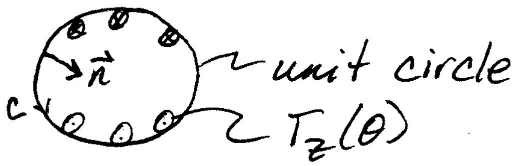
The external anti-plane problem. Traction BC's.



$\vec{n}$ points to left $\rightarrow$ $F_z = Re[\omega(z)]$

$$F_z = \int T_z \, ds = \int T_z \, d\theta \quad \text{on unit circle}$$

$$2F_z = \omega(z) + \bar{\omega}(\bar{z})$$

$$\frac{1}{2\pi i}\oint_c \frac{2F_z(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\oint_c \frac{\omega(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i}\oint_c \frac{\bar{\omega}(\xi)}{\xi - z} d\xi$$

$$\omega(\xi) = a_0 + \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \frac{a_3}{\xi^3} + \ldots \quad \text{b/c } \omega \text{ analytic outside unit circle.}$$

Note z outside unit circle.

$$\frac{1}{2\pi i}\oint_c \frac{\omega(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\oint_c \frac{a_0 + \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \ldots}{\xi - z} d\xi$$

$$\oint_c \frac{a_0}{\xi - z} d\xi = 0 \quad \text{b/c } \frac{a_0}{\xi - z} \text{ analytic in u.c.}$$

$$\oint_c \frac{a_n}{\xi^n}\frac{1}{\xi - z} d\xi = \oint_c \frac{-a_n}{\xi^n z}\frac{1}{1 - \frac{\xi}{z}} d\xi$$

$$= \oint_c \frac{-a_n}{z\xi^n}\left(1 + \frac{\xi}{z} + \frac{\xi^2}{z^2} + \ldots + \frac{\xi^{n-1}}{z^{n-1}} + \ldots\right) d\xi$$

$$= 2\pi i \, Res(\xi = 0) = -2\pi i \frac{a_n}{z^n}$$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{\omega(\xi)}{\xi - z} d\xi = -\frac{a_1}{z} - \frac{a_2}{z^2} - \frac{a_3}{z^3} \ldots = -\omega(z) + a_0$$

w/o loss of generality we can take $a_0 = \omega(\infty) = 0$

$$\therefore \frac{1}{\pi i} \oint_C \frac{F_z(\xi)}{\xi - z} d\xi = -\omega(z) + \frac{1}{2\pi i} \oint_C \frac{\bar{\omega}(\bar{\xi})}{\xi - z} d\xi$$

$$\oint_C \frac{\bar{\omega}(\bar{\xi})}{\xi - z} d\xi = \oint_C \frac{\bar{\omega}(\frac{1}{\xi})}{\xi - z} d\xi \quad \text{b/c } \bar{\xi} = \frac{1}{\xi} \text{ on the unit circle}$$

$$\bar{\omega}\left(\frac{1}{\xi}\right) = \bar{a}_0 + \bar{a}_1 \xi + \bar{a}_2 \xi^2 + \ldots$$

Since $\bar{\omega}(\frac{1}{\xi})$ is holomorphic in u.c. but z is outside u.c. $\Rightarrow$ $\oint_C \frac{\bar{\omega}(\frac{1}{\xi})}{\xi - z} d\xi = 0$

$$\therefore \boxed{\omega(z) = -\frac{1}{\pi i} \oint_C \frac{F_z(\xi)}{\xi - z} d\xi}$$

$\leftarrow$ external problem arbitrary traction applied to unit circle

Let's solve for $T_z = +\sigma \sin\theta$

$$\therefore F_z = \int +\sigma \sin\theta \, d\theta = -\sigma \cos\theta$$

$$\therefore \text{ on u.c. } F_z = \frac{-\sigma}{2}\left(\xi + \frac{1}{\xi}\right)$$

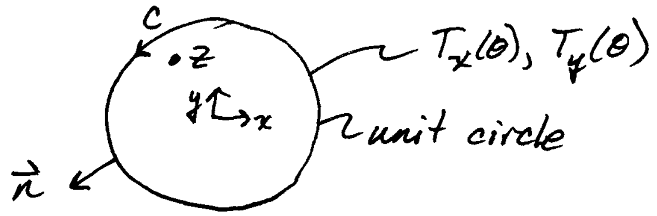
$$\therefore \omega(z) = \frac{+1}{\pi i} \frac{\sigma}{2} \oint_C \frac{\xi + \frac{1}{\xi}}{\xi - z} d\xi$$

$$= +\frac{\sigma}{2\pi i}\left[0 - 2\pi i \frac{1}{z}\right]$$

$$\boxed{\begin{aligned} \omega(z) &= \frac{-\sigma}{z} \\ \omega'(z) &= \frac{\sigma}{z^2} \end{aligned}}$$

* We could use this procedure with a conformal map to solve the loaded crack problem.

In-plane Elasticity – Internal Problem (tractions)



* Note the displacement problem follows exactly the same steps.

normal to right $\rightarrow$ $F_x + iF_y = -i\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_A^B$

where $\phi(z) = F(z)$ ← from previous notes
$\psi(z) = G'(z)$ ←

$$F_x + iF_y = \int_A^B T_x + i\,T_y\, ds = P(z)/i$$

$$\therefore\ P(z) = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}$$

$$\frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\oint_C \frac{\phi(\xi)}{\xi - z}\,d\xi + \frac{1}{2\pi i}\oint_C \frac{\xi\overline{\phi'(\xi)}}{\xi - z}\,d\xi + \frac{1}{2\pi i}\oint_C \frac{\overline{\psi(\xi)}}{\xi - z}\,d\xi$$

ϕ & ψ must be analytic (holomorphic) inside C (i.e. this is the internal problem)

since z is inside C & ϕ holomorphic in C → $\boxed{\frac{1}{2\pi i}\oint_C \frac{\phi(\xi)}{\xi - z}\,d\xi = \phi(z)}$

$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + \ldots \rightarrow \phi(\tfrac{1}{\xi}) = a_0 + a_1/\xi + \ldots$

on C $\bar{\xi} = \dfrac{1}{\xi}$

$$\therefore\ \frac{1}{2\pi i}\oint_C \frac{\xi\overline{\phi'(\xi)}}{\xi - z}\,d\xi = \frac{1}{2\pi i}\oint_C \frac{\xi\bar{a}_1 + 2\bar{a}_2 + 3\bar{a}_3\frac{1}{\xi} + \ldots}{\xi - z}\,d\xi$$

$$= \bar{a}_1 z + 2\bar{a}_2 + \oint_C \frac{\sum_{n=3}^{\infty} n\,\bar{a}_n\,\xi^{2-n}}{\xi - z}\,d\xi$$

$$\frac{\bar{a}_n \xi^{2-n}}{\xi - z} = \frac{\bar{a}_n}{\xi^{n-2}(\xi - z)} \quad \text{has poles at } \xi = 0 \text{ and } \xi = z$$

$$\xi = 0 \rightarrow \frac{\bar{a}_n}{\xi^{n-2}(\xi - z)} = \frac{-\bar{a}_n}{\xi^{n-2} z (1 - \frac{\xi}{z})} = \frac{-\bar{a}_n}{z} \frac{1}{\xi^{n-2}} \left(1 + \frac{\xi}{z} + \frac{\xi^2}{z^2} + \ldots \right)$$

$$\text{Res}[\xi = 0] = \frac{-\bar{a}_n}{z z^{n-3}} = \frac{-\bar{a}_n}{z^{n-2}} \quad \text{for } n \geq 3$$

$$\xi = 1 \rightarrow \frac{\bar{a}_n}{\xi^{n-2}(\xi - z)} = \frac{\bar{a}_n}{\xi - z} \left(\frac{1}{z^{n-2}} + \ldots \right)$$

$$\text{Res}[\xi = z] = \frac{\bar{a}_n}{z^{n-2}} \quad \text{for } n \geq 3$$

$$\therefore \boxed{\frac{1}{2\pi i} \oint_c \frac{\sum_{n=3}^{\infty} \bar{a}_n \xi^{2-n}}{\xi - z} d\xi = \sum_{n=3}^{\infty} \left(-\frac{\bar{a}_n}{z^{n-2}} + \frac{\bar{a}_n}{z^{n-2}} \right) = 0}$$

$$\therefore \boxed{\frac{1}{2\pi i} \oint_c \frac{\xi \bar{\phi}'(\bar{\xi})}{\xi - z} d\xi = \bar{a}_1 z + 2\bar{a}_2}$$

$$\psi(\xi) = b_0 + b_1 \xi + b_2 \xi^2 + \ldots$$

$$\therefore \frac{1}{2\pi i} \oint_c \frac{\bar{\psi}(\bar{\xi})}{\xi - z} d\xi = \frac{1}{2\pi i} \oint_c \frac{\bar{b}_0 + \bar{b}_1/\xi + \bar{b}_2/\xi^2 + \ldots}{\xi - z} d\xi$$

$$= \bar{b}_0 + \underbrace{\frac{1}{2\pi i} \oint_c \frac{\sum_{n=1}^{\infty} \bar{b}_n \xi^{-n}}{\xi - z} d\xi}_{\text{we have already shown this to be zero}}$$

$$\therefore \boxed{\frac{1}{2\pi i} \oint_c \frac{\bar{\psi}(\bar{\xi})}{\xi - z} d\xi = \bar{b}_0}$$

Recall that $2\mu(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}$

$\therefore$ without loss of generality we can set $\boxed{b_0 = 0}$ because it can be absorbed into ϕ without changing stresses or displacements.

$$\therefore \boxed{\frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi - z}\,d\xi = \phi(z) + \bar{a}_1 z + 2\bar{a}_2} \quad *$$

We still need $\bar{a}_1$ and $\bar{a}_2$. Note that

$\phi(z) = a_0 + a_1 z + a_2 z^2 + \ldots$

Differentiate wrt z and set $z=0$ and we get

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi^2}\,d\xi = a_1 + \bar{a}_1}$$

Differentiate 2 time wrt z and set $z=0$ and we get

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi^3}\,d\xi = a_2}$$

Note we can only determine the real part of a_1. $Im(a_1)$ can be set to zero since it represents a rigid body motion.

Note we could determine a_0 from

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi}\,d\xi = a_0 + 2\bar{a}_2}$$

We can use * to determine $\phi(z)$ then we can rearrange our initial equation and manipulate it ~~to~~ determine $\psi(z)$.

i.e. $\psi(z) = \overline{P(z)} - \overline{\phi(z)} - \bar{z}\phi'(z)$

$$\frac{1}{2\pi i}\oint_C \frac{\psi(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\oint_C \frac{\overline{P(\xi)}}{\xi - z}d\xi - \frac{1}{2\pi i}\oint_C \frac{\overline{\phi(\xi)}}{\xi - z}d\xi - \frac{1}{2\pi i}\oint_C \frac{\bar{\xi}\phi'(\xi)}{\xi - z}d\xi$$

by expanding ϕ into its Taylor series it can be shown that

$$\boxed{\begin{aligned}
\psi(z) &= -\bar{a}_0 - \frac{\phi'(z) - a_1}{z} + \frac{1}{2\pi i}\oint_C \frac{\overline{P(\xi)}}{\xi - z}d\xi \\
\phi(z) &= -\bar{a}_1 z - 2\bar{a}_2 + \frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi - z}d\xi \\
a_1 + \bar{a}_1 &= \frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi^2}d\xi \\
a_2 &= \frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi^3}d\xi \\
a_0 + 2\bar{a}_2 &= \frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi}d\xi
\end{aligned}}$$

– Note a_2 will not affect stresses.

Let's take a closer look at a_0, a_1, a_2 and b_0

$$\phi(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$
$$\psi(z) = b_0 + b_1 z + \ldots$$

We will only look at a_0, a_1, a_2, b_0

Stresses : $\sigma_{xx} + \sigma_{yy} = 2\phi'(z) + 2\overline{\phi'(z)}$
$$= 2a_1 + 2 \cdot 2a_2 z + 2\bar{a}_1 + 2 \cdot 2\bar{a}_2 \bar{z}$$
$$= 2(a_1 + \bar{a}_1) + 4(a_2 z + \overline{a_2 z})$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)]$$
$$= 2[\bar{z}\, 2a_2] = 4a_2\bar{z}$$

Displacements : $2\mu(u+iv) = \varkappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}$
$$= \varkappa(a_0 + a_1 z + a_2 z^2)$$
$$- z(\bar{a}_1 + 2\bar{a}_2\bar{z})$$
$$- \bar{b}_0$$

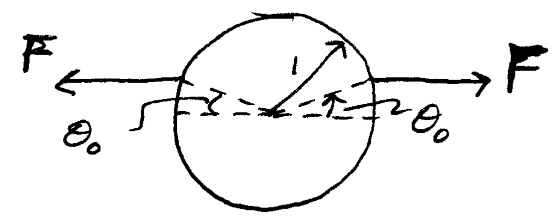
$$2\mu(u+iv) = \varkappa a_0 - \bar{b}_0 + z(a_1 - \bar{a}_1) + (\varkappa - 1) z a_1$$
$$+ a_2(\varkappa z^2) - 2\bar{a}_2 z\bar{z}$$

$\varkappa a_0 - \bar{b}_0$ represents a rigid body translation and we can set $b_0 = 0$ without loosing anything.

$z(a_1 - \bar{a}_1)$ represents a rigid body rotation.

For a traction problem $a_1 + \bar{a}_1$ and a_2 will be determined by the BCs and $a_1 - \bar{a}_1$, a_0, b_0 can be set to zero

Example :



$$P(z) = i\int T_x + iT_y \, ds$$

$$\begin{aligned} &= 0 \quad \text{if } 0 \le \theta \le \theta_0 \\ &= iF \quad \text{if } \theta_0 < \theta < \pi - \theta_0 \\ &= 0 \quad \text{if } \pi - \theta_0 < \theta \le 2\pi \end{aligned}$$

Recall that a_2 will not affect the stress solution $\therefore$ we will not evaluate it. Also to fix rigid body rotation we can take $a_1 - \bar{a}_1 = 0$

$$\therefore \bar{a}_1 + a_1 = 2a_1 = \frac{1}{2\pi i}\oint_C \frac{P(\xi)}{\xi^2}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{\theta_0}^{\pi-\theta_0} iFe^{-2i\theta}\, ie^{i\theta}\,d\theta = \frac{F}{\pi}\cos\theta_0$$

Note $\bar{a}_1 = \frac{F}{2\pi}\cos\theta_0$

$$\phi(z) = -\frac{F}{2\pi}\cos\theta_0 z + \frac{1}{2\pi i}\int_{\theta_0}^{\pi-\theta_0}\frac{iF}{\xi - z}\,d\xi$$

$$= -\frac{F}{2\pi}\cos\theta_0 z + \frac{iF}{2\pi i}\ln(\xi - z)\Big]_{e^{i\theta_0}}^{e^{i(\pi-\theta_0)} = -e^{-i\theta_0}}$$

$$\boxed{\phi(z) = \frac{-F}{2\pi} z\cos\theta_0 + \frac{F}{2\pi}\ln\left(\frac{z + e^{-i\theta_0}}{z - e^{i\theta_0}}\right)}$$

$$\psi(z) = -\frac{\phi'(z) - a_1}{z} + \frac{1}{2\pi i}\oint_c \frac{-iF}{\xi - z}\,d\xi$$

$$\boxed{\psi(z) = -\frac{F}{2\pi}\left[\frac{e^{i\theta_0}}{z+e^{i\theta_0}} + \frac{e^{-i\theta_0}}{z-e^{i\theta_0}}\right] - \frac{F}{2\pi}\ln\left(\frac{z+e^{-i\theta_0}}{z-e^{i\theta_0}}\right)}$$

How easy was that!!!

For the <u>external problem</u> $\phi(z)$ and $\psi(z)$ are analytic <u>outside</u> the unit circle such that

$$\left.\begin{aligned}\phi(z) &= c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots \\ \psi(z) &= d_0 + d_1/z + d_2/z^2 + \dots\end{aligned}\right\}\text{Taylor series}$$

Also $\vec{n}$ now points to the left <u>of the u.c. so</u> we will use $-P(z) = +\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)}$

note that since z is now outside the u.c. we have

$$\frac{1}{2\pi i}\oint_c \frac{\phi(\xi)}{\xi - z}\,d\xi = c_0 - \phi(z) = \phi(\infty) - \phi(z)$$

and a similar relationship holds for $\psi(z)$.

Elliptical Hole

From the conditions at $z \to \infty$ we have

$$\phi_\xi(\xi) = \frac{(\sigma_1+\sigma_2)(a+b)}{8}\xi + \sum_{n=1}^{\infty} \frac{A_n}{\xi^n}$$

$$\psi_\xi(\xi) = \left[\frac{(\sigma_2-\sigma_1)(a+b)}{4} + i\frac{\tau(a+b)}{2}\right]\xi + \sum_{n=1}^{\infty} \frac{B_n}{\xi^n}$$

We also have zero force on any part of the hole

$$\therefore \ \phi_\xi(\xi) + \frac{\omega(\xi)}{\overline{\omega'(\xi)}}\overline{\phi}_\xi'\left(\frac{1}{\xi}\right) + \overline{\psi}_\xi\left(\frac{1}{\xi}\right) = 0 \quad \text{on } \xi = e^{i\theta}$$

$$z = \omega(\xi) = \frac{a+b}{2}\left(\xi + \frac{m}{\xi}\right) \to \omega'(\xi) = \frac{a+b}{2}\left(1 - \frac{m}{\xi^2}\right)$$

$$\therefore \ \phi_\xi(\xi) + \frac{\xi + \frac{m}{\xi}}{1 - \frac{m}{\xi^2}}\overline{\phi}_\xi'\left(\frac{1}{\xi}\right) + \overline{\psi}_\xi\left(\frac{1}{\xi}\right) = 0$$

$$\phi_\xi(\xi) + \frac{\xi^2+m}{\xi - m\xi^3}\overline{\phi}_\xi'\left(\frac{1}{\xi}\right) + \overline{\psi}_\xi\left(\frac{1}{\xi}\right) = 0$$

$$\boxed{\frac{1}{2\pi i}\oint_c \frac{\phi_\xi(\sigma)}{\sigma-\xi}d\sigma + \frac{1}{2\pi i}\oint_c \frac{(\sigma^2+m)\overline{\phi}_\xi'\left(\frac{1}{\sigma}\right)}{(\sigma-m\sigma^3)(\sigma-\xi)}d\sigma + \frac{1}{2\pi i}\oint_c \frac{\overline{\psi}_\xi\left(\frac{1}{\sigma}\right)}{\sigma-\xi}d\sigma = 0}$$

Recall for functions analytic outside of the unit circle, including the point at ∞

$$\frac{1}{2\pi i}\oint_c \frac{f(\xi)}{\xi - z}d\xi = f(\infty) - f(z)$$

note: $f(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots$ where $A_0 = f(\infty)$

$$\therefore \boxed{\begin{aligned}\frac{1}{2\pi i}\oint_c \frac{\phi_\xi(\sigma)}{\sigma-\xi}\,d\sigma &= \frac{1}{2\pi i}\oint_c \frac{\frac{(\sigma_1+\sigma_2)(a+b)}{8}}{\sigma-\xi}\,d\sigma + \frac{1}{2\pi i}\oint_c \frac{\sum_{n=1}^{\infty}\frac{A_n}{\sigma^n}}{\sigma-\xi}\,d\sigma \\ &= 0 - \sum_{n=1}^{\infty}\frac{A_n}{\xi^n} \quad (\xi \text{ outside u.c.})\end{aligned}}$$

$$\frac{1}{2\pi i}\oint_c \frac{(\sigma^2+m)\,\bar{\phi}_\xi'\left(\frac{1}{\sigma}\right)}{(\sigma-m\sigma^3)(\sigma-\xi)}\,d\sigma = \frac{1}{2\pi i}\oint_c \frac{(\sigma^2+m)\left[\frac{(\sigma_1+\sigma_2)(a+b)}{8} - \sum_{n=1}^{\infty} n\bar{A}_n\sigma^{n+1}\right]}{\sigma(1-m\sigma^2)(\sigma-\xi)}\,d\sigma$$

poles only at $\sigma=0$, again ξ is outside contour

at $\sigma=0$ $\quad \dfrac{(\sigma^2+m)\frac{(\sigma_1+\sigma_2)(a+b)}{8}}{\sigma(1-m\sigma^2)(\sigma-\xi)} \rightarrow \text{Res} = -\dfrac{m}{\xi}\dfrac{(\sigma_1+\sigma_2)(a+b)}{8}$

at $\sigma=0$ $\quad \dfrac{n\bar{A}_n\sigma^{n+1}(\sigma^2+m)}{\sigma(1-m\sigma^2)(\sigma-\xi)} \rightarrow \text{Res} = 0$

$$\therefore \boxed{\frac{1}{2\pi i}\oint_c \frac{(\sigma^2+m)\,\bar{\phi}_\xi'\left(\frac{1}{\sigma}\right)}{(\sigma-m\sigma^3)(\sigma-\xi)}\,d\sigma = -\frac{m}{\xi}\frac{(\sigma_1+\sigma_2)(a+b)}{8}}$$

$$\frac{1}{2\pi i}\oint_c \frac{\bar{\psi}_\xi\left(\frac{1}{\sigma}\right)}{\sigma-\xi}\,d\sigma = \frac{1}{2\pi i}\oint_c \frac{\left[\frac{(\sigma_2-\sigma_1)(a+b)}{4} - i\frac{\tau(a+b)}{2}\right]\frac{1}{\sigma} + \sum_{n=1}^{\infty}\bar{B}_n\sigma^n}{\sigma-\xi}\,d\sigma$$

only has pole at $\sigma=0 \rightarrow \text{Res} = -\dfrac{1}{\xi}\left[\dfrac{(\sigma_2-\sigma_1)(a+b)}{4} - i\dfrac{\tau(a+b)}{2}\right]$

$$\therefore \boxed{\frac{1}{2\pi i}\oint_c \frac{\bar{\psi}_\xi\left(\frac{1}{\sigma}\right)}{\sigma-\xi}\,d\sigma = \frac{-1}{\xi}\left[\frac{(\sigma_2-\sigma_1)(a+b)}{4} - i\frac{\tau(a+b)}{2}\right]}$$

$$\therefore -\sum_{n=1}^{\infty}\frac{A_n}{\xi^n} - \frac{m}{\xi}\frac{(\sigma_1+\sigma_2)(a+b)}{8} - \frac{1}{\xi}\left[\frac{(\sigma_2-\sigma_1)(a+b)}{4} - i\frac{\tau(a+b)}{2}\right] = 0$$

$$\therefore \sum_{n=1}^{\infty}\frac{A_n}{\xi^n} = -\frac{m}{\xi}\frac{(\sigma_1+\sigma_2)(a+b)}{8} - \frac{1}{\xi}\left[\frac{(\sigma_2-\sigma_1)(a+b)}{4} - i\frac{\tau(a+b)}{2}\right]$$

$$= \frac{1}{\xi}\left[\frac{3b+a}{8}\sigma_1 - \frac{3a+b}{8}\sigma_2 + i\frac{\tau(a+b)}{2}\right]$$

$$\therefore \boxed{\Phi_\xi(\xi) = \frac{(\sigma_1+\sigma_2)(a+b)}{8}\xi + \left(\frac{3b+a}{8}\sigma_1 - \frac{3a+b}{8}\sigma_2 + i\frac{a+b}{2}\tau\right)\frac{1}{\xi}}$$

Let $X = \frac{(\sigma_1+\sigma_2)(a+b)}{8}$ and $Y = \frac{3b+a}{8}\sigma_1 - \frac{3a+b}{8}\sigma_2 + i\frac{a+b}{2}\tau$

then

$$\Phi_\xi(\xi) = X\xi + Y\frac{1}{\xi}\ ,\quad \Phi_\xi'(\xi) = X - Y\frac{1}{\xi^2}$$

From BC: $\Phi_\xi(\xi) + \frac{\xi^2+m}{\xi - m\xi^3}\bar{\Phi}_\xi'\left(\frac{1}{\xi}\right) + \bar{\Psi}_\xi\left(\frac{1}{\xi}\right) = 0$ on $e^{i\theta}=\xi$

$$\therefore \bar{\Phi}_\xi\left(\frac{1}{\xi}\right) + \frac{\xi+m\xi^3}{\xi^2-m}\Phi_\xi'(\xi) + \Psi_\xi(\xi) = 0 \text{ on } \xi = e^{i\theta}$$

$$\therefore \Psi_\xi(\xi) = -\bar{\Phi}_\xi\left(\frac{1}{\xi}\right) - \frac{\xi+m\xi^3}{\xi^2-m}\Phi_\xi'(\xi) \text{ on } \xi = e^{i\theta}$$

$$\boxed{\frac{1}{2\pi i}\oint_c \frac{\Psi_\xi(\sigma)}{\sigma-\xi}d\sigma = \frac{-1}{2\pi i}\oint_c \frac{\bar{\Phi}_\xi\left(\frac{1}{\sigma}\right)}{\sigma-\xi}d\sigma - \frac{1}{2\pi i}\oint_c \frac{\sigma+m\sigma^3}{\sigma^2-m}\frac{\Phi_\xi'(\sigma)}{\sigma-\xi}d\sigma}$$

$$\frac{1}{2\pi i}\oint_c \frac{\Psi_\xi(\sigma)}{\sigma-\xi}d\sigma = -\sum_{n=1}^{\infty}\frac{B_n}{\xi^n}$$

Recall ξ is outside u.c.

$$-\frac{1}{2\pi i}\oint_c \frac{\bar{\Phi}_\xi\left(\frac{1}{\sigma}\right)}{\sigma-\xi}d\sigma = \frac{-1}{2\pi i}\oint_c \frac{\bar{X}}{\sigma(\sigma-\xi)} + \frac{\bar{Y}\sigma}{\sigma-\xi}d\sigma = \frac{\bar{X}}{\xi} + 0$$

Note: $(\bar{X} = X)$

$$-\frac{1}{2\pi i}\oint_c \frac{\sigma+m\sigma^3}{\sigma^2-m}\frac{\Phi_\xi'(\sigma)}{\sigma-\xi}d\sigma = \frac{-1}{2\pi i}\oint_c \frac{\sigma+m\sigma^3}{(\sigma^2-m)(\sigma-\xi)}\left(X - \frac{Y}{\sigma^2}\right)d\sigma$$

poles at $\sigma = 0$ for Y/σ^2 term $\rightarrow$ Res $= \frac{Y}{m\xi}$

and $\sigma = \pm\sqrt{m}$ for both terms

$$\rightarrow \text{Res} = Y\frac{1+m^2}{2m(\sqrt{m}-\xi)} - X\frac{1+m^2}{2(\sqrt{m}-\xi)} - Y\frac{1+m^2}{2m(\sqrt{m}+\xi)} + X\frac{1+m^2}{2(\sqrt{m}+\xi)}$$

$$-\frac{1}{2\pi i}\oint_c \frac{\sigma+m\sigma^3}{\sigma^2-m}\frac{\Phi_\xi'(\sigma)}{\sigma-\xi}d\sigma = \frac{Y}{m\xi} + Y\frac{2\xi+2m^2\xi}{2m(m-\xi^2)} + X\frac{-2\xi-2m^2\xi}{2(m-\xi^2)}$$

$$= \frac{Y}{m}\left(\frac{1}{\xi} + \frac{\xi(1+m^2)}{m-\xi^2}\right) + X\frac{-\xi(1+m^2)}{m-\xi^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{B_n}{\xi^n} = -\left(X + \frac{Y}{m}\right)\frac{1}{\xi} + \left(X - \frac{Y}{m}\right)\frac{\xi(1+m^2)}{m-\xi^2}$$

$$= -\left(X+\frac{Y}{m}\right)\frac{1}{\xi} + \left(X - \frac{Y}{m}\right)\frac{(1+m^2)(\xi^2-m)+m(1+m^2)}{\xi(m-\xi^2)}$$

$$= -\left(X+\frac{Y}{m}\right)\frac{1}{\xi} + \left(X-\frac{Y}{m}\right)\frac{-(1+m^2)}{\xi} + \left(X-\frac{Y}{m}\right)\frac{m(1+m^2)}{\xi(m-\xi^2)}$$

$$= \left[-(2+m^2)X + mY\right]\frac{1}{\xi} + \left(\frac{Y}{m} - X\right)\frac{m(m^2+1)}{\xi^3 - m\xi}$$

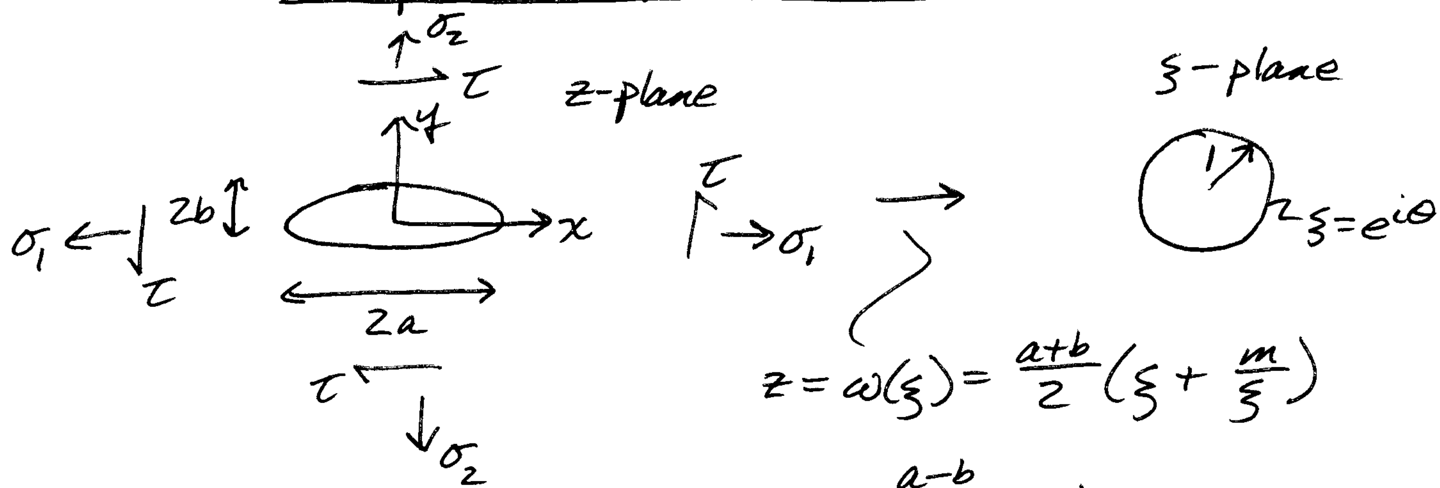
$$\sum_{n=1}^{\infty} \frac{B_n}{\xi^n} = \left(-\frac{a^2+3b^2}{4(a+b)}\sigma_1 - \frac{3a^2+b^2}{4(a+b)}\sigma_2 + i\,\frac{a-b}{2}\tau\right)\frac{1}{\xi}$$

$$+\left[\frac{b(a^2+b^2)}{(a+b)^2}\sigma_1 - \frac{a(a^2+b^2)}{(a+b)^2}\sigma_2 + i\,\frac{a^2+b^2}{a+b}\tau\right]\frac{1}{\xi^3-m\xi}$$

$$\boxed{\begin{aligned}
\therefore\ \psi_\xi(\xi) &= \left[\frac{(\sigma_2-\sigma_1)(a+b)}{4} + i\,\frac{\tau(a+b)}{2}\right]\xi \\
&\quad + \left(-\frac{a^2+3b^2}{4(a+b)}\sigma_1 - \frac{3a^2+b^2}{4(a+b)}\sigma_2 + i\,\frac{a-b}{2}\tau\right)\frac{1}{\xi} \\
&\quad + \left[\frac{b(a^2+b^2)}{(a+b)^2}\sigma_1 - \frac{a(a^2+b^2)}{(a+b)^2}\sigma_2 + i\,\frac{a^2+b^2}{a+b}\tau\right]\frac{1}{\xi^3-m\xi} \\[1em]
\phi_\xi(\xi) &= \frac{(\sigma_1+\sigma_2)(a+b)}{8}\xi + \left(\frac{3b+a}{8}\sigma_1 - \frac{3a+b}{8}\sigma_2 + i\,\frac{a+b}{2}\tau\right)\frac{1}{\xi} \\[1em]
z &= \frac{a+b}{2}\left(\xi + \frac{m}{\xi}\right) \\[1em]
\phi_z'(z) &= \phi_\xi'(\xi)\frac{1}{\left(\frac{dz}{d\xi}\right)}, \quad \psi_z'(z) = \psi_\xi'(\xi)\frac{1}{\left(\frac{dz}{d\xi}\right)}
\end{aligned}}$$

Elliptical Hole Solution

Ⓐ



$$z = \omega(\xi) = \frac{a+b}{2}\left(\xi + \frac{m}{\xi}\right)$$

$$-1 < m = \frac{a-b}{a+b} < 1$$

BCs as $|z| \to \infty$

$$\sigma_{xx} \to \sigma_1$$
$$\sigma_{yy} \to \sigma_2$$
$$\sigma_{xy} \to \tau$$

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}\left[\phi'(z)\right] = \sigma_1 + \sigma_2 \quad \text{as } |z| \to \infty$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\,\phi''(z) + \psi'(z)\right] = \sigma_2 - \sigma_1 + 2i\tau \quad \text{as } |z| \to \infty$$

$$\therefore \to \quad \phi(z) = \frac{\sigma_1 + \sigma_2}{4} z + \text{[redacted]} \sum_{n=1}^{\infty} \frac{a_n}{z^n}$$

$$\psi(z) = \left(\frac{\sigma_2 - \sigma_1}{2} + i\tau\right) z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

but note that a $|z| \to \infty$ $\quad z \to \frac{a+b}{2}\xi$

$$\therefore \quad \phi(\xi) = \frac{(\sigma_1 + \sigma_2)(a+b)}{8}\xi + \sum_{n=1}^{\infty} \frac{A_n}{\xi^n}$$

$$\psi(\xi) = \left[\frac{(\sigma_2 - \sigma_1)(a+b)}{4} + i\,\frac{\tau(a+b)}{2}\right]\xi + \sum_{n=1}^{\infty} \frac{B_n}{\xi^n}$$

B

Take $C_1 = \dfrac{(\sigma_1+\sigma_2)(a+b)}{8}$, $C_2 = \dfrac{(\sigma_2-\sigma_1)(a+b)}{4} + i\dfrac{\tau(a+b)}{2}$

BCs on ellipse boundary, i.e. on $\xi = e^{i\theta}$

No net force $\rightarrow$ $\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} = 0$ on $\xi = e^{i\theta}$

$$\phi'(z) = \phi'(\xi)\frac{1}{\frac{dz}{d\xi}} = \frac{1}{\omega'(\xi)}\phi'(\xi)$$

Then along with $\bar{\xi} = \frac{1}{\xi}$ on $\xi = e^{i\theta}$ we get

$$\therefore \quad \phi(\xi) + \frac{\omega(\xi)}{\overline{\omega'}(\frac{1}{\xi})}\overline{\phi'}\left(\frac{1}{\xi}\right) + \overline{\psi}\left(\frac{1}{\xi}\right) = 0 \text{ on } \xi = e^{i\theta}$$

$$\phi(\xi) + \frac{\xi + \frac{m}{\xi}}{1-m\xi^2}\overline{\phi'}\left(\frac{1}{\xi}\right) + \overline{\psi}\left(\frac{1}{\xi}\right) = 0$$

$$\frac{1}{2\pi i}\oint_C \frac{\phi(\sigma)}{\sigma-\xi}d\sigma + \frac{1}{2\pi i}\oint_C \frac{\sigma^2+m}{\sigma-m\sigma^3}\frac{\overline{\phi'}(\frac{1}{\sigma})}{\sigma-\xi}d\sigma + \frac{1}{2\pi i}\oint_C\frac{\overline{\psi}(\frac{1}{\sigma})}{\sigma-\xi}d\sigma = 0$$

<u>Consider</u>

$$\frac{1}{2\pi i}\oint_C \frac{\phi(\sigma)}{\sigma-\xi}d\sigma = \frac{1}{2\pi i}\oint_C \frac{C_1\sigma + \sum_{n=1}^{\infty}\frac{A_n}{\sigma^n}}{\sigma-\xi}d\sigma$$

$\dfrac{C_1\sigma}{\sigma-\xi}$ has a pole at $\sigma = \xi$ but ξ is outside the unit circle so this term does not contribute to the integral

$$\frac{A_n}{\sigma^n(\sigma-\xi)} = \frac{-A_n}{\sigma^n}\frac{1}{\xi(1-\frac{\sigma}{\xi})} = \frac{-A_n}{\sigma^n\xi}\left(1 + \frac{\sigma}{\xi} + \ldots + \left(\frac{\sigma}{\xi}\right)^{n-1} + \ldots\right)$$

$\therefore$ Residue at $\sigma = 0$ $= \dfrac{-A_n}{\xi^n}$

$$\therefore \quad -\sum_{n=1}^{\infty} \frac{A_n}{\xi^n} - \frac{m\bar{C}_1}{\xi} - \frac{\bar{C}_2}{\xi} = 0$$

$$\therefore \quad \sum_{n=1}^{\infty} \frac{A_n}{\xi^n} = -\frac{(m\bar{C}_1 + \bar{C}_2)}{\xi} = -\frac{C_3}{\xi}$$

$$C_3 = m\bar{C}_1 + \bar{C}_2$$

$$\therefore \quad \boxed{\phi(\xi) = C_1\xi - \frac{C_3}{\xi}}$$

Now to determine $\psi(\xi)$ we return to the net force equation, but take the complex conjugate.

$$\text{i.e.} \quad \overline{\phi(\xi)} + \frac{\bar{\xi} + \frac{m}{\bar{\xi}}}{1 - m\bar{\xi}^2} \phi'\left(\frac{1}{\bar{\xi}}\right) + \psi\left(\frac{1}{\bar{\xi}}\right) = 0$$

$$\bar{\phi}\left(\frac{1}{\xi}\right) + \frac{\frac{1}{\xi} + m\xi}{1 - \frac{m}{\xi^2}} \phi'(\xi) + \psi(\xi) = 0$$

$$\frac{1}{2\pi i} \oint_C \frac{\bar{\phi}\left(\frac{1}{\sigma}\right)}{\sigma - \xi} d\sigma + \frac{1}{2\pi i} \oint_C \frac{\sigma(1 + m\sigma^2)}{\sigma^2 - m} \frac{\phi'(\sigma)}{\sigma - \xi} d\sigma + \frac{1}{2\pi i} \oint_C \frac{\psi(\sigma)}{\sigma - \xi} d\sigma = 0$$

<u>Consider</u>

$$\frac{1}{2\pi i} \oint_C \frac{\psi(\sigma)}{\sigma - \xi} d\sigma = \frac{1}{2\pi i} \oint_C \frac{C_2\sigma + \sum_{n=1}^{\infty} \frac{B_n}{\sigma^n}}{\sigma - \xi} d\sigma$$

as ~~for~~ ϕ previously we can show that

$$\frac{1}{2\pi i} \oint_C \frac{\psi(\sigma)}{\sigma - \xi} d\sigma = \sum_{n=1}^{\infty} -\frac{B_n}{\xi^n} = -\sum_{n=1}^{\infty} \frac{B_n}{\xi^n}$$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{\phi(\sigma)}{\sigma-\xi}\,d\sigma = -\sum_{n=1}^{\infty}\frac{A_n}{\xi^n}$$

Consider,

$$\frac{1}{2\pi i}\oint_c \frac{\overline{\Psi}(\frac{1}{\sigma})}{\sigma-\xi}\,d\sigma = \frac{1}{2\pi i}\oint_c \frac{\overline{C}_2\frac{1}{\sigma}+\sum_{n=1}^{\infty}\frac{\overline{B}_n}{(\frac{1}{\sigma})^n}}{\sigma-\xi}\,d\sigma$$

$\dfrac{\overline{C}_2}{\sigma(\sigma-\xi)}$ has poles at $\underset{\text{inside}}{\sigma=0}$ and $\underset{\text{outside}}{\sigma=\xi}$

Simple pole $\rightarrow$ $\text{Res}(\sigma=0) = \dfrac{-\overline{C}_2}{\xi} = \lim_{\sigma\to 0}\sigma\dfrac{\overline{C}_2}{\sigma(\sigma-\xi)}$

$\therefore \dfrac{1}{2\pi i}\oint_c \dfrac{\overline{\Psi}(\frac{1}{\sigma})}{\sigma-\xi} = \dfrac{-\overline{C}_2}{\xi}$, b/c $\dfrac{\overline{B}_n\sigma^n}{\sigma-\xi}$ has no poles inside u.c.

Consider

$$\frac{1}{2\pi i}\oint_c \frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\frac{\overline{\Phi}'(\frac{1}{\sigma})}{\sigma-\xi}\,d\sigma$$

$$= \frac{1}{2\pi i}\oint_c \frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\frac{\overline{C}_1+\sum_{n=1}^{\infty}(-n)A_n\sigma^{n+1}}{\sigma-\xi}\,d\sigma$$

poles at $\sigma = 0, \xi, \pm\sqrt{\frac{1}{m}}$

0: inside; $\xi, \pm\sqrt{\frac{1}{m}}$: outside u.c.

$$\text{Res}(\sigma=0) = \lim_{\sigma\to 0}\sigma\frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\frac{\overline{C}_1+\sum_{n=1}^{\infty}(-n)A_n\sigma^{n+1}}{\sigma-\xi} = -\frac{m\overline{C}_1}{\xi}$$

$$\therefore \frac{1}{2\pi i}\oint_c \frac{\sigma^2+m}{\sigma-m\sigma^3}\frac{\overline{\Phi}'(\frac{1}{\sigma})}{\sigma-\xi}\,d\sigma = \frac{-m\overline{C}_1}{\xi}$$

E

<u>Consider</u>

$$\frac{1}{2\pi i}\oint_C \frac{\overline{\Phi}\left(\frac{1}{\sigma}\right)}{\sigma-\xi}\,d\sigma = \frac{1}{2\pi i}\oint_C \frac{\overline{C_1}\frac{1}{\sigma} - \overline{C_3}\sigma}{\sigma-\xi}\,d\sigma$$

poles at $\sigma = 0,\ \xi$

ξ outside u.c.

$$\operatorname{Res}(\sigma=0) = \lim_{\sigma\to 0} \sigma\,\frac{\overline{C_1}\frac{1}{\sigma} - \overline{C_3}\sigma}{\sigma-\xi} = -\frac{\overline{C_1}}{\xi}$$

$$\therefore\ \frac{1}{2\pi i}\oint_C \frac{\overline{\Phi}\left(\frac{1}{\sigma}\right)}{\sigma-\xi} = -\frac{\overline{C_1}}{\xi}$$

<u>Consider</u>

$$\frac{1}{2\pi i}\oint_C \frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\,\frac{C_1+\frac{C_3}{\sigma^2}}{\sigma-\xi} = \frac{1}{2\pi i}\oint_C \frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\,\frac{\phi'(\sigma)}{\sigma-\xi}\,d\sigma$$

poles at $\sigma = 0,\ \sqrt{m},\ -\sqrt{m},\ \xi$

$0,\ \sqrt{m},\ -\sqrt{m}$ inside u.c.

ξ outside u.c.

$$\operatorname{Res}(\sigma=0) = \lim_{\sigma\to 0} \sigma\,\frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\,\frac{C_1+\frac{C_3}{\sigma^2}}{\sigma-\xi} = \frac{C_3}{m\xi}$$

$$\operatorname{Res}(\sigma=\sqrt{m}) = \lim_{\sigma\to\sqrt{m}} (\sigma-\sqrt{m})\frac{\sigma(1+m\sigma^2)}{(\sigma-\sqrt{m})(\sigma+\sqrt{m})}\,\frac{C_1+\frac{C_3}{\sigma^2}}{\sigma-\xi}$$

$$= \frac{\sqrt{m}(1+m^2)}{2\sqrt{m}}\,\frac{C_1+\frac{C_3}{m}}{\sqrt{m}-\xi} = +\frac{(1+m^2)}{2m}\,\frac{C_1 m + C_3}{\sqrt{m}-\xi}$$

$$\operatorname{Res}(\sigma=-\sqrt{m}) = \lim_{\sigma\to-\sqrt{m}} (\sigma+\sqrt{m})\frac{\sigma(1+m\sigma^2)}{(\sigma-\sqrt{m})(\sigma+\sqrt{m})}\,\frac{C_1+\frac{C_3}{\sigma^2}}{\sigma-\xi}$$

$$= \frac{-\sqrt{m}(1+m^2)}{-2\sqrt{m}}\,\frac{C_1+\frac{C_3}{m}}{-\sqrt{m}-\xi} = -\frac{(1+m^2)}{2m}\,\frac{C_1 m + C_3}{\sqrt{m}+\xi}$$

(F)

$$\therefore \frac{1}{2\pi i}\oint_c \frac{\sigma(1+m\sigma^2)}{\sigma^2-m}\frac{\phi'(\sigma)}{\sigma-\xi}\,d\sigma = \frac{C_3}{m\xi} + \frac{1+m^2}{2m}C_4\left(\frac{1}{\sqrt{m}-\xi} - \frac{1}{\sqrt{m}+\xi}\right)$$

where $C_4 = mC_1 + C_3$

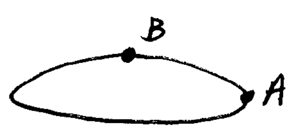
$$= \frac{C_3}{m\xi} + \frac{1+m^2}{2m}C_4\frac{2\xi}{m-\xi^2}$$

$$= \frac{C_3}{m\xi} + \frac{1+m^2}{m}C_4\frac{\xi}{m-\xi^2}$$

$$\therefore -\frac{\overline{C_1}}{\xi} + \frac{C_3}{m\xi} + \frac{1+m^2}{m}C_4\frac{\xi}{m-\xi^2} - \sum_{n=1}^{\infty}\frac{B_n}{\xi^n} = 0$$

$$\therefore \sum_{n=1}^{\infty}\frac{B_n}{\xi^n} = \left(C_3/m - \overline{C_1}\right)\frac{1}{\xi} + \frac{1+m^2}{m}C_4\frac{\xi}{m-\xi^2}$$

$$\therefore \boxed{\psi(\xi) = C_2\xi + \left(\frac{C_3}{m} - \overline{C_1}\right)\frac{1}{\xi} + \frac{1+m^2}{m}C_4\frac{\xi}{m-\xi^2}}$$



at A $\sigma_{xx} = 0 \rightarrow \sigma_{yy} = 4\,\mathrm{Re}[\phi'(z)] = 4\left(\mathrm{Re}\left[\phi'(\xi)\frac{1}{\omega'(\xi)}\right]\right)$

$$= 4\,\mathrm{Re}\left[\left(C_1 + \frac{C_3}{\xi^2}\right)\frac{1}{\frac{a+b}{2}\left(1-\frac{m}{\xi^2}\right)}\right]$$

ξ at $A = 1 \rightarrow$ $= 4\left[\left(C_1 + \mathrm{Re}[C_3]\right)\frac{2}{(a+b)\left(1-\frac{a-b}{a+b}\right)}\right]$

$\overline{C_1} = C_1 \rightarrow$ $= 4\left[C_1(1+m) + \mathrm{Re}[C_2]\right]\frac{2}{2b}$

$$= \frac{\cancel{4}}{b}\left\{\frac{(\sigma_1+\sigma_2)\cancel{(a+b)}}{8\cancel{4}}\frac{\cancel{2}a}{\cancel{a+b}} + \frac{(\sigma_2-\sigma_1)(a+b)}{\cancel{4}}\right\}$$

$$\sigma_{yy}(x=a, y=0) = (\sigma_1+\sigma_2)\frac{a}{b} + (\sigma_2-\sigma_1)\left(\frac{a}{b}+1\right) = \sigma_2\left(1+2\frac{a}{b}\right) - \sigma_1$$