3-D Elasticity

Singular Solutions in an Infinite Solid

We want to solve

$$\sigma_{ji,j} + b_i = 0$$
$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$
$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

in the volume and

$$\sigma_{ji} n_j = T_i \quad \text{on } S_T$$
$$u_i = u_i^{\circ} \quad \text{on } S_u$$

for the infinite solid we will assume that both $\sigma_{ij} \to 0$ and $u_i \to 0$ as $r \to \infty$.

$$\therefore \quad (C_{ijkl} u_{k,l})_{,j} + b_i = 0$$

(If $C_{ijkl} = C_{ijlk}$)

or for homogeneous C_{ijkl} $\to$ $\boxed{C_{ijkl} u_{k,lj} = -b_i}$

Consider the Dirac delta function $\delta(x_i)$ which has the property

$$\int_{V_x} f(x_i)\,\delta(x_i)\,dV_x = \begin{cases} f(0) & \text{if } V_x \text{ includes the origin} \\ 0 & \text{if } V_x \text{ does not include the origin} \end{cases}$$

for any $f(x_i)$ that is continuos at the origin.

We can generalize this to

$$\int_{V_x} f(x_i)\,\delta(x_i - \xi_i)\,dV_x = \begin{cases} f(\xi_i) & \text{if } V_x \text{ includes } \xi_i \\ 0 & \text{if } V_x \text{ does not include } \xi_i \end{cases}$$

Note that $\delta(x_i)$ is not really well defined, but its operation under an integral is.

$$\int_{V_x} \delta(x_i - \xi_i)\, dV_x = \begin{cases} 1 & V_x \text{ includes } \xi_i \\ 0 & \xi_i \notin V_x \end{cases}$$

Hence a point load P_i located at the origin can be represented as a body force

$$b_i = P_i\, \delta(x_i)$$

$$\therefore \int_{V_x} b_i\, dV_x = \begin{cases} P_i & O_i \in V_x \\ 0 & O_i \notin V_x \end{cases}$$

$\therefore$ The point load at the origin has the following governing equation (homogeneous but anisotropic).

$$\boxed{C_{ijk\ell}\, u_{k,\ell j} = -P_i\, \delta(x_i)}$$

Recall Fourier Transforms:

1-D: $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) \exp[i\xi x]\, d\xi$$

$$F(\xi) = \int_{-\infty}^{\infty} f(x) \exp[-i\xi x]\, dx$$

3-D: $$f(x_i) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} F(\xi_i) \exp[i\xi_k x_k]\, dV_\xi$$

$$F(\xi_i) = \int_{\infty} f(x_i) \exp[-i\xi_k x_k]\, dV_x$$

$$f_{,j} = \frac{1}{(2\pi)^3} \int_\infty i\xi_j F(\xi_i) \exp[i\xi_k x_k] dV_\xi$$

$$\therefore\ i\xi_j F(\xi_i) = \int_\infty f_{,j}(x_i) \exp[-i\xi_k x_k] dV_x$$

but $F_{,j}(\xi_i) = \int_\infty f_{,j}(x_i) \exp[-i\xi_k x_k]$

$$- i\xi_j f(x_i) \exp[-i\xi_k x_k] dV_x = 0$$

$$\therefore\ \int_\infty f_{,j}(x_i) \exp[-i\xi_k x_k] dV_x = \int_\infty i\xi_j f(x_i) \exp[i\xi_k x_k] dV_x$$

$$= i\xi_j F(\xi_i)\ \checkmark$$

So what? We just confirmed what we knew already.

$$f_{,jk} = \frac{1}{(2\pi)^3} \int_\infty -\xi_j \xi_k F(\xi_i) \exp[i\xi_m x_m] dV_\xi$$

$$\therefore\ -\xi_j \xi_k F(\xi_i) = \int_\infty f_{,jk}(x_i) \exp[-i\xi_m x_m] dV_x$$

Return to $\quad C_{ijkl} u_{k,lj} = -P_i \delta(x_i)$

Fourier Transform this equation

$$\int_\infty C_{ijkl} u_{k,lj} \exp[-i\xi_k x_k] dV_x = \int_\infty -P_i \delta(x_i) \exp[-i\xi_k x_k] dV_x$$

Let $\quad U_i(\xi_i) = \int_\infty u_i(x_i) \exp[-i\xi_k x_k] dV_x$

$$\therefore\ -C_{ijkl} \xi_l \xi_j U_k = -P_i$$

$$A_{ij} = C_{ikjl} \xi_k \xi_l \rightarrow \boxed{A_{ij} U_j = P_i}$$

$\rho = (\xi_i \xi_i)^{1/2}$ analogous to $r = (x_i x_i)^{1/2}$

$f(x_i)$	$F(\xi_i)$
$F(x_i)$	$(2\pi)^3 f(-\xi_i)$
$f_{,k}(x_i)$	$i\xi_k F(\xi_i)$
$x_k f(x_i)$	$i F_{,k}(\xi_i)$
$\exp[i a_k x_k]$	$F(\xi_i - a_i)$
$f(x_i - a_i)$	$\exp[-i\xi_k a_k] F(\xi_i)$
$\delta(x_i)$	1
$\dfrac{1}{4\pi r}$	$\dfrac{1}{\rho^2}$
$\dfrac{1}{\pi r^2}$	$\dfrac{2\pi}{\rho}$
$\dfrac{x_k}{r^2}$	$\dfrac{-2\pi^2 i \xi_k}{\rho^3}$
$\ln r$	$\dfrac{-2\pi^2}{\rho^3}$
$\dfrac{1}{r^3}$	$-4\pi \ln \rho$
$\dfrac{x_k}{r}$	$\dfrac{-8\pi i \xi_k}{\rho^4}$
$\delta_{,k}(x_i)$	$i\xi_k$
x_i	$-i(2\pi)^3 \delta_{,i}(\xi_j)$
$\dfrac{x_k}{r^3}$	$-\dfrac{4\pi i \xi_k}{\rho^2}$
r	$\dfrac{8\pi}{\rho^4}$

For the isotropic case with $C_{ijkl} = C_{ijlk}$

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) + \lambda\delta_{ij}\delta_{kl}$$

$$\therefore \left[\mu(\delta_{ik}\xi_l\xi_l + \xi_i\xi_k) + \lambda\xi_i\xi_k\right]U_k = P_i$$

$$\left[\mu\xi_k\xi_k\delta_{ij} + (\lambda+\mu)\xi_i\xi_j\right]U_j = P_i$$

invert: $\left[\mu\xi_m\xi_m\delta_{ij} + (\lambda+\mu)\xi_i\xi_j\right]\left[A\delta_{kj} + B\xi_k\xi_j\right] = \delta_{ik}$

$$A\mu\xi_m\xi_m\delta_{ik} + A(\lambda+\mu)\xi_i\xi_k + B\mu\xi_m\xi_m\xi_i\xi_k + B(\lambda+\mu)\xi_i\xi_k\xi_j\xi_j = \delta_{ik}$$

$$\therefore \quad A\mu\xi_m\xi_m = 1 \qquad A(\lambda+\mu) + B\xi_m\xi_m(\lambda+2\mu) = 0$$

$$A = \frac{1}{\mu\xi_m\xi_m} \qquad B = -\frac{\lambda+\mu}{\mu(\lambda+2\mu)(\xi_m\xi_m)^2}$$

note $\lambda = \dfrac{2\mu\nu}{1-2\nu} \rightarrow \lambda+\mu = \dfrac{\mu}{1-2\nu}$, $\lambda+2\mu = \dfrac{2\mu(1-\nu)}{1-2\nu}$

$$A = \frac{1}{\mu}\frac{1}{\xi_m\xi_m}, \qquad B = -\frac{1}{\mu}\frac{1}{(\xi_m\xi_m)^2}\frac{1}{2(1-\nu)}$$

$$\therefore \boxed{U_i = \frac{1}{\mu}\frac{P_i}{\xi_k\xi_k} - \frac{1}{2\mu(1-\nu)}\frac{\xi_i\xi_j}{(\xi_k\xi_k)^2}P_j}$$

$$u_i = \frac{1}{(2\pi)^3}\int_\infty U_i(\xi_j)\exp[i\xi_m x_m]\,dV_\xi$$

→ Inverse transforms are not always easy to determine. Some useful pairs are:

$$F = \frac{1}{\xi_k \xi_k} = \frac{1}{\rho^2} \rightarrow f = \frac{1}{4\pi r}$$

$$F = \frac{\xi_i \xi_j}{(\xi_k \xi_k)^2} = \frac{\xi_i \xi_j}{\rho^4} \rightarrow f = \frac{1}{8\pi}\left(\frac{x_i}{r}\right)_{,j}$$

recall $f_{,k}(x_i) = i\xi_k F(\xi_i)$

$$\therefore \left(\frac{x_i}{r}\right)_{,k} = i\xi_k\left(\frac{-8\pi i \xi_i}{\rho^4}\right) = \frac{8\pi \xi_i \xi_k}{\rho^4}$$

$$\left(\frac{x_i}{r}\right)_{,j} = \left(\frac{x_i}{(x_k x_k)^{1/2}}\right)_{,j} = \frac{\delta_{ij}}{r} - \frac{1}{2}\frac{x_i \, 2x_j}{r^3}$$

$$\left(\frac{x_i}{r}\right)_{,j} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}$$

$$\therefore u_i = \frac{1}{4\pi\mu}\frac{P_i}{r} - \frac{1}{16\pi\mu(1-\nu)}\left[\frac{P_i}{r} - \frac{x_i x_j}{r^3}P_j\right]$$

$$\boxed{u_i = \frac{1}{16\pi\mu(1-\nu)}\left[(3-4\nu)\frac{P_i}{r} + \frac{x_i x_j}{r^3}P_j\right]}$$

Solution for point load P_i applied at the origin in an infinite space.

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{x_k}{r^3}P_k\delta_{ij} - \frac{3x_i x_j x_k}{r^5}P_k - (1-2\nu)\frac{x_i\delta_{jk} + x_j\delta_{ik}}{r^3}P_k\right]$$

$$\sigma_{ij} = \frac{-1}{8\pi(1-\nu)}\left[\frac{3x_i x_j x_k}{r^5}P_k + (1-2\nu)\frac{x_j P_i + x_i P_j - x_k P_k \delta_{ij}}{r^3}\right]$$

Let's call $u_i^k = \frac{1}{16\pi\mu(1-\nu)}\left[\frac{3-4\nu}{r}\delta_{ik} + \frac{x_i x_k}{r^3}\right]$

the displacement solution for a unit point force applied in the k direction.
We can take spatial derivatives of this solution to generate more 3-D solutions with some "funny business" going on at the origin.

Recall for points outside of the origin we had

$$C_{ijkl}\, u^i_{k,lj} = 0 \quad (C_{ijkl} \text{ is homogeneous, isotropic and } C_{ijkl}=C_{ijlk})$$

then $C_{ijkl}\, u^i_{k,ljp} = 0_{,p} = 0$

$$C_{ijkl}\, u^i_{k,ljpq} = 0_{,pq} = 0$$

etc.

$\therefore$ $u^i_{k,p}$, $u^i_{k,pq}$, ... are solutions to the field equations for an infinite body with no body force but some type of singularity at the origin.

The 9 solutions represented by $u^k_{i,j}$ $(k=1,3,\ j=1,3)$ are called the doublet states.

Some notation: $u_i^k(x_\ell, \xi_m)$ is the displacement field as a function of the coordinates x_ℓ for a unit point force in the k direction applied at the point ξ_m.

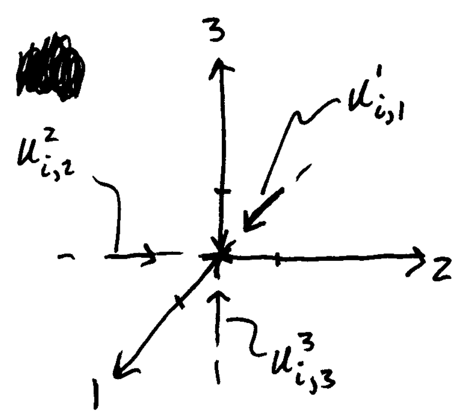
Note that $u_i^k(x_\ell + a_\ell, \xi_m) = u_i^k(x_\ell, \xi_m - a_m)$

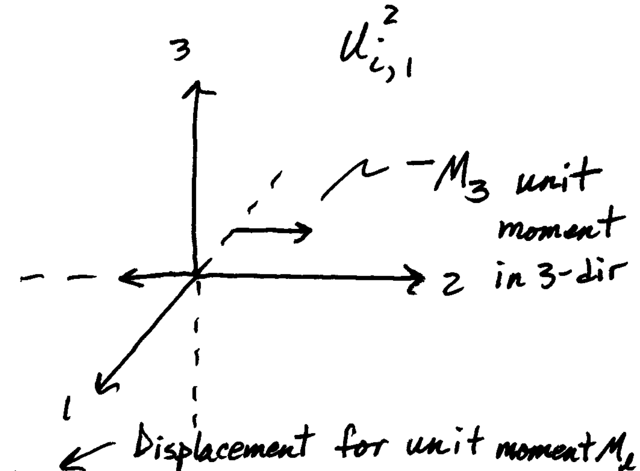
Now investigate $u_{i,j}^k$ to determine the physical situation represented by the doublet states.

$$u_{i,j}^k = \lim_{\Delta x_j \to 0} \frac{u_i^k(x_j + \Delta x_j, 0) - u_i^k(x_j, 0)}{\Delta x_j}$$

$$u_{i,j}^k = \lim_{\Delta x_j \to 0} \frac{u_i^k(x_j, -\Delta x_j) - u_i^k(x_j, 0)}{\Delta x_j}$$

= Solutions for opposing point loads of magnitude $\frac{1}{|\Delta x_j|}$ located at a distance of $|\Delta x_j|$ from one another.





$u_{i,k}^k$ = center of compression

Displacement for unit moment M_ℓ

$u_i^{M_\ell} = \epsilon_{\ell k j}\, u_{i,j}^k = -\epsilon_{jk\ell}\, u_{i,j}^k$

Papkovitch - Neuber Potentials

Isotropic, no body forces

$$\mu u_{i,kk} + (\lambda+\mu) u_{k,ki} = 0$$

$$u_i = \phi_{,i} + \epsilon_{ijk} \psi_{k,j}$$

$$u_{i,i} = \phi_{,ii} + \underbrace{\epsilon_{ijk} \psi_{k,ji}}_{0}$$

$$\therefore \quad \mu u_{i,kk} + (\lambda+\mu) \phi_{,ikk} = 0$$

$$\left[\mu u_i + (\lambda+\mu)\phi_{,i}\right]_{,kk} = 0$$

$$\therefore \quad \mu u_i + (\lambda+\mu)\phi_{,i} = \mu \Phi_i \quad \text{where } \Phi_{i,kk} = 0$$

$$\therefore \quad u_i = \Phi_i - \frac{\lambda+\mu}{\mu} \phi_{,i}$$

$$u_{i,i} = \Phi_{i,i} - \frac{\lambda+\mu}{\mu} \phi_{,ii} = \phi_{,ii}$$

$$\therefore \left(1 + \frac{\lambda+\mu}{\mu}\right) \phi_{,ii} = \Phi_{i,i} = \frac{1}{2}(x_i \Phi_i)_{,jj}$$

Show: $\frac{1}{2}(x_i\Phi_i)_{,jj} = \frac{1}{2}\left(x_{i,j}\Phi_i + x_i \Phi_{i,j}\right)_{,j}$

$$= \frac{1}{2}\left(\Phi_j + x_i \Phi_{i,j}\right)_{,j} = \frac{1}{2}\left(\Phi_{j,j} + x_{i,j}\Phi_{i,j} + x_i \Phi_{i,jj}\right)$$

$$= \frac{1}{2}\left(\Phi_{j,j} + \Phi_{j,j} + 0\right) = \Phi_{j,j} = \Phi_{i,i}$$

$$\therefore \quad \left(2+\frac{\lambda}{\mu}\right)\phi_{,jii} = \frac{1}{2}(x_i \Phi_i)_{,jjj} = \Phi_{i,i}$$

$$\therefore \quad \phi = \frac{\mu}{2(\lambda+2\mu)}\left[x_i \Phi_i + \Phi_0\right]$$

where $\Phi_{0,ii} = 0$

$$\therefore \quad \boxed{\begin{aligned} u_i &= \Phi_i - \frac{1}{4(1-\nu)}\left[\Phi_{0,i} + (x_j \Phi_j)_{,i}\right] \\ &\text{with } \Phi_{i,jj} = 0 \text{ and } \Phi_{0,jj} = 0 \end{aligned}}$$

Or in vector notation

$$\boxed{\begin{aligned} \vec{u} &= \vec{\Phi} - \frac{1}{4(1-\nu)}\left[\vec{\nabla}\Phi_0 + \vec{\nabla}(\vec{x}\cdot\vec{\Phi})\right] \\ &= \vec{\Phi} - \frac{1}{4(1-\nu)}\cdot\vec{\nabla}\left[\Phi_0 + \vec{x}\cdot\vec{\Phi}\right] \\ &\text{with } \nabla^2\vec{\Phi} = \vec{0} \text{ and } \nabla^2\Phi_0 = 0 \end{aligned}}$$

Note that Φ_i and Φ_0 are not unique since there are 4 potentials for 3 displacements.

Stresses: $\dfrac{2(1-\nu)}{\mu}\sigma_{ij} = 2\nu\,\Phi_{k,k}\,\delta_{ij} + (1-2\nu)(\Phi_{i,j} + \Phi_{j,i})$

$$- x_k \Phi_{k,ij} + \Phi_{0,ij}$$

<u>Some Solutions generated from a harmonic potential ϕ</u>

A) $\Phi_i = 0$, $\Phi_0 = \frac{2(1-\nu)}{\mu}\phi$ where $\phi_{,ii} = 0$

then : $\phi_{,ii} = 0$, $2\mu u_i = \phi_{,i}$, $\sigma_{ij} = \phi_{,ij}$

B) $\Phi_0 = 0$, $\Phi_i = \frac{-2(1-\nu)}{\mu}\phi\delta_{i3}$

then: $\phi_{,ii} = 0$, $2\mu u_i = x_3\phi_{,i} - (3-4\nu)\delta_{i3}\phi$

$$\sigma_{ij} = x_3\phi_{,ij} - (1-2\nu)(\delta_{i3}\phi_{,j} + \delta_{j3}\phi_{,i}) - 2\nu\delta_{ij}\phi_{,3}$$

C) $\Phi_0 = 0$, $\Phi_i = \frac{-2(1-\nu)}{\mu}\phi\delta_{i2}$ see B - replace 3 with 2

D) $\Phi_0 = 0$, $\Phi_i = -\frac{2(1-\nu)}{\mu}\phi\delta_{i1}$ see B - replace 3 with 1

E) $\Phi_0 = -x_i\Phi_i$, $\Phi_i = \frac{1}{\mu}\epsilon_{ijk}(\phi\delta_{k3})_{,j}$

then: $\phi_{,ii} = 0$, $2\mu u_i = \epsilon_{ij3}\phi_{,j}$

$$\sigma_{ij} = \epsilon_{ik3}\phi_{,kj} + \epsilon_{jk3}\phi_{,ki}$$

F) $\Phi_0 = -x_i\Phi_i$, $\Phi_i = \frac{1}{\mu}\epsilon_{ijk}(\phi\delta_{k2})_{,j}$, see E - replace 3 w/ 2

G) $\Phi_0 = -x_i\Phi_i$, $\Phi_i = \frac{1}{\mu}\epsilon_{ijk}(\phi\delta_{k1})_{,j}$, see E - replace 3 w/ 1

<u>Back to the point load</u> (Use P-N potentials)

Solution A: $u_{i,i} = \frac{1}{2\mu}\phi_{,ii} = 0$

Solutions E, F, G: $u_{i,i} = 0$

Not likely for point force

Try to use B, C, D. Dimensional analysis $\rightarrow \sigma_{ij} \approx \frac{P}{R^2}$
$\rightarrow u_i \approx \frac{P}{R}$

k: 1 2 3
D, C, B: $\Phi_0 = 0$, $\Phi_i = \frac{-2(1-\nu)}{\mu}\phi\,\delta_{ik}$

Try $\phi = \frac{A}{R}$, $R = (x_i x_i)^{1/2}$, $R_{,i} = \frac{x_i}{R}$

$$\phi_{,i} = \frac{-A}{R^2} R_{,i} = -\frac{A x_i}{R^3}$$

$$\phi_{,ij} = \frac{-A\,\delta_{ij}}{R^3} + \frac{3A\,x_i x_j}{R^5}$$

$$\therefore\ 2\mu u_i^k = -\frac{A\,x_i x_k}{R^3} - (3-4\nu)\frac{A\,\delta_{ik}}{R}$$

$$\sigma_{ij}^k = -x_k \frac{A\,\delta_{ij}}{R^3} + \frac{3A\,x_i x_j x_k}{R^5} + (1-2\nu)\frac{A\,x_j\delta_{ik}}{R^3} + (1-2\nu)\frac{A x_i \delta_{jk}}{R^3}$$
$$+ 2\nu\frac{A\,x_k\delta_{ij}}{R^3}$$

$$\sigma_{ij}^k = (1-2\nu)A\left(\frac{x_j\delta_{ik}}{R^3} + \frac{x_i\delta_{jk}}{R^3} - \frac{x_k\delta_{ij}}{R^3}\right) + 3A\frac{x_i x_j x_k}{R^5}$$

We need to evaluate A by integrating T_k around some surface enclosing the origin and setting this integral equal to -1.

Let's intergrate around a spherical surface centered on the origin of radius R. Note that

$n_j = \frac{x_j}{R}$. Also take the 3 coordinate

direction to be aligned with the point force, i.e. $k=3$.

$$x_1 = R\sin\theta\cos\phi \;,\; x_2 = R\sin\theta\sin\phi \;,\; x_3 = R\cos\theta$$

$$\sigma_{ij} n_j = (1-2\nu)A\left(\frac{\overbrace{x_j x_j}^{R^2} \delta_{i3}}{R^4} + \cancel{\frac{x_i x_3}{R^4}} - \cancel{\frac{x_3 x_i}{R^4}} \right) + 3A\frac{x_i x_3 \overbrace{x_j x_j}^{R^2}}{R^6}$$

$$\therefore \sigma_{ij} n_j = (1-2\nu)A\frac{\delta_{i3}}{R^2} + 3A\frac{x_i x_3}{R^4}$$

$$T_3 = T_k = \sigma_{3j} n_j = (1-2\nu)\frac{A}{R^2} + 3A\frac{x_3^2}{R^4}$$

$$\int_0^{2\pi}\int_0^{\pi} T_3 R^2 \sin\theta \, d\theta \, d\phi = -1$$

$$\int_0^{2\pi}\int_0^{\pi} \left[(1-2\nu)A + 3A\cos^2\theta\right]\sin\theta \, d\theta \, d\phi = -1$$

$$\Rightarrow 4\pi(1-2\nu)A + 4\pi A = -1$$

$$8\pi(1-\nu)A = -1$$

$$A = \frac{-1}{8\pi(1-\nu)}$$

$$\therefore \boxed{\phi = \frac{-1}{8\pi(1-\nu)}\frac{1}{R}}$$

Solution for point load not at origin has form:

$$\Phi_i^{\,k} = \frac{\delta_{ik}}{4\pi\,|x_j - \xi_j|}\ , \quad \Phi_o^{\,k} = \frac{-\xi_k}{4\pi\,|x_j - \xi_j|}$$

because $u_i = \Phi_i - \frac{1}{4(1-\nu)}\left[\Phi_{o,i} + (x_j \Phi_j)_{,i}\right]$

and the term $(x_j \Phi_j)_{,i}$ will give a term like $x_j \Phi_{j,i}$ instead of $(x_j - \xi_j)\Phi_{j,i}$ so the extra $-\xi_j \Phi_{j,i}$ must come from a non-vanishing Φ_o potential.

These potentials are very use-ful because they can be integrated to give the potetials for an arbitrary distribution of body force.

This of course can also be done with displacement or stress components.

So for a general body force distribution the Papkovitch-Neuber potentials are

$$\Phi_i(x_j) = \int_V \frac{b_i(\xi_j)}{4\pi\,|x_j - \xi_j|}\, dV$$

$$\Phi_o(x_j) = \int_V \frac{-\xi_i\, b_i(\xi_j)}{4\pi\,|x_j - \xi_j|}\, dV$$

Micromechanics – Eigenstrains and Eshelby Methods

The point force solution can be used to derive solutions due to eigenstrains in an infinite solid.

Consider an unbounded infinite solid free of stress. Now introduce an inelastic, i.e. eigenstrain, strain due to thermal strain, plastic strain, transformation strain or whatever. The total strain must be compatible:

i.e. $$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{*} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

ε_{ij}^{*} = the eigenstrain

ε_{ij}^{e} = the elastic strain

Stresses arise only due to the elastic strain i.e.

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^{e} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{*}) = C_{ijkl}(u_{k,l} - \varepsilon_{kl}^{*})$$

In the absence of body forces $\sigma_{ji,j} = 0$

$$\therefore \quad C_{ijkl}u_{k,lj} - C_{ijkl}\varepsilon_{kl,j}^{*} = 0$$

Let $$b_i^{*} = -C_{ijkl}\varepsilon_{kl,j}^{*}$$

Then $C_{ijkl} u_{k,lj} + b_i^* = 0$

the solution to this equation in terms of Papkovitch-Neuber potentials is

$$\Phi_i(x_i) = \int_V \frac{b_i^*(\xi_i)}{4\pi |x_j - \xi_j|} dV$$

$$\Phi_0(x_i) = \int_V \frac{-\xi_i b_i^*(\xi_j)}{4\pi |x_j - \xi_j|} dV$$

In general the solutions to eigenstrain problems are not interesting unless they are used as an aid to solve inclusion or void problems.

Consider an ellipsoidal region bounded by the surface

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 = 1$$

subjected to a <u>uniform</u> eigenstrain ε_{ij}^*.

Therefore the body force b_i^* due to the eigenstrain is zero everywhere except at the elliptical surface.

i.e. $b_i^* = C_{ijkl}\, \varepsilon_{kl}^*\, \delta\left[\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 - 1\right] n_j$

where n_j is the normal to the ellipsoid

We could also use our point force solution, $u_i^k(x_j - \xi_j)$, as a Green's function to obtain the solution.

These integrals can be evaluated exactly for an isotropic solid with the uniform eigenstrain in a spherical region but they can be reduced to elliptic integrals. Remarkably, the strains and stresses inside the ellipsoid are <u>uniform</u>.

The total strain is related to the eigenstrain through the so-called Eshelby tensor.

$$\text{i.e.} \qquad \varepsilon_{ij}^c = \varepsilon_{ij}^e + \varepsilon_{ij}^* = S_{ijkl}\, \varepsilon_{kl}^*$$

where S_{ijkl} is a function of the elastic properties of the solid and the shape of the ellipsoid. For an isotropic solid its components are.

$$S_{1111} = \frac{3}{8\pi(1-\nu)} a_1^2 I_{11} - \frac{1-2\nu}{8\pi(1-\nu)} I_1$$
$$S_{2222} = \frac{3}{8\pi(1-\nu)} a_2^2 I_{22} - \frac{1-2\nu}{8\pi(1-\nu)} I_2$$
$$S_{3333} = \frac{3}{8\pi(1-\nu)} a_3^2 I_{33} - \frac{1-2\nu}{8\pi(1-\nu)} I_3$$
$$S_{1122} = \frac{3}{8\pi(1-\nu)} a_2^2 I_{12} - \frac{1-2\nu}{8\pi(1-\nu)} I_1$$
$$S_{2211} = \frac{3}{8\pi(1-\nu)} a_1^2 I_{12} - \frac{1-2\nu}{8\pi(1-\nu)} I_2$$
$$S_{1133} = \frac{3}{8\pi(1-\nu)} a_3^2 I_{13} - \frac{1-2\nu}{8\pi(1-\nu)} I_1$$
$$S_{3311} = \frac{3}{8\pi(1-\nu)} a_1^2 I_{13} - \frac{1-2\nu}{8\pi(1-\nu)} I_3$$
$$S_{2233} = \frac{3}{8\pi(1-\nu)} a_3^2 I_{23} - \frac{1-2\nu}{8\pi(1-\nu)} I_2$$
$$S_{3322} = \frac{3}{8\pi(1-\nu)} a_2^2 I_{23} - \frac{1-2\nu}{8\pi(1-\nu)} I_3$$

$$S_{1212} = S_{1221} = S_{2121} = S_{2112} = \frac{a_1^2 + a_2^2}{16\pi(1-\nu)} I_{12} - \frac{1-2\nu}{16\pi(1-\nu)}(I_1 + I_2)$$
$$S_{1313} = S_{1331} = S_{3131} = S_{3113} = \frac{a_1^2 + a_3^2}{16\pi(1-\nu)} I_{13} - \frac{1-2\nu}{16\pi(1-\nu)}(I_1 + I_3)$$
$$S_{2323} = S_{2332} = S_{3232} = S_{3223} = \frac{a_2^2 + a_3^2}{16\pi(1-\nu)} I_{23} - \frac{1-2\nu}{16\pi(1-\nu)}(I_2 + I_3)$$

all other $S_{ijkl} = 0$

$$I_1 = \frac{4\pi a_1 a_2 a_3}{(a_1^2 - a_2^2)(a_1^2 - a_3^2)} F(\theta, k)$$

$$I_3 = \frac{4\pi a_1 a_2 a_3}{(a_2^2 - a_3^2)(a_1^2 - a_3^2)} \left[\frac{a_2 (a_1^2 - a_3^2)^{1/2}}{a_1 a_3} - E(\theta, k) \right]$$

$$F(\theta, k) = \int_0^\theta \frac{1}{(1 - k^2 \sin^2 t)^{1/2}} dt$$

$$E(\theta, k) = \int_0^\theta (1 - k^2 \sin^2 t)^{1/2} dt$$

$$\theta = \arcsin\left(1 - \frac{a_3^2}{a_1^2}\right), \quad k^2 = \frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}$$

$$I_1 + I_2 + I_3 = 4\pi$$
$$3I_{11} + I_{12} + I_{13} = 4\pi / a_1^2$$
$$3a_1^2 I_{11} + I_{12} a_2^2 + a_3^2 I_{13} = 3I_1$$
$$I_{12} = (I_2 - I_1) / (a_1^2 - a_3^2)$$

For a spherical region

$$S_{1111} = S_{2222} = S_{3333} = \frac{7 - 5\nu}{15(1-\nu)}$$
$$S_{1122} = S_{2211} = S_{1133} = S_{3311} = S_{2233} = S_{3322} = \frac{5\nu - 1}{15(1-\nu)}$$
$$S_{1212} = S_{1313} = S_{2323} = \text{etc.} = \frac{4 - 5\nu}{15(1-\nu)}$$

Stress state in the region that transformed

$$\sigma_{ij}^{c} = C_{ijkl}\,\varepsilon_{kl}^{e} = C_{ijkl}\left(\varepsilon_{kl}^{c} - \varepsilon_{kl}^{*}\right)$$

$$= C_{ijkl}\left[S_{klmn}\,\varepsilon_{mn}^{*} - \varepsilon_{kl}^{*}\right]$$

$$\boxed{\sigma_{ij}^{c} = C_{ijkl}\left(S_{klmn} - \delta_{km}\delta_{ln}\right)\varepsilon_{mn}^{*}}$$

How can we use this to investigate a spherical void in an isotropic solid loaded by some stress state at infinity?

$$C_{ijkl} = \mu\,\delta_{ik}\delta_{jl} + \mu\,\delta_{il}\delta_{kj} + \lambda\,\delta_{ij}\delta_{kl}$$

$$S_{ijkl} = \frac{4-5\nu}{15(1-\nu)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{5\nu-1}{15(1-\nu)}\delta_{ij}\delta_{kl}$$

$$\sigma_{ij}^{c} = \underbrace{C_{ijkl}\left(S_{klmn} - \tfrac{1}{2}\delta_{km}\delta_{ln} - \tfrac{1}{2}\delta_{kn}\delta_{lm}\right)}_{P_{ijmn}}\varepsilon_{mn}^{*}$$

$$P_{ijmn} = \frac{E(5\nu-7)}{30(1+\nu)(1-\nu)}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) - \frac{E(1+5\nu)}{15(1+\nu)(1-\nu)}\delta_{ij}\delta_{mn}$$

$$\therefore\ \sigma_{ij}^{c} = P_{ijmn}\,\varepsilon_{mn}^{*} \rightarrow \varepsilon_{ij}^{*} = Q_{ijkl}\,\sigma_{kl}^{c}$$

where $Q_{ijkl} = P_{ijkl}^{-1}$

Note that if $A_{ijkl} = A(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + B\delta_{ij}\delta_{kl}$

then $A^{-1}_{ijkl} = C(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + D\delta_{ij}\delta_{kl}$

where $C = \dfrac{1}{4A}$ and $D = \dfrac{-B}{2A(2A+3B)}$

with the 4th rank identity tensor $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ which is only valid for relating <u>symmetric</u> 2nd rank tensors.

$$\therefore\ Q_{ijkl} = \frac{15(1-\nu^2)}{2E(7-5\nu)}\left[-(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{1+5\nu}{5(1+\nu)}\delta_{ij}\delta_{kl}\right]$$

then $\varepsilon^c_{ij} = S_{ijkl}\,\varepsilon^*_{kl} = \underbrace{S_{ijkl}\,Q_{klmn}}_{R_{ijmn}}\,\sigma^c_{mn}$

$$R_{ijmn} = \frac{1+\nu}{2E(7-5\nu)}\left[-2(4-5\nu)(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + (3-5\nu)\delta_{ij}\delta_{mn}\right]$$

<u>Superposition Scheme</u>

No loading at ∞ σ^A_{ij} applied at ∞

σ^c_{ij} +   (Uniform stress field)

σ^A_{ij} applied at ∞

= $\sigma_{ij} = 0$ if $\sigma^c_{ij} = -\sigma^A_{ij}$

$$\therefore \quad \sigma_{ij}^{C} = -\sigma_{ij}^{A}$$

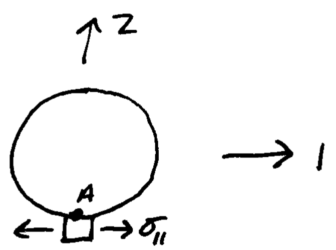
$$\varepsilon_{ij}^{C} = -R_{ijkl}\,\sigma_{ij}^{A}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{A} + \varepsilon_{ij}^{C}$$

<u>Ex</u>) $\sigma_{ij}^{A} = \sigma\,\delta_{1i}\,\delta_{1j}$

then $\varepsilon_{11}^{C} = \dfrac{(1+\nu)(13-15\nu)}{2E\,(7-5\nu)}\,\sigma$

$$\varepsilon_{22}^{C} = \varepsilon_{33}^{C} = \frac{(1+\nu)(5\nu-3)}{2E\,(7-5\nu)}\,\sigma$$



At point A we know $\sigma_{22} = \sigma_{12} = \sigma_{13} = 0$ due to traction free conditions, $\sigma_{13} = 0$ due to symmetry $\sigma_{11} \neq 0$, $\sigma_{33} \neq 0$, $\varepsilon_{11} = \varepsilon_{11}^{A} + \varepsilon_{11}^{C}$,
$\varepsilon_{33} = \varepsilon_{33}^{A} + \varepsilon_{33}^{C}$

$$\therefore \quad \varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{33}, \quad \varepsilon_{33} = \frac{1}{E}\sigma_{33} - \frac{\nu}{E}\sigma_{11}$$

$$\rightarrow \quad \sigma_{11} = \frac{E}{1-\nu^2}\left(\varepsilon_{11} + \nu\varepsilon_{33}\right)$$

$$= \sigma + \frac{E}{1-\nu^2}\left(\varepsilon_{11}^{C} + \nu\varepsilon_{33}^{C}\right)$$

$$= \sigma + \frac{\cancel{E}}{\cancel{(1+\nu)}(1-\nu)} \frac{\cancel{1+\nu}}{2\cancel{E}(7-5\nu)} \left[13 - 15\nu + 5\nu^2 - 3\nu \right] \sigma$$

$$= \sigma \left[\frac{2(1-\nu)(7-5\nu) + 13 - 18\nu + 5\nu^2}{2(1-\nu)(7-5\nu)} \right]$$

$$= \sigma \left[\frac{2(7 - 12\nu + 5\nu^2) + 13 - 18\nu + 5\nu^2}{2(1-\nu)(7-5\nu)} \right]$$

$$\boxed{\sigma_{11} = \sigma \, \frac{3}{2} \, \frac{9-5\nu}{7-5\nu}}$$

ν	σ_{11}/σ	σ_{33}/σ
0	1.929	-0.214
.25	2.022	0.065
.3	2.045	0.136
.5	2.167	0.5

Similarly $\boxed{\sigma_{33} = \sigma \, \frac{3}{2} \, \frac{5\nu - 1}{7-5\nu}}$

Ex | $\sigma_{ij}^A = \sigma \delta_{ij}$ (hydrostatic stress)

Then $\varepsilon_{ij}^c = -\frac{1+\nu}{2E(7-5\nu)} \left[-2(4-5\nu)(\delta_{ij} + \delta_{ij}) + (3-5\nu) 3 \delta_{ij} \right] \sigma$

$$= -\frac{1+\nu}{2E(7-5\nu)} \left[-7 + 5\nu \right] \sigma \delta_{ij}$$

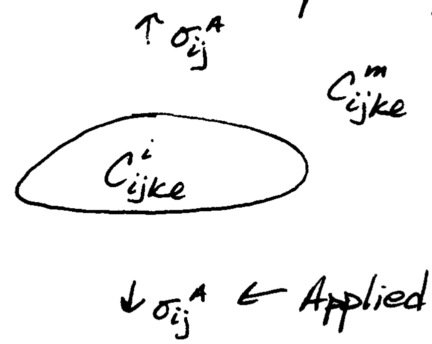
$$\left.\begin{aligned} \varepsilon_{ij}^c &= \frac{1+\nu}{2E} \sigma \delta_{ij} \\ \varepsilon_{ij}^A &= \frac{1-2\nu}{E} \sigma \delta_{ij} \end{aligned}\right\} \varepsilon_{ij} = \varepsilon_{ij}^c + \varepsilon_{ij}^A = \frac{3(1-\nu)}{2E} \sigma \delta_{ij}$$

$$\therefore \sigma_{11} = \sigma_{33} = \frac{\cancel{E}}{\cancel{1-\nu^2}} \left[\cancel{(1+\nu)} \frac{3\cancel{(1-\nu)}}{2\cancel{E}} \sigma \right]$$

$$\boxed{\sigma_{11} = \sigma_{33} = \frac{3}{2} \sigma}$$ ✓

Eshelby Methods Continued

The ellipsoidal inhomogeneity



← Applied Stress at infinity.

Consider the uniform solid with properties C_{ijkl}^m with an ellipsoidal region with the same shape as the inhomogeneity subjected to an eigenstrain ε_{ij}^*.

Recall that the constrained strain, ε_{ij}^c, is related to the eigenstrain through the Eshelby tensor.

$$\varepsilon_{ij}^c = S_{ijkl}\,\varepsilon_{kl}^*$$

where the components of S_{ijkl} depend only on the elastic properties of the <u>matrix</u> and the aspect ratios of the ellipsoid.

The constrained stress in the region is then

$$\sigma_{ij}^c = C_{ijkl}^m\left(\varepsilon_{kl}^c - \varepsilon_{kl}^*\right) = C_{ijkl}^m\left(S_{klmn}\,\varepsilon_{mn}^* - \varepsilon_{kl}^*\right)$$

$$\sigma_{ij}^c = P_{ijmn}\,\varepsilon_{mn}^* \quad , \quad P_{ijmn} = C_{ijkl}^m\left(S_{klmn} - \tfrac{1}{2}\delta_{km}\delta_{ln} - \tfrac{1}{2}\delta_{kn}\delta_{lm}\right)$$

Now include the homogeneous applied stress state σ_{ij}^A to get the total stress, σ_{ij}, in the region.

$$\sigma_{ij} = \sigma_{ij}^c + \sigma_{ij}^A \quad \text{also:} \quad \varepsilon_{ij} = \varepsilon_{ij}^c + \varepsilon_{ij}^A$$

Now, if we can choose ε_{ij}^* just right then we can get the stress and strain in the "eigenregion" to have the same stress and strain as the inhomogeneity.

The inhomogeneity has no transformation strain.

$$\therefore \quad \sigma_{ij}^i = C_{ijkl}^i \, \varepsilon_{kl}^i \quad ①$$

and we want $\sigma_{ij}^i = \sigma_{ij}$ and $\varepsilon_{ij}^i = \varepsilon_{ij}$

Ultimately we want the inhomogeneity stress and strain as a function of the applied stress and strain

$$① \rightarrow \sigma_{ij}^c + \sigma_{ij}^A = C_{ijkl}^i \left(\varepsilon_{kl}^c + \varepsilon_{kl}^A \right)$$

$$P_{ijmn} \varepsilon_{mn}^* + \sigma_{ij}^A = C_{ijkl}^i \left(S_{klmn} \varepsilon_{mn}^* + \text{[redacted]} \, \varepsilon_{kl}^A \right)$$

$$\underbrace{\left(P_{ijmn} - C_{ijkl}^i S_{klmn} \right)}_{M_{ijmn}} \varepsilon_{mn}^* = \left(C_{ijkl}^i - C_{ijkl}^m \right) \varepsilon_{kl}^A$$

$$\therefore \quad \varepsilon_{ij}^* = M_{ijmn}^{-1} \left(C_{mnkl}^i - C_{mnkl}^m \right) \varepsilon_{kl}^A$$