10/11/01

## In-plane Complex Potentials

$$\left.\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} &= 0\end{aligned}\right\} \quad \begin{aligned}\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} \\ \sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}\end{aligned}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1+\nu'}{E'}\sigma_{xx} - \frac{\nu'}{E'}(\sigma_{xx}+\sigma_{yy}) = \frac{1}{E'}(\sigma_{xx}+\sigma_{yy}) - \frac{1+\nu'}{E'}\sigma_{yy}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1+\nu'}{E'}\sigma_{yy} - \frac{\nu'}{E'}(\sigma_{xx}+\sigma_{yy}) = \frac{1}{E'}(\sigma_{xx}+\sigma_{yy}) - \frac{1+\nu'}{E'}\sigma_{xx}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1+\nu')}{E'}\sigma_{xy}$$

All of these $\longrightarrow$ $\boxed{\nabla^4 \phi = \nabla^2(\nabla^2 \phi) = 0}$

Recall: $$\left.\begin{aligned} z &= x+iy \\ \bar{z} &= x-iy \end{aligned}\right\} \quad \begin{aligned} x &= \tfrac{1}{2}z + \tfrac{1}{2}\bar{z} \\ y &= \tfrac{1}{2i}z - \tfrac{1}{2i}\bar{z}\end{aligned}$$

$$\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial y^2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = -\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

$$\nabla 4 = \nabla^2(\nabla^2) = 16\frac{\partial^4}{\partial z^2 \partial \bar{z}^2}$$

$$\therefore \quad \nabla^4\phi = 16\frac{\partial^4\phi}{\partial z^2\partial\bar{z}^2} = 0$$

$$\therefore \phi = \bar{z}f(z) + zg(\bar{z}) + h(z) + k(\bar{z})$$

However $\phi$ is real $\rightarrow$ $zg(\bar{z}) = \overline{\bar{z}f(z)} = z\overline{f(z)}$

$$\therefore g(\bar{z}) = \overline{f(z)}$$

$$k(\bar{z}) = \overline{h(z)}$$

$$\therefore \phi = \bar{z}f(z) + \overline{\bar{z}f(z)} + h(z) + \overline{h(z)}$$

$$\phi = \frac{1}{2}\left[\bar{z}F(z) + z\overline{F(z)} + G(z) + \bar{G}(\bar{z})\right]$$

$$\boxed{\phi = Re\left[\bar{z}F(z) + G(z)\right]}, \quad F(z) = 2f(z), \quad G(z) = 2h(z)$$

<u>Aside</u>

$f(z) = (1+i)z^2 + (1+2i)$ analytic

$f(\bar{z}) = (1+i)\bar{z}^2 + (1+2i)$ not analytic

$\bar{f}(z) = (1-i)z^2 + (1-2i)$ analytic

$\bar{f}(\bar{z}) = \overline{f(z)} = (1-i)\bar{z}^2 + (1-2i)$ not analytic

$\overline{f(\bar{z})} = \bar{f}(z)$, $\overline{\bar{f}(z)} = f(\bar{z})$, etc.

As long as F & G are analytic then $\phi$ as defined here satisfies all of the field equations of in-plane elasticity.

— How do we get stresses?

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial z^2} + 2\frac{\partial^2\phi}{\partial z\partial\bar{z}} + \frac{\partial^2\phi}{\partial\bar{z}^2}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -\frac{\partial^2\phi}{\partial z^2} + 2\frac{\partial^2\phi}{\partial z\partial\bar{z}} - \frac{\partial^2\phi}{\partial\bar{z}^2}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right)i\left(\frac{\partial\phi}{\partial z}-\frac{\partial\phi}{\partial\bar{z}}\right) = -i\frac{\partial^2\phi}{\partial z^2}+i\frac{\partial^2\phi}{\partial\bar{z}^2} = -i\left(\frac{\partial^2\phi}{\partial z^2}-\frac{\partial^2\phi}{\partial\bar{z}^2}\right)$$

$$\therefore \boxed{\sigma_{xx}+\sigma_{yy} = 4\frac{\partial^2\phi}{\partial z\partial\bar{z}} = \frac{1}{2}\left[4F'(z)+4\overline{F'(z)}\right] = 4\,\mathrm{Re}\left[F'(z)\right]}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2\frac{\partial^2\phi}{\partial z^2}+2\frac{\partial^2\phi}{\partial\bar{z}^2}+2\frac{\partial^2\phi}{\partial z^2}-2\frac{\partial^2\phi}{\partial\bar{z}^2} = 4\frac{\partial^2\phi}{\partial z^2}$$

$$\therefore \boxed{\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = \frac{1}{2}\left[4\bar{z}F''(z)+4G''(z)\right] = 2\left[\bar{z}F''(z)+G''(z)\right]}$$

— How about displacements?

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E'}\nabla^2\phi - \frac{1+\nu'}{E'}\frac{\partial^2\phi}{\partial x^2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1}{E'}\nabla^2\phi - \frac{1+\nu'}{E'}\frac{\partial^2\phi}{\partial y^2}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = \frac{2(1+\nu')}{E'}\frac{\partial^2\phi}{\partial x\partial y}$$

Let's deal with $\nabla^2\phi$. $\nabla^2\phi = \sigma_{xx}+\sigma_{yy} = 4\,\mathrm{Re}[F'(z)]$

let $F(z) = p(x,y)+iq(x,y)$, $p=\mathrm{Re}[F(z)]$
$q=\mathrm{Im}[F(z)]$

$$F'(z) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)$$

$$\therefore\ \mathrm{Re}[F'(z)] = \frac{1}{2}\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}\right)$$

However, the Cauchy-Riemann conditions

$$\rightarrow \quad \frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$$

$$\therefore \quad \mathrm{Re}\left[F'(z)\right] = \frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}$$

$$\nabla^2 \phi = 4\,\mathrm{Re}\left[F'(z)\right] = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y}$$

$$\rightarrow u = \frac{4}{E'} p(x,y) - \frac{1+\nu'}{E'}\frac{\partial \phi}{\partial x} + f(y)$$

$$v = \frac{4}{E'} q(x,y) - \frac{1+\nu'}{E'}\frac{\partial \phi}{\partial y} + g(x)$$

$$\therefore \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{4}{E'}\frac{\partial p}{\partial y} - \frac{1+\nu'}{E'}\frac{\partial^2 \phi}{\partial x \partial y} + f'(y)$$

$$+ \frac{4}{E'}\frac{\partial q}{\partial x} - \frac{1+\nu'}{E'}\frac{\partial^2 \phi}{\partial x \partial y} + g'(x)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{4}{E'}\underbrace{\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right)}_{=0 \text{ due to C-R}} - \cancel{\frac{2(1+\nu')}{E'}\frac{\partial^2 \phi}{\partial x \partial y}} + f'(y) + g'(x) = \cancel{\frac{-2(1+\nu')}{E'}\frac{\partial^2 \phi}{\partial x \partial y}}$$

$$\therefore \quad f'(y) + g'(x) = 0 \rightarrow f'(y) = -g'(x) = c$$

$$\left.\begin{array}{l} f(y) = cy + d \\ g(x) = -cx + e \end{array}\right\}$$

Rigid body motion
$d$ = $x$ translation
$e$ = $y$ translation
$c$ = rotation in $x$-$y$ plane

To fix the body we can take $f = g = 0$

$$\therefore \quad u = \frac{4}{E'} p(x,y) - \frac{1+\nu'}{E'}\frac{\partial \phi}{\partial x}$$

$$v = \frac{4}{E'} q(x,y) - \frac{1+\nu'}{E'}\frac{\partial \phi}{\partial y}$$

$$p(x,y) = Re[F(z)] \quad , \quad q(x,y) = Im[F(z)]$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} \quad , \quad \frac{\partial \phi}{\partial y} = i\left(\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial \bar{z}}\right)$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{2}\left[\bar{z} F'(z) + \overline{F(z)} + G'(z)\right]$$

$$\frac{\partial \phi}{\partial \bar{z}} = \frac{1}{2}\left[F(z) + z\overline{F'(z)} + \overline{G}'(\bar{z})\right]$$

$$u + iv = \frac{4}{E'} F(z) - \frac{1+\nu'}{2E'}\left[\cancel{\bar{z} F'(z)} + z\overline{F'(z)} + F(z) + \cancel{\overline{F(z)}} + \cancel{G'(z)} + \overline{G}'(\bar{z})\right]$$
$$- \frac{1+\nu'}{2E'}\left[-\cancel{\bar{z} F'(z)} + z\overline{F'(z)} + F(z) - \cancel{\overline{F(z)}} + \overline{G}'(\bar{z}) - \cancel{G'(z)}\right]$$

$$= \frac{4}{E'} F(z) - \frac{1+\nu'}{\cancel{2}E'}\left[2z\overline{F'(z)} + \cancel{2}F(z) + \cancel{2}\overline{G}'(\bar{z})\right]$$

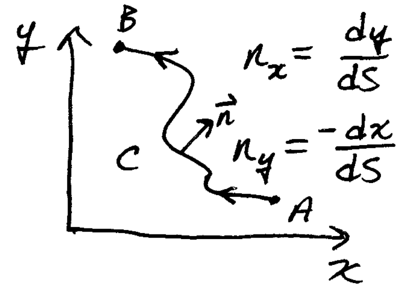
$$2\mu(u+iv) = \frac{4}{1+\nu'} F(z) - z\overline{F'(z)} - F(z) - \overline{G}'(\bar{z})$$

$$2\mu(u+iv) = \frac{3-\nu'}{1+\nu'} F(z) - z\overline{F'(z)} - \overline{G'(z)}$$

$$\boxed{2\mu(u+iv) = \varkappa F(z) - z\overline{F'(z)} - \overline{G'(z)}}$$

$$\varkappa = \begin{cases} \frac{3-\nu}{1+\nu} & \text{plane stress} \\ 3-4\nu & \text{plane strain} \end{cases}$$

## – Net Force on an Arc




$$F_x = \int_C \sigma_{xx} n_x + \sigma_{xy} n_y \, dS$$

$$F_y = \int_C \sigma_{yy} n_y + \sigma_{xy} n_x \, dS$$

$$F_x = \int_C \frac{\partial^2 \phi}{\partial y^2} \frac{dy}{dS} + \frac{\partial^2 \phi}{\partial x \partial y} \frac{dx}{dS} \, dS$$

$$F_x = \int_C \frac{d}{dS}\left(\frac{\partial \phi}{\partial y}\right) dS$$

$$F_y = \int_C -\frac{\partial^2 \phi}{\partial x^2} \frac{dx}{dS} - \frac{\partial^2 \phi}{\partial x \partial y} \frac{dy}{dS} dS = \int_C -\frac{d}{dS}\left(\frac{\partial \phi}{\partial x}\right) dS$$

$$F_x + i F_y = \int_C \frac{d}{dS}\left(\frac{\partial \phi}{\partial y} - i \frac{\partial \phi}{\partial x}\right) dS$$

$$= \int_C \frac{d}{dS}\left[\cancel{i \frac{\partial \phi}{\partial z}} - i \frac{\partial \phi}{\partial \bar{z}} - \cancel{i \frac{\partial \phi}{\partial z}} - i \frac{\partial \phi}{\partial \bar{z}}\right] dS = \int_C \frac{d}{dS}\left(-2i \frac{\partial \phi}{\partial \bar{z}}\right) dS$$

$$= \int_C \frac{d}{dS} (-2i) \frac{1}{2} \left[ F(z) + z \overline{F'(z)} + \overline{G'(z)} \right] dS$$

$$\boxed{F_x + i F_y = -i \left[ F(z) + z \overline{F'(z)} + \overline{G'(z)} \right]_A^B}$$

* Note $\vec{n}$ is always pointing to the right of the arc.

$$F_x + i F_y = i \left[ F(z) + z \overline{F'(z)} + \overline{G'(z)} \right]_A^B$$

if $\vec{n}$ points to the left of the arc.

10/18/01

## Polar / Cylindrical Components

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - 2\sigma_{r\theta}\sin\theta\cos\theta$$

$$\sigma_{yy} = \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + 2\sigma_{r\theta}\sin\theta\cos\theta$$

$$\sigma_{xy} = (\sigma_{rr} - \sigma_{\theta\theta})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = (\sigma_{\theta\theta} - \sigma_{rr})\cos 2\theta + \overset{-2i(i)}{2}\sigma_{r\theta}\sin 2\theta$$
$$-2i(\sigma_{\theta\theta} - \sigma_{rr})\tfrac{1}{2}\sin 2\theta + 2i\sigma_{r\theta}\cos 2\theta$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = [(\sigma_{\theta\theta} - \sigma_{rr}) + 2i\sigma_{r\theta}]e^{-2i\theta}$$

$$u_x = u_r\cos\theta - u_\theta\sin\theta$$
$$u_y = u_r\sin\theta + u_\theta\cos\theta$$
$$u_x + iu_y = u_r(\cos\theta + i\sin\theta) + iu_\theta(\cos\theta + i\sin\theta)$$
$$= (u_r + iu_\theta)e^{i\theta}$$

$$\boxed{\begin{aligned} \therefore\ \sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} &= 2[\bar{z}F''(z) + G''(z)]e^{2i\theta} \\ \sigma_{rr} + \sigma_{\theta\theta} &= 4\,\mathrm{Re}[F'(z)] \\ 2\mu(u_r + iu_\theta) &= [\kappa F(z) - z\overline{F'(z)} - \overline{G'(z)}]e^{-i\theta} \end{aligned}}$$

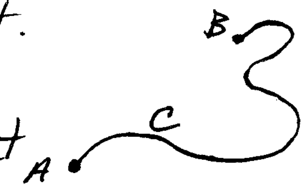
# Complex Analysis Handout



## Complex Integration

– Assertion: The contour integral of an analytic function $f(z)$ depends only on its endpoints, i.e. it is path independent.

$$\int_C f(z)\,dz \text{ is path independent}$$



– Proof: $f(z) = u(x,y) + i\,v(x,y)$

$$\text{C-R} \rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$dz = dx + i\,dy$$

$$\int_C f(z)dz = \int_C \left[(u\,dx - v\,dy) + i(u\,dy + v\,dx)\right]$$

but due to C-R $u$ and $v$ can be derived from the harmonic functions $\phi$ and $\psi$ as

$$u = \frac{\partial \phi}{\partial x} \qquad u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \phi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

$$\nabla^2\phi = 0 \qquad \nabla^2\psi = 0$$

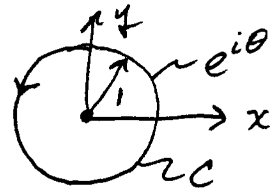
then $$\int_C f(z)dz = \int_C \left[\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right) + i\left(\frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx\right)\right]$$

$$= \int_C d\phi + i\,d\psi \leftarrow \text{total differential}$$

$$\int_C f(z)dz = \phi + i\psi \Big]_A^B \qquad \text{QED}$$

Note that $f(z)$ must be analytic at all points on $C$.

## Closed contour integrals




$dz = ie^{i\theta}d\theta$

$r = 1$

$$\oint_C z^n\, dz = \int_0^{2\pi} \cancel{r^n} e^{in\theta}\, i e^{i\theta} d\theta$$

$$= \int_0^{2\pi} i\, e^{i(n+1)\theta}\, d\theta$$

$$= i \int_0^{2\pi} \cos(n+1)\theta + i\sin(n+1)\theta \; d\theta$$

$$= i \left[\frac{1}{n+1}\sin(n+1)\theta - \frac{i}{n+1}\cos(n+1)\theta\right]_0^{2\pi}$$

$$\boxed{\begin{array}{l} \oint_C z^n dz = 0 \quad \text{if } n \neq -1 \;,\; n \in \text{Integers} \\ \oint_C z^{-1} dz = i\int_0^{2\pi} d\theta = 2\pi i \quad \text{for } n = -1 \end{array}}$$

We could have also noted that

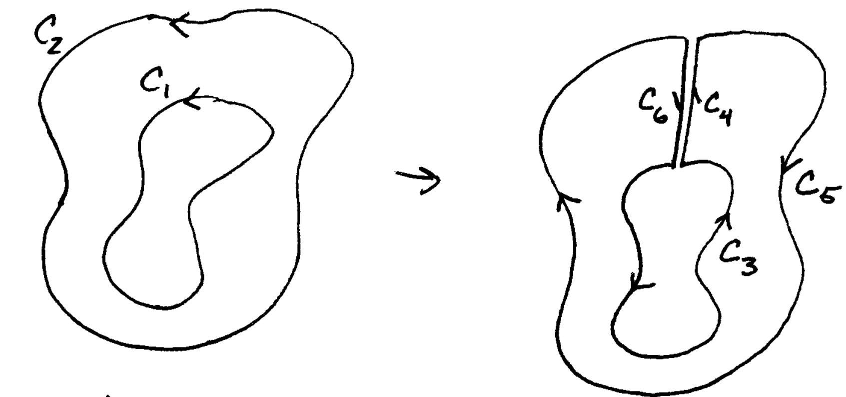
$$\oint_C z^n\, dz = \begin{cases} \frac{1}{n+1} z^{n+1} \Big]_0^{e^{i2\pi}} & \text{for } n \neq -1 \\ \ln z \Big]_{e^{i0}}^{e^{i2\pi}} & \text{for } n = -1 \end{cases}$$

$$= \begin{cases} 0 & \text{for } n \neq -1 \,,\; n \in I \\ \ln e^{2\pi i} - \ln 1 = 2\pi i & \text{for } n = -1 \end{cases}$$

– Assertion: The integral of $f(z)$ around any two closed contours is equal if $f(z)$ is analytic on both contours and within the region bounded by the contours.

CA3

Proof :




Since $f(z)$ is analytic on and between $C_1$ and $C_2$ and since $\int_C f(z)dz$ depends only on endpoints

$$\oint_{C_3+C_4+C_5+C_6} f(z)dz = 0$$

if $C_4$ and $C_6$ are differentially close together then $\int_{C_4} f(z)\,dz = -\int_{C_6} f(z)dz$

$$\oint_{C_3+C_4+C_5+C_6} f(z)dz = \int_{C_3} f(z)dz + \cancel{\int_{C_4} f(z)dz} + \int_{C_5} f(z)dz + \cancel{\int_{C_6} f(z)dz}$$

$$= 0$$

$$\therefore \int_{C_3} f(z)\,dz = -\int_{C_5} f(z)\,dz$$

but again if $C_4$ & $C_6$ are differentially close then $\int_{C_3} f(z)dz = \oint_{C_1} f(z)\,dz$ and $-\int_{C_5} f(z)dz = \oint_{C_2} f(z)dz$

$$\boxed{\therefore \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz}$$

if $f(z)$ holomorphic on $C_1$ & $C_2$ and within the region between them.

QED

$$\therefore \oint_C z^n \, dz = \begin{cases} 0 & \text{if } n \neq -1, \; n \in \text{Integers} \\ 2\pi i & \text{if } n = -1 \end{cases}$$

for *any* closed contor $C$ that *encloses* $z=0$

- We can generalize this to

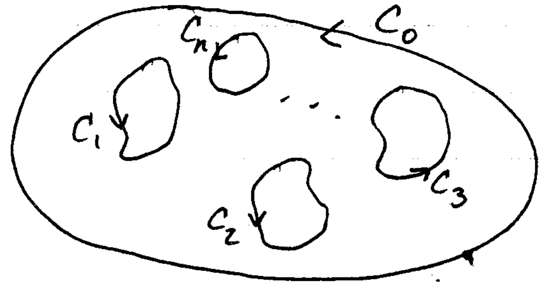
$$\oint_C (z-a)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

for any closed contour $C$ that encloses $z=a$

- If $C$ does not enclose $z=a$ then $(z-a)^n$ is holomorphic on and within $C$ and $\oint_C (z-a)^n dz = 0$ any $n$

- Another Generalization: $\oint_{C_0} f(z)dz = \sum_{i=1}^{n} \oint_{C_i} f(z) dz$

where $C_0$ encloses all of the $C_i$



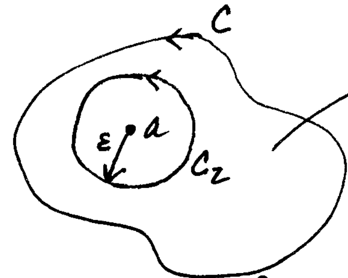

and $f(z)$ is holomorphic on all contors and in the domain between all of the contours.

## Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$f(z)$ is analytic on and within $C$ and $a$ lies within $C$.

Proof:




$\frac{f(z)}{z-a}$ analytic between $C$ & $C_2$

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_2} \frac{f(z)}{z-a}\,dz \qquad C_2 = a+\varepsilon e^{i\theta}$$

$$\oint_{C_2} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\cancel{a}+\varepsilon \cancel{e^{i\theta}}-\cancel{a}}\, i\varepsilon \cancel{e^{i\theta}}\, d\theta$$

$$= i\int_0^{2\pi} f(a+\varepsilon e^{i\theta})\,d\theta$$

$$\oint_{C_2} \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$ by taking the limit as $\varepsilon \to 0$

$$\therefore \boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz} \quad a \text{ inside } C$$

. If we know $f(z)$ on some boundary $C$ and we know that $f(z)$ is analytic within $C$ then we can figure out what $f$ is at <u>any</u> point within $C$. We can use this to solve boundary value problems with analytic functions, i.e. boundary conditions are prescribed on some surface and we need to figure out what the solution is inside.

Note that if $a$ lies outside of $C$ then

$$\oint_C \frac{f(z)}{z-a}\,dz = 0 \qquad a \text{ outside } C$$

## Derivatives

$$f'(a) = \frac{d}{da} f(a) = \frac{d}{da} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$$

Similarly : $f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

## Taylor Series

If $f(z)$ is analytic inside a circle $C$ about $a$ and $z$ is inside $C$, then $f(z)$ has the uniformly convergent expansion:

$$\boxed{f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n \; ; \; A_n = \frac{f^n(a)}{n!}}$$

Proof: $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi$

but $$\frac{1}{\xi - z} = \frac{1}{(\xi - a) - (z-a)} = \frac{1}{\xi - a} \frac{1}{1 - \underbrace{\frac{z-a}{\xi - a}}_{\varphi}}$$

$$= \frac{1}{\xi - a} \frac{1}{1-\varphi} = \frac{1}{\xi - a}(1 + \varphi + \varphi^2 + \varphi^3 + \dots)$$

$$= \frac{1}{\xi - a} \sum_{n=0}^{\infty} \varphi^n = \frac{1}{\xi - a} \sum_{n=0}^{\infty} \frac{1}{(\xi - a)^n} (z-a)^n$$

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} (z-a)^n \frac{1}{(\xi - a)^{n+1}}$$

$$\therefore f(z) = \frac{1}{2\pi i} \oint_C f(\xi) \sum_{n=0}^{\infty} (z-a)^n \frac{1}{(\xi - a)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left[ \oint_C \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi \, (z-a)^n \right]$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (z-a)^n \text{, inside } C$$

# Laurent Series / Singular Points

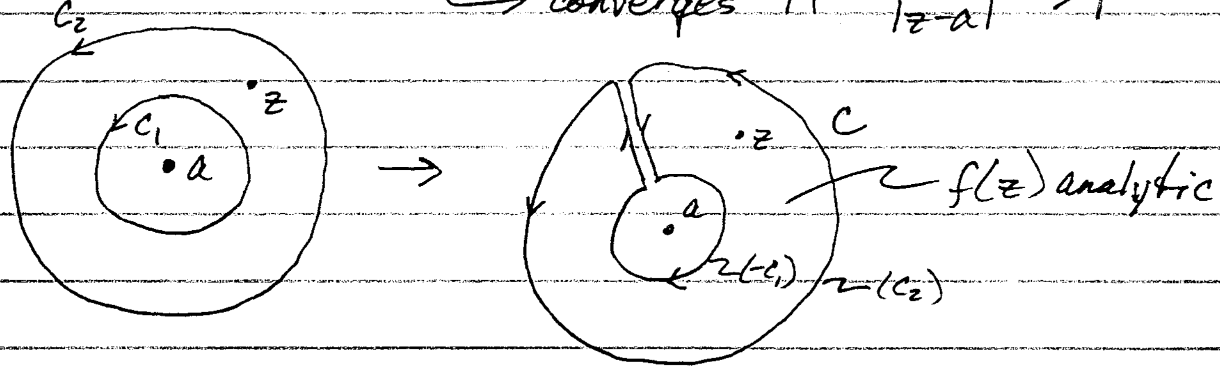
Recall $$\frac{1}{\xi - z} = \frac{1}{(\xi - a) - (z-a)} = \frac{-1}{z-a}\,\frac{1}{1-\frac{\xi-a}{z-a}}$$

$$\frac{1}{\xi - z} = -\frac{1}{z-a}\left[1 + \frac{\xi-a}{z-a} + \left(\frac{\xi-a}{z-a}\right)^2 + \dots\right]$$

$\hookrightarrow$ converges if $\frac{|\xi-a|}{|z-a|} < 1$

$$\frac{1}{\xi - z} = \frac{1}{\xi-a}\left[1 + \frac{z-a}{\xi-a} + \left(\frac{z-a}{\xi-a}\right)^2 + \dots\right]$$

$\hookrightarrow$ converges if $\frac{|\xi-a|}{|z-a|} > 1$




$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)}{\xi - z}\,d\xi - \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)}{\xi - z}$$

on $C_2$ $\quad |z-a| < |\xi - a| \rightarrow \frac{|\xi-a|}{|z-a|} > 1$

$$\therefore \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)}{\xi - z}\,d\xi = \sum_{n=0}^{\infty} A_n (z-a)^n \; ; \quad \boxed{A_n = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)}{(\xi-a)^{n+1}}\,d\xi = \frac{f^n(a)}{n!}}$$

on $C_1$ $\quad |z-a| > |\xi - a| \rightarrow \frac{|\xi-a|}{|z-a|} < 1$

$$\therefore \frac{-1}{2\pi i}\oint_{C_1} \frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\,\frac{1}{z-a}\sum_{n=1}^{\infty}\oint_{C_1} f(\xi)\left(\frac{\xi-a}{z-a}\right)^{n-1} d\xi$$

$$= B_n (z-a)^{-n} \; ; \quad \boxed{B_n = \frac{1}{2\pi i}\oint_{C_1} f(\xi)(\xi-a)^{n-1}\,d\xi}$$

$$\therefore \boxed{f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n + \sum_{n=1}^{\infty} B_n (z-a)^{-n}}$$

1) If all of the $B_n$ coefficients are zero for $n > p$ and $B_p \neq 0$ with $p$ finite then $f(z)$ has a pole of order $p$ at $z = a$. $p = 1$ is called a simple pole.

2) If the $B_n$ series does not truncate after a finite number of terms then $f(z)$ has an essential singularity at $z = a$. For example, $e^{1/z}$ has an essential singularity at $z = 0$.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \ldots$$

- $B_1$ is called the residue of $f(z)$ at $z = a$

recall $B_1 = \frac{1}{2\pi i} \oint_{C_1} f(\xi)\, d\xi$

<u>Residue Theorem</u>

If $C$ encloses only 1 pole of order $p$ then

$$\oint_C f(z)\,dz = \oint_C \frac{B_p}{(z-a)^p} + \frac{B_{p-1}}{(z-a)^{p-1}} + \ldots \frac{B_1}{(z-a)} + A_0 + A_1(z-a) + \ldots dz$$

$$= 0 + 0 + \ldots B_1(2\pi i) + 0 + 0 + \ldots$$

$$= 2\pi i \cdot B_1 = 2\pi i \operatorname{Res}[f(z=a)]$$

Generalization: If $C$ encloses $n$ singularities, $a_n$,

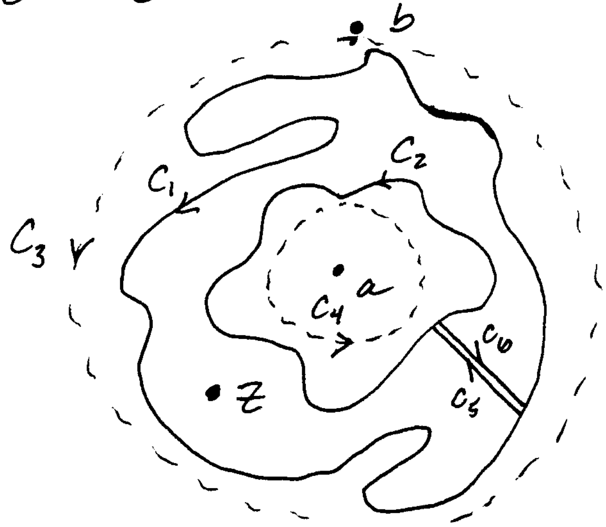
then $\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{m=1}^{n} \operatorname{Res}_m [f(z = a_m)]}$

# Careful Laurent Series Proof



$f(z)$ has a singularity at $z = a$

If there is more than one singularity then the next closest one is located at $z = b$




$C_3$ and $C_4$ are circular contours with the radius of $C_3$ greater than that of $C_4$.

- $C_3$ encloses $C_1$ encloses $C_2$ encloses $C_4$
- $b$ is outside of $C_3$
- $z$ is between $C_1$ and $C_2$
- $a$ is inside $C_4$

$$\oint_{C_3} \frac{f(\xi)}{\xi - z}\, d\xi = \oint_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi$$

b/c $\dfrac{f(\xi)}{\xi - z}$ has singularities at

$\xi = a, b$ and $z$ none of which are between $C_3$ and $C_1$

By a similar argument

$$\oint_{C_4} \frac{f(\xi)}{\xi - z} d\xi = \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi$$

Now, $C = C_1 + C_5 - C_2 + C_6$

$$\therefore f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} d\xi$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi$$

b/c $\oint_{C_5} + \oint_{C_6} = 0$

$$= \frac{1}{2\pi i} \oint_{C_3} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \oint_{C_4} \frac{f(\xi)}{\xi - z} d\xi$$

$$\frac{1}{\xi - z} = \frac{1}{(\xi - a) - (z - a)}$$

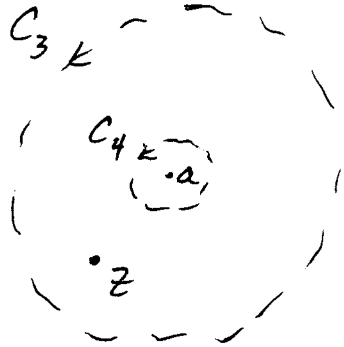
converges if $\frac{|z-a|}{|\xi - a|} < 1$

$$= \frac{1}{\xi - a} \left( \frac{1}{1 - \frac{z-a}{\xi - a}} \right) = \frac{1}{\xi - a} \left[ 1 + \frac{z-a}{\xi - a} + \left( \frac{z-a}{\xi - a} \right)^2 + \dots \right]$$

or

$$= \frac{-1}{z - a} \left( \frac{1}{1 - \frac{\xi - a}{z - a}} \right) = \frac{-1}{z - a} \left[ 1 + \frac{\xi - a}{z - a} + \left( \frac{\xi - a}{z - a} \right)^2 + \dots \right]$$

converges if $\frac{|\xi - a|}{|z - a|} < 1$




for $\xi$ on $C_3$ $\quad |\xi-a| > |z-a|$

$$\rightarrow \frac{|z-a|}{|\xi-a|} < 1$$

for $\xi$ on $C_4$ $\quad |\xi-a| < |z-a|$

$$\rightarrow \frac{|\xi-a|}{|z-a|} < 1$$

$$\therefore f(z) = \frac{1}{2\pi i} \oint_{C_3} \frac{f(\xi)}{\xi-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\xi-a}\right)^n d\xi$$

$$+ \frac{1}{2\pi i} \oint_{C_4} \frac{f(\xi)}{z-a} \sum_{n=0}^{\infty} \left(\frac{\xi-a}{z-a}\right)^n d\xi$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \oint_{C_3} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi}_{f^n(a)/n! = A_n} (z-a)^n$$

$$+ \underbrace{\sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_4} f(\xi)(\xi-a)^n d\xi \, (z-a)^{-(n+1)}}_{= \sum_{n=1}^{\infty} \underbrace{\frac{1}{2\pi i} \oint_{C_4} f(\xi)(\xi-a)^{n-1} d\xi}_{B_n} (z-a)^{-n}}$$

$$= \sum_{n=0}^{\infty} A_n (z-a)^n + \sum_{n=1}^{\infty} B_n (z-a)^{-n}$$