega}{dz} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial \psi}{\partial x} + i\mu\frac{\partial w}{\partial x}\right) + \frac{1}{2i}\left(\frac{\partial \psi}{\partial y} + i\mu\frac{\partial w}{\partial y}\right)$$

$$\omega' = \frac{d\omega}{dz} = \frac{1}{2}\left(\frac{\partial \psi}{\partial x} + \mu\frac{\partial w}{\partial y}\right) + i\frac{1}{2}\left(\mu\frac{\partial w}{\partial x} - \frac{\partial \psi}{\partial y}\right)$$

Recall: $\mu \frac{\partial w}{\partial x} = \mu \gamma_{xz} = \sigma_{xz}$

$\mu \frac{\partial w}{\partial y} = \mu \gamma_{yz} = \sigma_{yz}$

$$-\frac{\partial \psi}{\partial y} = \sigma_{xz} , \quad \frac{\partial \psi}{\partial x} = \sigma_{yz}$$

$$\therefore \quad \omega'(z) = \frac{d\omega}{dz} = \frac{1}{2}(\sigma_{yz} + \sigma_{yz}) + i\,\frac{1}{2}(\sigma_{xz} + \sigma_{xz})$$

$$\omega'(z) = \sigma_{yz} + i\,\sigma_{xz} = \frac{1}{2\mu}(\varepsilon_{yz} + i\,\varepsilon_{xz})$$

$$w(x,y) = Im\,[\omega(z)] = \frac{1}{2i}\left[\omega(z) - \overline{\omega(z)}\right]$$

$$\omega(z) = \dots \underbrace{\frac{A_{-3}}{z^3} + \frac{A_{-2}}{z^2} + \frac{A_{-1}}{z} + A_{ln} \ln z}_{\text{not analytic at } z=0} + A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \dots$$

Note: the A_i are complex

Coming up with other functions takes insight and experience.

Ex) Uniform Stress field

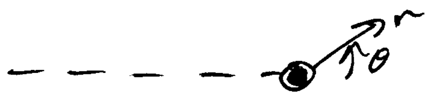
$$\omega'(z) = \sigma_{yz} + i\,\sigma_{xy}$$

does not change anything

$$\therefore \quad \omega(z) = (\sigma_{yz} + i\,\sigma_{xy})\, z + A + iB$$

rigid body disp.

Screw Dislocation



BCs: $w(r,\pi) - w(r,-\pi) = b$

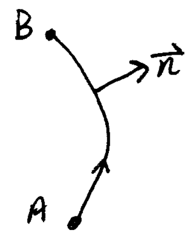
Recall for the in-plane $\perp$ $u_i \sim \ln r$

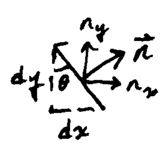
$$\therefore \text{ Try } \quad \omega(z) = (A+Bi)\ln z$$
$$= (A+Bi)\ln re^{i\theta}$$
$$= (A+Bi)\ln r + i\theta(A+Bi)$$

$$\therefore \quad w = B\ln r + A\theta$$
$$w(r,\pi) - w(r,-\pi) = \frac{1}{\mu}(A\pi - -A\pi) = b$$
$$\therefore \quad A = \frac{b\mu}{2\pi}$$

What about B?

Net Force on an arc: (*Note $\vec{n}$ to right $\rightarrow$ A to B limits
$\vec{n}$ to left $\rightarrow$ B to A limits)





$$n_x = \frac{dy}{ds}, \quad n_y = \frac{-dx}{ds}$$

$$T_z = \sigma_{xz} n_x + \sigma_{yz} n_y$$

$$T_z = -\frac{\partial\psi}{\partial y}\frac{dy}{ds} - \frac{\partial\psi}{\partial x}\frac{dx}{ds} = \text{[redacted]} -\frac{d\psi}{ds}$$

$$\text{Net force} = \int_A^B -\frac{d\psi}{ds}\,ds = -\psi\Big]_A^B \text{ [redacted]}$$

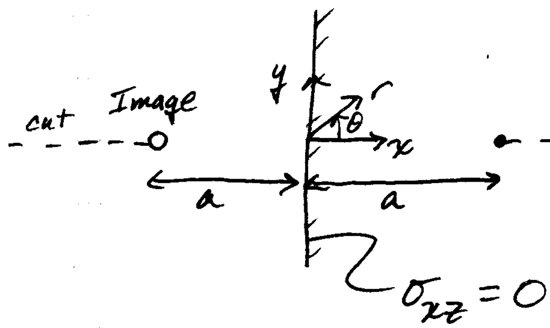
$$\therefore \quad F_z = -\mathrm{Re}\left[\omega(z)\right]\Big]_A^B \text{ [redacted]}$$

$\subset$ — $\rightarrow$ $\vec{n}$ to right of arc

+ $\rightarrow$ $\vec{n}$ to left of arc

Page 63 finishes the • solution

Screw Dislocation Near a free Surface



Infinite solution:
$$\sigma_{yz} = \frac{\mu b}{2\pi} \frac{x}{x^2+y^2}$$
$$\sigma_{xz} = \frac{-\mu b}{2\pi} \frac{y}{x^2+y^2}$$

• Solution:
$$\sigma_{yz} = \frac{\mu b}{2\pi} \frac{x-a}{(x-a)^2+y^2}$$
$$\sigma_{xz} = \frac{-\mu b}{2\pi} \frac{y}{(x-a)^2+y^2}$$

o Solution:
$$\sigma_{yz} = \frac{-\mu b\,(x+a)}{2\pi\left[(x+a)^2+y^2\right]}$$
$$\sigma_{xz} = \frac{\mu b}{2\pi} \frac{y}{(x+a)^2+y^2}$$

Total:
$$\sigma_{yz} = \frac{\mu b}{2\pi}\left[\frac{x-a}{(x-a)^2+y^2} - \frac{x+a}{(x+a)^2+y^2}\right]$$
$$\sigma_{xz} = \frac{\mu b}{2\pi}\left[\frac{-y}{(x-a)^2+y^2} + \frac{y}{(x+a)^2+y^2}\right]$$

$$\sigma_{xz}(x=0, y) = 0 \checkmark$$

$$U = \frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{1}{\mu}\left(\sigma_{yz}^2 + \sigma_{xz}^2\right) dx\,dy$$

$$= \frac{1}{2}\int_{a}^{\infty} b\,\sigma_{yz}(y=0)\,dx = \frac{1}{2}\int_{a}^{\infty} \frac{\mu b^2 a}{\pi(x^2-a^2)}\,dx$$

$$U = \frac{-\mu b^2 a}{2\pi}\left[\frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|\right]_{a+r_0}^{\infty} = \frac{\mu b^2}{4\pi}\ln\left|\frac{2a}{r_0}\right| \text{ b/c } a \gg r_0$$

$$\boxed{F = -\frac{\partial U}{\partial a} = -\frac{\mu b^2}{4\pi}\frac{r_0}{2a}\frac{2}{r_0} = \frac{-\mu b^2}{4\pi a}}$$

$$\therefore \quad F_z = -A \ln r + B\theta \Big]_{-\pi}^{\pi}$$

$$= 2B\pi = 0 \rightarrow B = 0$$

$$\therefore \quad \omega(z) = \frac{\mu b}{2\pi} \ln z$$

$$\omega'(z) = \frac{\mu b}{2\pi}\frac{1}{z} = \frac{\mu b}{2\pi} r^{-1} e^{-i\theta}$$
$$= \frac{\mu b}{2\pi}\frac{1}{r}(\cos\theta - i\sin\theta)$$

$$\sigma_{yz} + i\sigma_{xz} = \frac{\mu b}{2\pi r}(\cos\theta - i\sin\theta)$$

$$\sigma_{rz} = \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta = \frac{\mu b}{2\pi r}\left[-\sin\theta\cos\theta + \sin\theta\cos\theta\right] = 0$$
$$\sigma_{\theta z} = -\sigma_{xz}\sin\theta + \sigma_{yz}\cos\theta = \frac{\mu b}{2\pi r}\left[\sin^2\theta + \cos^2\theta\right] = \frac{\mu b}{2\pi r}$$

$$\therefore \quad \sigma_{rz} = 0, \quad \sigma_{\theta z} = \frac{\mu b}{2\pi r}$$

$$w(r,\theta) = \frac{1}{\mu}\frac{\mu b}{2\pi}\theta = \frac{b}{2\pi}\theta$$

10/4/01 **Mode III Asymptotic Crack Tip Field**

In-plane $\rightarrow \sigma_{ij} \sim \frac{1}{\sqrt{r}}, \; u_i \sim \sqrt{r}$

try $\omega(z) = (A + Bi)\, z^p$

BCs: $\sigma_{yz}(x<0, y=0) = 0$, or $\sigma_{yz}(r, \theta = \pm\pi) = 0$

$$\omega'(z) = p(A + Bi)\, z^{p-1}$$

$$\therefore \quad \sigma_{yz} + i\sigma_{xz} = p(A+Bi)\, r^{p-1}\left[\cos(p-1)\theta + i\sin(p-1)\theta\right]$$

$$\sigma_{yz} = p(A)\, r^{p-1}\cos(p-1)\theta - p(B)\, r^{p-1}\sin(p-1)\theta$$

$$\sigma_{yz}(r, \theta=\pm\pi) = 0 \rightarrow A\cos(p-1)\pi - B\sin(p-1)\pi = 0$$

$$\therefore \quad p-1 = \pm 1, \pm 2, \pm 3, \ldots \text{ and } A = 0$$
$$\text{or } \quad p-1 = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots \text{ and } B = 0$$

Singular $\rightarrow p-1 < 0$, Exclusion $\rightarrow p-1 > -1$

$$\therefore \quad 0 < p < 1$$
$$\rightarrow p = \tfrac{1}{2} \text{ and } B = 0$$

furthermore $\sigma_{yz}(x>0, y=0) = \dfrac{K_{III}}{\sqrt{2\pi r}}$

$$\sigma_{yz} = \tfrac{1}{2} A\, r^{-1/2}\cos\frac{\theta}{2}$$

$$\sigma_{yz}(r, \theta=0) = \frac{A}{2\sqrt{r}} = \frac{K_{III}}{\sqrt{2\pi r}} \rightarrow A = K_{III}\sqrt{\frac{2}{\pi}}$$

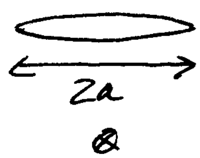
$$\therefore \quad \boxed{\omega(z) = K_{III}\sqrt{\frac{2}{\pi}}\,\sqrt{z}} = K_{III}\sqrt{\frac{2r}{\pi}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$\boxed{\begin{aligned} \sigma_{yz} + i\sigma_{xz} &= \frac{K_{III}}{\sqrt{2\pi}}\, z^{-1/2} = \frac{K_{III}}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) \\ w &= \frac{1}{\mu} K_{III}\sqrt{\frac{2r}{\pi}}\sin\frac{\theta}{2} \end{aligned}}$$

$$\mathcal{G}\,\delta a = \frac{1}{2}\int_0^{\delta a} \sigma_{yz}(x)\, CSD(x)\, dx = \frac{1}{2}\int_0^{\delta a} \frac{K_{III}}{\sqrt{2\pi x}}\,\frac{2}{\mu} K_{III}\sqrt{\frac{2(\delta a - x)}{\pi}}\, dx$$

$$= \frac{K_{III}^2}{\pi\mu}\int_0^{\delta a}\sqrt{\frac{\delta a - x}{x}}\, dx = \frac{K_{III}^2}{2\mu}\,\delta a \rightarrow \boxed{\mathcal{G} = \frac{K_{III}^2}{2\mu}}$$

$\odot\ \sigma_{yz}(r\to\infty)=\sigma$



$$\sigma_{yz}(|x|<a,\ y=0)=0$$

$\therefore$ as $z\to\infty$ $\quad \omega'(z)=\sigma$

as $z\to a$ $\quad \omega'(z)\to \dfrac{K_{III}^{r}}{\sqrt{2\pi}}\dfrac{1}{\sqrt{z-a}}$

as $z\to -a$ $\quad \omega'(z)\to \dfrac{K_{III}^{\ell}}{\sqrt{2\pi}}\dfrac{1}{\sqrt{z+a}}$

$$\omega'(z)=\frac{\sigma z}{\sqrt{(z-a)(z+a)}}=\frac{\sigma z}{\sqrt{z^2-a^2}}$$

Check BC on crack : $z=x\ ,\ |x|<a$

$$\omega'(z)=\frac{\sigma x}{\sqrt{x^2-a^2}}=-\frac{i\sigma x}{\sqrt{a^2-x^2}}\quad \text{for } |x|<a$$

$$\therefore\ \sigma_{\cancel{x}z}=\frac{-\sigma x}{\sqrt{a^2-x^2}}\ ,\ \sigma_{yz}=0\quad \text{for } |x|<a,\ y=0$$

$$\lim_{z\to a}\omega'(z)=\frac{\sigma a}{\sqrt{(z-a)\,2a}}=\sigma\sqrt{\frac{a}{2}}\frac{1}{\sqrt{z-a}}=\frac{K_{III}}{\sqrt{2\pi}}\frac{1}{\sqrt{z-a}}$$

$$\therefore\ \boxed{K_{III}=\sigma\sqrt{\pi a}}$$

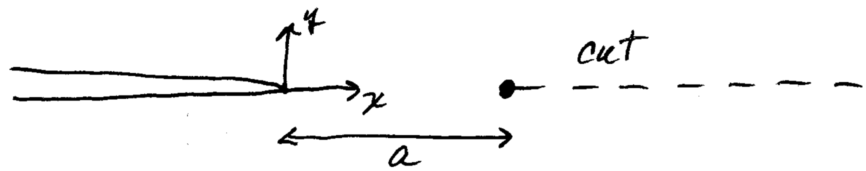
The following branch cuts need to be used

$\rightarrow\ z=re^{i\theta}$

$\sqrt{z-a}=\sqrt{r_1}\,e^{i\theta_1/2}$

$\sqrt{z+a}=\sqrt{r_2}\,e^{i\theta_2/2}$

Screw Dislocation near a crack



try $\quad \omega'(z) = \dfrac{A}{\sqrt{z}\,(z-a)}$

as $\quad z \to a \quad \omega'(z) \to \dfrac{\mu b}{2\pi (z-a)} = \dfrac{A}{\sqrt{a}\,(z-a)}$

$\therefore \quad A = \dfrac{\mu b \sqrt{a}}{2\pi}$

$$\omega'(z) = \frac{\mu b \sqrt{a}}{2\pi \sqrt{z}\,(z-a)}$$

Check BCs

$$\underset{x<0}{\omega'(z=x)} = \frac{\mu b \sqrt{a}}{2\pi \sqrt{x}\,(x-a)} = \frac{\mu b \sqrt{a}}{2\pi i \sqrt{|x|}\,(x-a)} = \sigma_{yz} + i\sigma_{xz}$$

$\therefore \; \sigma_{yz} = 0 \;$ for $\; x<0 \quad \checkmark$

$\omega(z) = -\dfrac{\mu b}{\pi} \operatorname{arctanh} \sqrt{\dfrac{z}{a}} \quad$ (Integrate $\omega'(z)$ with Mathematica)

for $z = x,\; x > a,\; y = 0^+ \quad \omega(z) = \dfrac{-\mu b}{\pi} \operatorname{arctanh} \sqrt{\dfrac{x}{a}}$

$$= -\frac{\mu b}{\pi}\left[\frac{1}{2}\ln\left(\frac{x+a}{x-a}\right) + i\frac{\pi}{2}\right]$$

for $y = 0^- \quad \omega(z) = -\dfrac{\mu b}{\pi}\left[\dfrac{1}{2}\ln\left(\dfrac{x+a}{x-a}\right) - i\dfrac{\pi}{2}\right]$

$\therefore \; w(x>a, y=0^-) - w(x>a, y=0^+) = \dfrac{1}{\mu}\left[\dfrac{\mu b}{\pi}\dfrac{\pi}{2} - \dfrac{-\mu b}{\pi}\dfrac{\pi}{2}\right]$

$= b \quad \checkmark$

What is K_{III} for this situation?

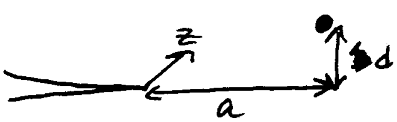
$$\omega'(z) \to \frac{K_{III}}{\sqrt{2\pi z}} \quad \text{as } z \to 0$$

$$\therefore \quad \frac{\mu b \sqrt{a}}{2\pi \sqrt{z}\,(-a)} = \frac{K_{III}}{\sqrt{2\pi z}}$$

$$\therefore \quad K_{III} = \frac{-\mu b}{\sqrt{2\pi a}}$$

Note that the − sign appears because $\sigma_{yz} < 0$ to the left of the •.

This type of "trick" does not always work. For example

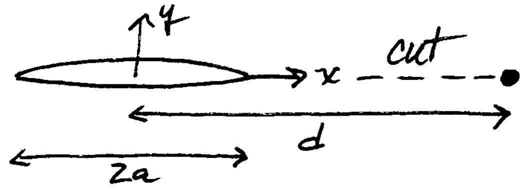


$$\omega'(z) \cancel{=} \frac{\mu b \sqrt{a+id}}{\sqrt{z}\,[z-(a+id)]}$$

This is not the correct complex potential.

What I have noticed is that this "tends" to work when all of the singularities lie on the x-axis. However, I would expect that this is not a general rule.

Another example with singularities on the x-axis



No applied stress at $z \to \infty$.

$$\omega'(z) = \frac{\mu b \sqrt{d^2-a^2}}{2\pi\sqrt{z^2-a^2}\,(z-d)} \quad , \; d>a$$

Now, what if we want applied stress $\sigma_{yz} = \sigma$ at $z \to \infty$. We need to superpose a solution that has no traction on the crack faces and in general no displacement jumps elsewhere.

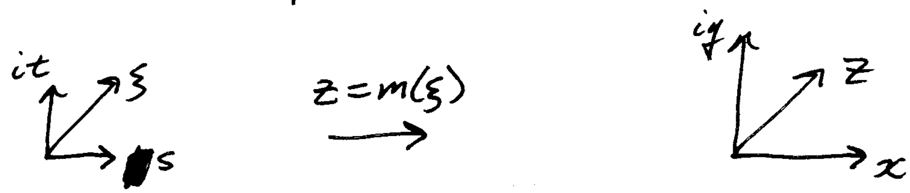
$$\therefore \; \omega'(z) = \frac{\mu b \sqrt{d^2-a^2}}{2\pi\sqrt{z^2-a^2}\,(z-d)} + \frac{\sigma z}{\sqrt{z^2-a^2}}$$

superposition implies addition of solutions.

Note: I figured out where the cut was after integrating $\omega'(z)$ to get $\omega(z)$ and investigating. I'm not entirely sure how to get the cut on the other side. I will have to look at this some more.

Appropriate manipulations can move the cut to other angles.

Conformal Mapping



$m(\xi)$ maps points from the ξ-plane onto the z-plane. m must be analytic.

$$\omega_z'(z) = \frac{\partial \omega_\xi}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial \omega_\xi}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}$$

ω analytic $\rightarrow \frac{\partial \omega}{\partial \bar{\xi}} = 0$

$$\left.\begin{aligned} dz &= \frac{\partial z}{\partial \xi}d\xi + \frac{\partial z}{\partial \bar{\xi}}d\bar{\xi} \\ d\bar{z} &= \frac{\partial \bar{z}}{\partial \xi}d\xi + \frac{\partial \bar{z}}{\partial \bar{\xi}}d\bar{\xi} \end{aligned}\right\} \quad \begin{Bmatrix} dz \\ d\bar{z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \bar{\xi}} \\ \frac{\partial \bar{z}}{\partial \xi} & \frac{\partial \bar{z}}{\partial \bar{\xi}} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\bar{\xi} \end{Bmatrix}$$

$$\frac{\partial z}{\partial \xi} = m'(\xi), \quad \frac{\partial z}{\partial \bar{\xi}} = 0$$

$$\bar{z} = \bar{m}(\bar{\xi}) \rightarrow \frac{\partial \bar{z}}{\partial \xi} = 0, \quad \frac{\partial \bar{z}}{\partial \bar{\xi}} = \bar{m}'(\bar{\xi})$$

$$\therefore \begin{Bmatrix} d\xi \\ d\bar{\xi} \end{Bmatrix} = \begin{bmatrix} 1/m'(\xi) & 0 \\ 0 & 1/\bar{m}'(\bar{\xi}) \end{bmatrix} \begin{Bmatrix} dz \\ d\bar{z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial \bar{z}} \\ \frac{\partial \bar{\xi}}{\partial z} & \frac{\partial \bar{\xi}}{\partial \bar{z}} \end{bmatrix} \begin{Bmatrix} dz \\ d\bar{z} \end{Bmatrix}$$

$$\therefore \frac{\partial \xi}{\partial z} = 1/m'(\xi), \quad \frac{\partial \bar{\xi}}{\partial z} = 0$$

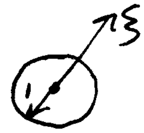
$$\therefore \omega_z'(z) = \frac{d\omega_\xi}{d\xi}\frac{1}{m'(\xi)} = \frac{d\omega_\xi}{d\xi} M'(z)$$

where $M = m^{-1}$ note $\omega_\xi(\xi) = \omega_z(m(\xi))$

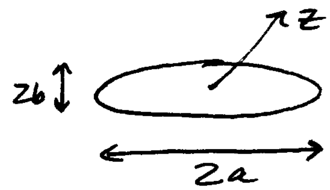
Depending on the problem it may be easier to use $\frac{1}{m'}$ instead of M'.

10/9/01

The elliptical hole



$$z = \frac{a+b}{2}\xi + \frac{a-b}{2}\frac{1}{\xi} \quad \xrightarrow{\text{map}}$$



Note $\xi = \frac{a+b}{a^2+b^2} z - \frac{ab(a-b)}{a^2+b^2}\frac{1}{z}$ is not a good map
i.e. it is not $m^{-1}(\xi)$

BCs: $\omega_z'(z\to\infty) = \sigma$
$\mathrm{Re}\left[\omega_\xi(\xi = e^{i\theta})\right] = 0$ (no force on any part of the arc)

$$m'(\xi) = \frac{a+b}{2} - \frac{a-b}{2}\frac{1}{\xi^2}$$

$z\to\infty \rightarrow \xi\to\infty$

$$\omega_z'(z) = \omega_\xi'(\xi)\frac{1}{m'(\xi)} \rightarrow \omega_\xi'(\xi)\frac{2}{a+b} = \sigma$$

$$\therefore \omega_\xi'(\xi) \rightarrow \frac{\sigma(a+b)}{2} \quad \text{as } z,\xi\to\infty$$

$$\omega_\xi(\xi) \rightarrow \frac{\sigma(a+b)}{2}\xi \quad \text{as } z,\xi\to\infty$$

$$\omega_\xi(\xi = e^{i\theta}) = \frac{\sigma(a+b)}{2}e^{i\theta} \underbrace{- \frac{\sigma(a+b)}{2}e^{-i\theta}}_{\text{subtract complex conjugate}}$$

$$-\frac{\sigma(a+b)}{2}e^{-i\theta} = -\frac{\sigma(a+b)}{2}\frac{1}{\xi}$$

We could get this in terms of z but it is a long mess and not any clearer.

$$\therefore \omega_\xi(\xi) = \frac{\sigma(a+b)}{2}\left(\xi - \frac{1}{\xi}\right)$$

$$\boxed{\omega_z'(z) = \sigma_{yz} + i\sigma_{xz} = \frac{\sigma(a+b)}{2}\left(1+\frac{1}{\xi^2}\right)\frac{1}{\frac{a+b}{2} - \frac{a-b}{2}\frac{1}{\xi^2}}}$$

$$\omega_z'(z=\pm a) = \omega_z'(\xi=\pm 1) = \frac{\sigma(a+b)}{\cancel{2}}(\cancel{1+1})\frac{1}{\frac{a+b}{2} - \frac{a-b}{2}}$$

$$\sigma_{yz}(x=\pm a, y=0) = \sigma\left(1+\frac{a}{b}\right), \quad \sigma_{xz}(x=\pm a, y=0) = 0$$

Do we recover the crack solution.

Investigate $b \to 0 \to z = \frac{a}{2}\xi + \frac{a}{2}\frac{1}{\xi}$

$$\to \frac{a}{2}\xi^2 - \xi z + \frac{a}{2} = 0 \to \xi = \frac{z \pm \sqrt{z^2-a^2}}{a}$$

$$\omega_z'(z) = \frac{\sigma \cancel{a}}{\cancel{2}}\left(1+\frac{1}{\xi^2}\right)\frac{\cancel{2}}{\cancel{a}\left(1-\frac{1}{\xi^2}\right)} = \sigma\,\frac{\xi^2+1}{\xi^2-1}$$

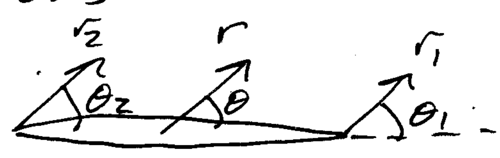
$$\xi^2 = \frac{1}{a^2}\left[z^2 \pm 2z\sqrt{z^2-a^2} + z^2 - a^2\right]$$

$$\therefore\ \omega_z'(z) = \frac{\sigma\left(\cancel{2}z^2 \pm \cancel{2}z\sqrt{z^2-a^2}\right)}{\cancel{2}z^2 \pm \cancel{2}z\sqrt{z^2-a^2} - \cancel{2}a^2}$$

$$= \frac{\sigma z\left(z \pm \sqrt{z^2-a^2}\right)}{\pm\sqrt{z^2-a^2}\left(z \pm \sqrt{z^2-a^2}\right)}$$

Choose + sign $\to \omega_z'(z) = \dfrac{\sigma z}{\sqrt{z^2-a^2}}$

+ sign is valid for the standard branch cut conventions



$$z \to r e^{i\theta}$$
$$z - a \to r_1 e^{i\theta_1}$$
$$z + a \to r_2 e^{i\theta_2}$$