Notation

Throughout this course standard index notation will be used. The following definitions and conventions will be used.

Kronecker delta (identity tensor): δ_{ij}

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$

Permutation tensor: ϵ_{ijk}

where $\epsilon_{ijk} = 1$ if $ijk = 123, 231, 312$ (even permutations of 123)

$\epsilon_{ijk} = -1$ if $ijk = 321, 132, 213$ (odd permutations of 123)

$\epsilon_{ijk} = 0$ otherwise

Vectors: The vector $\mathbf{A} = \sum_{i=1}^{3} A_i \mathbf{e}_i$ where A_i are the components of $\mathbf{A}$ in an orthogonal Cartesian basis with coordinates x_i and associated unit vectors along these coordinate directions $\mathbf{e}_i$. Use of the Einstein summation convention allows us to use a shorthand notation, where the summation symbol is dropped and summation from 1 to 3 is assumed over all repeated indices, i.e. $\mathbf{A} = A_i \mathbf{e}_i = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3$.

Example, del operator: $\nabla = \mathbf{e}_1 \dfrac{\partial}{\partial x_1} + \mathbf{e}_2 \dfrac{\partial}{\partial x_2} + \mathbf{e}_3 \dfrac{\partial}{\partial x_3} = \mathbf{e}_i \dfrac{\partial}{\partial x_i}$

Example: $A_i \delta_{ij} = A_1 \delta_{1j} + A_2 \delta_{2j} + A_3 \delta_{3j}$

$= A_1$ if $j = 1$

$= A_2$ if $j = 2$

$= A_3$ if $j = 3$

$= A_j$

Dot product: $\mathbf{A} \cdot \mathbf{B} = A_i B_i$

Cross product: $\mathbf{C} = \mathbf{A} \times \mathbf{B} \Rightarrow C_i = \epsilon_{ijk} A_j B_k$

Derivatives are represented by a subscript ,*j*, e.g. $\dfrac{\partial A}{\partial x_i} = A_{,i}$.

Example, divergence: $\nabla \cdot \mathbf{A} = \left(\mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \right) \cdot \left(A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_2 \mathbf{e}_3 \right)$

$$= \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(A_j \mathbf{e}_j \right)$$

$$= \frac{\partial A_j}{\partial x_i} \mathbf{e}_i \cdot \mathbf{e}_j$$

$$= \frac{\partial A_j}{\partial x_i} \delta_{ij}$$

$$= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_2}{\partial x_2} = A_{i,i}$$

Note that the third line above is only valid for a non-rotating Cartesian basis. Also note that it was necessary to use separate i and j indices on the second line since having i appearing 4 times would have no meaning.

Rotation of basis

To determine how the components of a given vector change under a rotation of the Cartesian coordinate system we note that the vector can be written with reference to either system.

$$\mathbf{A} = A_i \mathbf{e}_i = A'_j \mathbf{e}'_j$$

$$\mathbf{A} \cdot \mathbf{e}'_j = A_i \mathbf{e}_i \cdot \mathbf{e}'_j = A'_i \mathbf{e}'_i \cdot \mathbf{e}'_j = A'_i \delta_{ij} = A'_j$$

$$A'_j = a_{ij} A_i$$

Where $a_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$ is the cosine of the angle between the x_i direction and the x'_j direction. By performing the dot product of the first equation with $\mathbf{e}_j$ and recognizing the equivalence with the result of inverting the third equation, it can be shown that $\left(a_{ij} \right)^{-1} = \mathbf{e}'_i \cdot \mathbf{e}_j = a_{ji}$, or $\mathbf{a}^{-1} = \mathbf{a}^T$, and hence $\mathbf{a}$ is an orthogonal matrix.

Tensors

Tensors of arbitrary rank can be represented with dyadic notation. For example, the second rank stress tensor can be written as $\underset{\approx}{\boldsymbol{\sigma}} = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$. Transformations of tensor components can then be carried out as follows.

$$\underset{\approx}{\boldsymbol{\sigma}} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \sigma'_{ij}\mathbf{e}'_i \otimes \mathbf{e}'_j$$

$$\mathbf{e}'_k \cdot \underset{\approx}{\boldsymbol{\sigma}} \cdot \mathbf{e}'_l = \sigma_{ij}\mathbf{e}'_k \cdot \mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}'_l = \sigma'_{ij}\mathbf{e}'_k \cdot \mathbf{e}'_i \otimes \mathbf{e}'_j \cdot \mathbf{e}'_l$$

$$\sigma'_{ij}\delta_{ki}\delta_{jl} = \sigma_{ij}a_{ik}a_{jl}$$

$$\sigma'_{kl} = a_{ik}a_{jl}\sigma_{ij}$$

This procedure can be applied to tensors of arbitrary rank, $\underset{\sim}{\mathbf{b}} = b_{ijk\ldots p}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \ldots \otimes \mathbf{e}_p$, and the transformation formulas for the components follow as,

$$b'_{lmn\ldots q} = a_{il}a_{jm}a_{kn}\ldots a_{pq}b_{ijk\ldots p}.$$

Electrostatics

The fundamental quantity in electrostatics is charge. The purpose of electrostatics is to determine the forces and energies of charges given some configuration of their positions. The theory of electrostatics can be built from the simple expression for the force between two charges.

Coulomb's Law: $\mathbf{F}_{ij} = -\dfrac{1}{4\pi\epsilon_0}\dfrac{Q_i Q_j}{r^2}\mathbf{e}_{ij}$ (no summation implied)

Throughout this section, subscripts will refer to the numbering of charges, *not* to the components of vectors or tensors. Here, Q_i and Q_j are the two charges under consideration, $\mathbf{F}_{ij}$ is the force on Q_i due to Q_j, $\mathbf{e}_{ij}$ is the unit vector in the direction from Q_i towards Q_j, r is the distance between the two charges, and ϵ_0 is the permittivity of free space, $\epsilon_0 = 8.854\times10^{-12}\ \mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$. This expression indicates that the force between two charges is inversely proportional to the square of the distance between them and directed along the line joining their locations. Furthermore, the sign indicates that like charges repel one another and opposite charges attract one another.

The force on a given charge due to some set $(1\ldots N)$ of other charges is a simple sum of the contributions from each charge interaction, i.e. there are no three, four, ... N-body interactions.

$$\mathbf{F}_i = \sum_{\substack{j=1\\ j\neq i}}^{N}\mathbf{F}_{ij} = -\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{4\pi\epsilon_0}\frac{Q_i Q_j}{r^2}\mathbf{e}_{ij}$$

Note that a given charge does not contribute a force on itself.

Next, we would like to compute how much work it takes to set up some distribution of charges. We first recognize that since a charge does not place a force on itself it does not take any work to place it at some location in space. (We will not concern ourselves with the fact that the energy of the electric field that the charge sets up is infinite.) We will take the location of this first charge to be the origin of our coordinate system. Now consider bringing a second charge from a distance far from the first charge (formally an infinite distance so the force is zero) to its final location. The increment of work done on this charge during some part of this journey is,

$$dW = \mathbf{F}\cdot d\mathbf{x}$$

Setting up a spherical coordinate system centered on Q_1, the force on the second charge and the incremental displacement can be written as,

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}\mathbf{e}_r$$

$$d\mathbf{x} = \mathbf{e}_r dr + \mathbf{e}_\theta r d\theta + \mathbf{e}_\phi r \sin\theta d\phi$$

Then, the increment of work is simply, $dW = -\frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}dr$, and the total work done to take this second charge from infinity to its final location is,

$$W = \int dW = \int_\infty^r -\frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r'^2}dr' = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r}$$

Notice that this integral was entirely independent of the path that this second charge took to get to its final location. Hence, it is possible to interpret this total work as the change in the potential energy of the second charge. We define the potential energy of two charges as,

$$V_{ij} = \frac{1}{4\pi\epsilon_0}\frac{Q_i Q_j}{r_{ij}}.$$

Here r_{ij} is the distance between Q_i and Q_j. We can now consider bringing a third charge into place, and then a fourth and so on. Since the forces between multiple charges are additive, it can be shown that the total work done in this process is simply,

$$W = \tfrac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} V_{ij} = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{8\pi\epsilon_0}\frac{Q_i Q_j}{r_{ij}}$$

Note that the factor of $\frac{1}{2}$ arises due to the double counting inherent in the double summation process. Also, this equation suggests, and it is true, that it does not matter in which order the charge configuration is set up, i.e. conceptually we can choose any of the charges to be the "first" charge. Returning to the expression for the potential energy, it is possible to define the *electric potential* field that is set up due to a given charge as,

$$\phi_i = \frac{1}{4\pi\epsilon_0}\frac{Q_i}{r}$$

Here r must be interpreted as the distance from the charge Q_i. Then the potential energy can be written as,

$V_{ij} = \phi_i Q_j$.

Returning to our expression for the force between two charges and noting that

$$\begin{aligned}\mathbf{F}_{ji} &= -\nabla V_{ij} = \underbrace{-\nabla \phi_i}_{\mathbf{E}_i} Q_j \\ &= \left(-\frac{\partial \phi_i}{\partial r}\mathbf{e}_r - \frac{1}{r}\frac{\partial \phi_i}{\partial \theta}\mathbf{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial \phi_i}{\partial \phi}\mathbf{e}_\phi \right) Q_j \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q_i Q_j}{r^2}\mathbf{e}_r = \frac{1}{4\pi\epsilon_0}\frac{Q_i Q_j}{r^2}\mathbf{e}_{ij} = -\frac{1}{4\pi\epsilon_0}\frac{Q_i Q_j}{r^2}\mathbf{e}_{ji} \\ &= \mathbf{E}_i Q_j\end{aligned}$$

Here we have defined the *electric field* as the opposite of the gradient of the electric potential. This definition allows us to identify the electric field set up by a point charge and rewrite the expression for the force due to this charge on the other charges in the system.

If we know the electric potential field in the vicinity of a given point due to some distribution of charges then we can calculate the electric field in this vicinity as $\mathbf{E} = -\nabla\phi$. If we then place some charge Q at this location it will experience a force on it given by $\mathbf{F} = Q\mathbf{E}$.

Let's take a closer look at the vector field $\mathbf{E}$. First notice that since the components of $\mathbf{E}$ can be derived from one scalar field ϕ, these components cannot be truly independent of one another. In fact, we can show that the vector field $\mathbf{E}$ must be curl-free.

$\nabla \times \mathbf{E} = \nabla \times \nabla\phi = 0$

This can be demonstrated with index notation as follows (here our indices do represent components of vectors).

$$\begin{aligned}\left(\nabla\times\mathbf{E}\right)_i &= \epsilon_{ijk}E_{k,j}\\ &= -\epsilon_{ijk}\phi_{,kj} \quad \text{definition } E_i = -\phi_{,i}\\ &= \epsilon_{ikj}\phi_{,kj} \quad \text{sign change due to change of permutation}\\ &= \epsilon_{ikj}\phi_{,jk} \quad \text{change order of differentiation of } \phi\\ &= \epsilon_{ijk}\phi_{,kj} \quad \text{change dummy indices } j\Rightarrow p\Rightarrow k,\ k\Rightarrow q\Rightarrow j\\ &= 0 \qquad a = \text{-}a \ \Rightarrow\ a = 0\end{aligned}$$

Next let's consider the divergence of the electric field due to an arbitrary distribution of charges. To do this we first consider a single charge and then we can build up the arbitrary distribution using superposition. Consider the integral,

$$\int_S \mathbf{E}\cdot\mathbf{n}\ dS$$

where S is some arbitrarily shaped closed surface in space and $\mathbf{n}$ is the unit outward pointing normal to this surface. Again, we will consider the contribution to this integral from one "piece" of charge dQ, and we will set up a spherical coordinate system to perform the integration.

$$\mathbf{E}\cdot\mathbf{n}\ dS = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2}\mathbf{e}_r\cdot\mathbf{n}\ \frac{r^2}{\left|\mathbf{e}_r\cdot\mathbf{n}\right|}\sin\theta\, d\theta\ d\phi$$

The factor of $1/\left|\mathbf{e}_r\cdot\mathbf{n}\right|$ in the expression for dS arises when the normal of the differential surface element is inclined with respect to the radial direction. The existence of the absolute value sign suggests two possibilities for our differential element. The differential element pyramid crosses the surface an even number of times with $\mathbf{e}_r\cdot\mathbf{n}/\left|\mathbf{e}_r\cdot\mathbf{n}\right|$ being plus and minus one an equal number of times, or the element crosses the surface an odd number of times in which case there is always one more occurrence of $\mathbf{e}_r\cdot\mathbf{n}/\left|\mathbf{e}_r\cdot\mathbf{n}\right| = 1$. The case for the even number of crossings occurs when the surface does not enclose the point charge (you must enter and exit the surface n times), and the case for the odd number of crossings occurs when the surface does enclose the point charge (you must first exit the surface and then you can enter and exit m times).

Hence, our integral becomes,

$$\begin{aligned}\int_S \mathbf{E}\cdot\mathbf{n}\ dS &= \frac{dQ}{4\pi\epsilon_0}\int_0^{2\pi}\left(\int_0^{\pi}\sin\theta\, d\theta\right)d\phi = \frac{dQ}{\epsilon_0} \quad \text{if } S \text{ encloses } dQ\\ &= 0 \quad \text{if } S \text{ does not enclose } dQ\end{aligned}$$

We can then perform this same integral for each "piece" of charge, the sum of all of the charges enclosed by S denoted as $Q_{enclosed}$. For a distribution of charges we can now write,

$$\int_S \mathbf{E}\cdot\mathbf{n}\; dS = \frac{Q_{enclosed}}{\epsilon_0}$$

If we introduce the concept of a continuous charge distribution with charge density per unit volume q (ρ in many electrostatics texts, but we will use ρ for mass density), the charge enclosed by a given surface is simply,

$$Q_{enclosed} = \int_V q\; dV$$

This then gives us,

$$\int_S \mathbf{E}\cdot\mathbf{n}\; dS = \int_V \frac{q}{\epsilon_0}\; dV$$

Applying the divergence theorem to the left-hand side of this equation yields,

$$\int_V \nabla\cdot\mathbf{E}\; dV = \int_V \frac{q}{\epsilon_0}\; dV \quad \text{or in index notation} \quad \int_S E_i n_i\; dS = \int_V E_{i,i}\; dV = \int_V \frac{q}{\epsilon_0}\; dV$$

Finally, this equation must be valid for any and every surface, and the volume that it encloses, that we choose. Given that this volume is arbitrary, the integrands must therefore be equivalent.

$$\nabla\cdot\mathbf{E} = \frac{q}{\epsilon_0} \quad \Rightarrow \quad E_{i,i} = \frac{q}{\epsilon_0}$$

This is also referred to as Gauss' Law. Finally, we can introduce the electric potential once again to obtain a governing partial differential equation for the electric potential.

$$-\nabla\cdot\nabla\phi = \frac{q}{\epsilon_0} \quad \Rightarrow \quad \left(-\phi_{,i}\right)_{,i} = \frac{q}{\epsilon_0}$$

$$\nabla^2\phi = -\frac{q}{\epsilon_0} \quad \Rightarrow \quad \phi_{,ii} = -\frac{q}{\epsilon_0}$$

This is of course Poisson's equation with the source term $-q/\epsilon_0$, and in the absence of a charge density reduces to Laplace's equation. This equation is useful for determining the fields in free space around a set of conductors. We will briefly discuss such situations and introduce the boundary conditions imposed on this equation by the

conductors. However, ultimately we are more concerned with the fields in matter. Before we get into this let's take another look at the electrostatic energy.

$$W = \tfrac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} V_{ij} = \tfrac{1}{2}\sum_{i=1}^{N} Q_i \underbrace{\sum_{\substack{j=1\\ j\neq i}}^{N} \phi_j}_{\phi(\mathbf{x}_i)} = \tfrac{1}{2}\sum_{i=1}^{N} Q_i \phi(\mathbf{x}_i)$$

Here $\phi(\mathbf{x}_i)$ is the electric potential that exists at the location of the charge Q_i and is due to the other charges in the system. For a continuum distribution of charges we can generalize this equation as,

$$W = \tfrac{1}{2}\int_V q\phi \; dV$$

Note that the volume V must at least contain all of the charge in the system. This can then be manipulated with Gauss' law to obtain,

$$\begin{aligned} W &= \tfrac{1}{2}\int_V q\phi \; dV \\ &= \tfrac{1}{2}\int_V \epsilon_0 E_{i,i}\phi \; dV \\ &= \tfrac{1}{2}\int_V \epsilon_0 \left[(E_i\phi)_{,i} - E_i\phi_{,i}\right] dV \\ &= \tfrac{1}{2}\int_V \epsilon_0 E_i E_i \; dV + \tfrac{1}{2}\int_S \epsilon_0 E_i n_i \phi \; dS \end{aligned}$$

If there is no charge at infinity then the electric field and electric potential have the following characteristics, $E_i \sim 1/r^2$ and $\phi \sim 1/r$, and in the limit as the bounding surface becomes infinitely extended the surface integral vanishes and we obtain,

$$W = \int_{\text{all space}} \tfrac{1}{2}\epsilon_0 E_i E_i \; dV \, .$$

An apparent contradiction exists between this equation and the previous double summation equation for the point charges since this equation must be positive while the prior one can be negative. This difference arises due to the accounting of the singularities associated with the point charges themselves which are thrown out in the double summation, but do not even exist in a smeared out continuum charge distribution. Consider the electric field and potential due to some distribution of point charges.

$$\mathbf{E}(\mathbf{x}) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{Q_i(\mathbf{x}-\mathbf{x}_i)}{\left|\mathbf{x}-\mathbf{x}_i\right|^3} \quad \text{and} \quad \phi(\mathbf{x}) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{Q_i}{\left|\mathbf{x}-\mathbf{x}_i\right|}$$

The energy of this field is then,

$$W = \tfrac{1}{2}\int_V \epsilon_0 \mathbf{E}\cdot\mathbf{E}\, dV + \tfrac{1}{2}\int_S \epsilon_0 \phi \mathbf{E}\cdot\mathbf{n}\, dS$$
$$= \tfrac{1}{2}\int_V \epsilon_0 \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{Q_i Q_j}{16\pi^2\epsilon_0^2} \frac{(\mathbf{x}-\mathbf{x}_i)\cdot(\mathbf{x}-\mathbf{x}_j)}{\left|\mathbf{x}-\mathbf{x}_i\right|^3\left|\mathbf{x}-\mathbf{x}_j\right|^3}\, dV + \tfrac{1}{2}\int_S \epsilon_0 \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{Q_i Q_j}{16\pi^2\epsilon_0^2} \frac{(\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{n}}{\left|\mathbf{x}-\mathbf{x}_i\right|^3\left|\mathbf{x}-\mathbf{x}_j\right|}\, dS$$

Recall that this was the rigorous form for W before we made the arguments allowing us to take the bounding surface out to infinity. To analyze these integrals we will instead take our bounding surface to be small spheres surrounding each charge all connected by small tubes. The integrals around the tubes vanish since there are no singularities in these tubes and the surface area and volume of the tubes vanish. To analyze the integrals around the small spheres we note that $\mathbf{x}-\mathbf{x}_i = \rho\mathbf{e}_\rho$ and $\mathbf{x}-\mathbf{x}_j \Rightarrow \mathbf{x}_i-\mathbf{x}_j$ when $i \neq j$ for the sphere surrounding Q_i.

$$W = \tfrac{1}{2}\int_V \epsilon_0 \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{Q_i Q_j}{16\pi^2\epsilon_0^2} \frac{(\mathbf{x}-\mathbf{x}_i)\cdot(\mathbf{x}_i-\mathbf{x}_j)}{\left|\mathbf{x}-\mathbf{x}_i\right|^3\left|\mathbf{x}_i-\mathbf{x}_j\right|^3}\, dV + \tfrac{1}{2}\int_S \epsilon_0 \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{Q_i Q_j}{16\pi^2\epsilon_0^2} \frac{(\mathbf{x}-\mathbf{x}_i)\cdot\mathbf{n}}{\left|\mathbf{x}-\mathbf{x}_i\right|^3\left|\mathbf{x}_i-\mathbf{x}_j\right|}\, dS$$
$$= \tfrac{1}{2}\epsilon_0 \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{Q_i Q_j}{16\pi^2\epsilon_0^2} \frac{(\mathbf{x}_i-\mathbf{x}_j)}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^3}\cdot \underbrace{\int_{\rho,\theta,\phi} \mathbf{e}_\rho \sin\theta'\, d\rho'\, d\theta'\, d\phi'}_{0} + \tfrac{1}{2}\epsilon_0 \sum_{i=1}^{N} \frac{Q_i Q_i}{16\pi^2\epsilon_0^2} \underbrace{\int_{\rho,\theta,\phi} \frac{1}{\rho'^2}\sin\theta'\, d\rho'\, d\theta'\, d\phi'}_{-4\pi/\rho + 4\pi/r_{ii}}$$
$$+ \tfrac{1}{2}\epsilon_0 \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{Q_i Q_j}{16\pi^2\epsilon_0^2} \frac{1}{\left|\mathbf{x}_i-\mathbf{x}_j\right|} \underbrace{\int_{\theta,\phi} \sin\theta'\, d\theta'\, d\phi'}_{4\pi} + \tfrac{1}{2}\epsilon_0 \sum_{i=1}^{N} \frac{Q_i Q_i}{16\pi^2\epsilon_0^2} \underbrace{\int_{\theta,\phi} \frac{1}{\rho}\sin\theta'\, d\theta'\, d\phi'}_{4\pi/\rho}$$
$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{8\pi\epsilon_0}\frac{Q_i Q_j}{r_{ij}}$$

Here we see that this expression agrees with our previous expression for W, except for the fact that it also includes the terms when $i = j$. This term is singular since $r_{ii} = 0$ and represents the energy to create the charges themselves. Since we can assume that we are "given" the charges and the energy of the charges does not change, we will not concern ourselves with this issue and we can safely remove this energy from our considerations. For the case where there are no point charges and the representation of the charge distribution is a continuous field, this problem does not arise.

We will now discuss boundary conditions for the PDE $\nabla^2\phi = -q/\epsilon_0$ for a set of conductors in free space. The fundamental difference between conductors and insulators is that electrons (at least those in the conduction band) are relatively free to move in conductors, whereas in insulators they are bound to a specific nucleus. By "free to move" we are implying that if a free electron is exposed to an electrostatic force then it

will move in the direction of that force. This consideration can be used to reveal the following conditions:

1. The electric field inside a conductor is zero, and hence the electric potential of a given conductor is constant. Since a conductor always has charges available to move, the existence of any electric field would rearrange the charges until the field is nullified.

2. Since the electric field in the conductor is zero, Gauss' law implies that the charge density inside the conductor is zero. Therefore, any unbalanced charge in a conductor must reside at its surface.

3. Since the electric potential is a continuous field, and the potential on the surface of a conductor is constant, the components of the electric field *tangential* to the conductor must be zero. Again, if tangential fields did exist, then the charges on the surface would move until such fields were cancelled.

4. Across a layer of surface charge, Gauss' law can be used to show that the jump in the normal component of the electric field is,

   $$\left|\mathbf{E}\cdot\mathbf{n}\right| = \frac{\omega}{\epsilon_0} \quad \text{or with index notation} \quad E_i n_i = \frac{\omega}{\epsilon_0}$$

   Here ω is the surface charge density (sometimes denoted σ but we will reserve σ for stresses). For this equation $\mathbf{n}$ is directed out from a conductor. If $\mathbf{n}$ is taken to point out of the region of free space, then a negative sign is required. This supplies the Neumann boundary conditions for the governing PDE.

Example

Determine the electric field for two infinite cylindrical tubes, one of radius a embedded in a second of radius b, both with a common center. The inner tube is maintained at zero potential and the outer tube at ϕ_0. Also determine the charge densities on each tube.

This problem has no volume charge and radial symmetry, so we are faced with solving Laplace's equation with radial symmetry.

$$\nabla^2\phi = 0 \quad \Rightarrow \quad \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0$$

This is an equi-dimensional ordinary differential equation which has the solution form,

$$\phi = Ar^p \quad \Rightarrow \quad p(p-1)Ar^{p-2} + pAr^{p-2} = 0 \quad \Rightarrow \quad p^2 = 0$$

The double root at zero implies that the two general solutions are combined as,

$\phi = A + B \ln r$

Applying boundary conditions provides the following two algebraic equations required to solve for the coefficients A and B.

$$\left.\begin{aligned} \phi(r=a) &= A + B\ln a = 0 \\ \phi(r=b) &= A + B\ln b = \phi_0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -\phi_0 \ln(a) / \ln(b/a) \\ B &= \phi_0 / \ln(b/a) \end{aligned}$$

Finally, our solution can be given as,

$$\phi = \phi_0 \frac{\ln(r/a)}{\ln(b/a)} \quad \text{with} \quad E_r = -\frac{d\phi}{dr} = \frac{\phi_0}{\ln(b/a)} \frac{1}{r}$$

This is the solution *between* the two cylinders. To determine the charge on the cylinders we also need the solutions within the inner cylinder and outside the outer cylinder. The solution in these regions still takes the form $\phi = A + B \ln r$. The inner region contains the point $r = 0$ and so the coefficient B must be zero. Furthermore ϕ is continuous throughout space and so the coefficient A must be equal to zero since this inner region shares the boundary at $r = a$. The problem in the outer region is a bit more subtle. If there is a net charge on the two cylinders then the potential far from the cylinders cannot be defined, but varies as $\phi \sim \ln r$ which is singular as $r \to \infty$. This feature arises since the charges on the cylinders extend out infinitely in the axial direction of the cylinders. In any case, this example is for illustrative purposes, so let's assume that there is no *net* charge on the cylinders. So, outside of the outer cylinder we have the solution $\phi = \phi_\infty$, and the continuity of the electric potential at $r = b$ implies that $\phi_\infty = \phi_0$. Then, at $r = a$ the jump in the normal component of the electric field is, $\phi_0 \big/ a \ln(b/a)$, and at $r = b$ the jump is, $-\phi_0 \big/ b \ln(b/a)$. The net *charge* per unit length on the two cylinders is $2\pi a \epsilon_0 \phi_0 \big/ a \ln(b/a) - 2\pi b \epsilon_0 \phi_0 \big/ b \ln(b/a) = 0$. Note that we had to account for the surface area of the two cylinders.

Electrostatic Fields in Matter

We now turn our attention to electrostatic fields in *insulating* matter. The simplest situation is to think about how charges move around under the influence of an external electric field in a relatively simple material. The key idea is to realize that all matter is in the arrangement of positive charge (nuclei or positive ions) and negative charge (the electrons or negative ions). When an electric field is applied the positive charge tends to move in the direction of the electric field and the negative charge moves in the opposite direction (at least if the solid is isotropic, in anisotropic solids the charge motions can be directed differently). This change of location of the positive and negative charge creates electric dipoles in the material. Perhaps the simplest system to analyze in a qualitative manner is the hydrogen atom in its ground state. Quantum mechanics gives the electron distribution as,

$$q = -\frac{Q}{\pi a^3} e^{-2r/a}$$

where Q is the charge of a proton, a is the Bohr radius, and r is the distance from the nucleus. Application of an electric field will tend to perturb the center of this distribution away from the nucleus. To a first approximation we can assume that the electron distribution retains its spherically symmetric shape but simply moves a distance d away from the nucleus. The applied electric field of magnitude E tends to push the nucleus in one direction, but since it is now off-center the electron cloud pulls the nucleus in the opposite direction. These two forces on the nucleus will balance out when the electric field due to the electron cloud at the location of the nucleus is equal and opposite to the applied electric field. Since we are assuming that the electron cloud retains its spherical symmetry we can apply Gauss' law to determine the field.

$$\int_S \mathbf{E}\cdot\mathbf{n}\, dS = \frac{Q_{enclosed}}{\epsilon_0}$$

$$\begin{aligned} 4\pi d^2 E_r &= -\frac{4\pi Q}{\pi a^3 \epsilon_0}\int_0^d e^{-2r/a} r^2\, dr = \frac{Q}{a^2\epsilon_0}\left(a^2 + 2ar + 2r^2\right)e^{-2r/a}\Big]_0^d \\ &= \frac{Q}{a^2\epsilon_0}\left[\left(a^2 + 2ad + 2d^2\right)\left(1 - 2\tfrac{d}{a} + 2\tfrac{d^2}{a^2} - \tfrac{4}{3}\tfrac{d^3}{a^3} + \ldots\right) - a^2\right] \\ &\approx -\frac{4Qd^3}{3a^3\epsilon_0} \text{ for } d \ll a \end{aligned}$$

Therefore, $E_r = \dfrac{-Qd}{3\pi a^3\epsilon_0}$ for $d \ll a$.

Defining the *polarizability* as, $\mathbf{p} = \alpha \mathbf{E}$, and $\mathbf{p} = Qd\,\mathbf{e}$, we get $\alpha = 3\pi a^3 \epsilon_0$. This is a rigorous result for the given static charge density, however a real hydrogen atom has a dynamic electron described by Schrödinger's equation, and the full quantum mechanical result is 6 times greater than this qualitative classical model. In any case the model illustrates one mechanism for polarization changes in matter.

Of course we are not interested in the dipoles created by individual atoms, but rather the polarization density for a representative region of material. A rigorous definition of material polarization based on the charge distributions in a sample of matter would at first glance seem to be possible. However, careful considerations of the possible definitions make it clear that none are satisfactory. In fact, in the laboratory it is only possible to measure *changes* in the polarization of a material sample from the current passing through the sample, $\Delta \mathbf{P} = \mathbf{J}\,dt$. At this juncture we will not concern ourselves with such issues, and instead we will assume that we know the distribution of dipoles in the material. The total dipole moment of a volume of material is then just the sum of all dipoles contained in that volume. At the continuum level we identify the polarization as the density of dipoles per unit volume, such that the dipole moment of the volume can be written as,

$$\mathbf{p} = \int_V \mathbf{P}\, dV$$

We would like to determine the electrical effects due to some distribution of polarization in matter. We can do this in two ways. First we will consider the electric potential due to some polarization field. Recall that the potentials at some position $\mathbf{x}$ due to a point charge or point dipole located at $\mathbf{x}'$ are

$$V_Q = \frac{Q}{4\pi\epsilon_0} \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|}$$

$$V_\mathbf{p} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \left(\mathbf{x} - \mathbf{x}'\right)}{\left|\mathbf{x} - \mathbf{x}'\right|^3}$$

Then, the potential due to a polarization distribution is,

$$V_\mathbf{P} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\mathbf{P} \cdot \left(\mathbf{x} - \mathbf{x}'\right)}{\left|\mathbf{x} - \mathbf{x}'\right|^3} dV'$$

Note that, $\dfrac{\partial}{\partial x_i'}\left[\left(x_j - x_j'\right)\left(x_j - x_j'\right)\right]^{-1/2} = -\dfrac{1}{2}\left(-2\right)\left(x_i - x_i'\right)\left[\left(x_j - x_j'\right)\left(x_j - x_j'\right)\right]^{-3/2}$, or in vector notation,

$$\nabla' \frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|} = \frac{\mathbf{x}-\mathbf{x}'}{\left|\mathbf{x}-\mathbf{x}'\right|^3} \rightarrow V_{\mathbf{P}} = \frac{1}{4\pi\epsilon_0}\int_V \frac{\mathbf{P}\cdot\left(\mathbf{x}-\mathbf{x}'\right)}{\left|\mathbf{x}-\mathbf{x}'\right|^3} dV' = \frac{1}{4\pi\epsilon_0}\int_V \mathbf{P}\cdot\nabla' \frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|} dV'$$

Applying the divergence theorem to this result yields,

$$\begin{aligned} V_{\mathbf{P}} &= \frac{1}{4\pi\epsilon_0}\int_V \nabla'\cdot\left(\mathbf{P}\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}\right) - \frac{\nabla'\cdot\mathbf{P}}{\left|\mathbf{x}-\mathbf{x}'\right|} dV' \\ &= \frac{1}{4\pi\epsilon_0}\int_V \frac{-\nabla'\cdot\mathbf{P}}{\left|\mathbf{x}-\mathbf{x}'\right|} dV' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\mathbf{P}\cdot\mathbf{n}}{\left|\mathbf{x}-\mathbf{x}'\right|} dS' \\ &= \frac{1}{4\pi\epsilon_0}\int_V \frac{q_b}{\left|\mathbf{x}-\mathbf{x}'\right|} dV' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\omega_b}{\left|\mathbf{x}-\mathbf{x}'\right|} dS' \end{aligned}$$

On the last line we have defined the *b*ound charge density as $q_b = -\nabla\cdot\mathbf{P}$, and the bound surface charge density as $\omega_b = \mathbf{P}\cdot\mathbf{n}$. This last line also demonstrates that the potential of the polarization distribution can be represented through these associated charge distributions. Finally, if we recognize that it is really only polarization changes that can be determined through the measurement of currents, then these two equations represent the conservation of charge.

The physical pictures associated with ω_b and q_b are as follows. First, consider a material surface inclined to the material polarization. As the material is poled (note we are conceptually considering changes in polarization from some initial state) charge separation occurs and at the surface charge appears in an amount that is equal to the normal component of the polarization (change). This accumulation of surface charge is exactly $\omega_b = \mathbf{P}\cdot\mathbf{n}$. Next, consider some differential volume element. The accumulation of bound charge within this volume is computed by accounting for the amount of charge coming into the volume and then subtracting off the charge leaving the volume.

$$\begin{aligned} q_b dx\,dy\,dz = {} & P_x dy\,dz - \left(P_x + \frac{\partial P_x}{\partial x}dx + O\left(dx_i dx_j\right)\right)dy\,dz \\ & + P_y dx\,dz - \left(P_x + \frac{\partial P_y}{\partial y}dy + O\left(dx_i dx_j\right)\right)dx\,dz \\ & + P_z dx\,dy - \left(P_x + \frac{\partial P_z}{\partial z}dz + O\left(dx_i dx_j\right)\right)dx\,dy \end{aligned}$$

In the limit that the spatial dimensions of this volume element vanish, this bound charge accumulation equation takes the form $q_b = -\nabla\cdot\mathbf{P}$. We emphasize that bound

charge is in fact real charge. However, it is distinguished from free charge in the sense that we have no control over it. We are able to move free charges around through the use of conductors and batteries, but the bound charge distributes itself according to the physical behavior of the material.

Let us now consider how we can use these results to perhaps simplify the equations governing the electrostatic fields in matter and the computation of the energy. First, we decompose the total charge densities into *f*ree and *b*ound contributions.

$$q = q_f + q_b \quad \text{and} \quad \omega = \omega_f + \omega_b$$

Gauss' law in the volume and on a surface become,

$$\epsilon_0 \nabla \cdot \mathbf{E} = q_f + q_b \quad \text{and} \quad \epsilon_0 \mathbf{E} \cdot \mathbf{n} = -\omega_f - \omega_b$$

Here, the unit normal $\mathbf{n}$ points out of the material. Applying the definitions for the bound charge densities yields,

$$\epsilon_0 \nabla \cdot \mathbf{E} = q_f - \nabla \cdot \mathbf{P} \quad \text{and} \quad \epsilon_0 \mathbf{E} \cdot \mathbf{n} = -\omega_f - \mathbf{P} \cdot \mathbf{n}$$

Finally, a rearrangement of terms gives,

$$\nabla \cdot \left(\epsilon_0 \mathbf{E} + \mathbf{P} \right) = q_f \quad \text{and} \quad \left(\epsilon_0 \mathbf{E} + \mathbf{P} \right) \cdot \mathbf{n} = -\omega_f$$

These equations suggest the definition of a new quantity within the parentheses, $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$. The quantity $\mathbf{D}$ is called the *electric displacement*, which is governed by the balance laws,

$$\nabla \cdot \mathbf{D} = q_f \text{ in } V \quad \text{and} \quad \mathbf{D} \cdot \mathbf{n} = -\omega_f \text{ on } S$$

Let's look at the work done on a dielectric. Generally we can think of the electrical work being done by a battery, and we must recognize that batteries push around *free* charge, and so only do work on the free charges. Hence, the increment of work done to bring a free charge into place is,

$$\delta W = \int_V \phi \, \delta q_f \, dV + \int_S \phi \, \delta \omega_f \, dS$$

Applying our new balance laws for the electric displacement and manipulating with the divergence theorem yields,

$$\begin{aligned}\delta W &= \int_V \phi \nabla \cdot \delta \mathbf{D}\, dV - \int_S \phi\, \mathbf{n} \cdot \delta \mathbf{D}\, dV \\ &= \int_V \Big[\nabla \cdot \big(\phi\, \delta \mathbf{D}\big) - \delta \mathbf{D} \cdot \nabla \phi\Big] dV - \int_S \phi\, \mathbf{n} \cdot \delta \mathbf{D}\, dS \\ &= \int_V \delta \mathbf{D} \cdot \mathbf{E}\, dV + \int_S \mathbf{n} \cdot \phi\, \delta \mathbf{D}\, dS - \int_S \phi\, \mathbf{n} \cdot \delta \mathbf{D}\, dS \\ &= \int_V \mathbf{E} \cdot \delta \mathbf{D}\, dV\end{aligned}$$

In summary, our most significant results for the electrostatic relationships in matter are,

$\mathbf{E} = -\nabla \phi$	$E_i = -\phi_{,i}$
$\nabla \cdot \mathbf{P} = -q_b$	$P_{i,i} = -q_b$
$\mathbf{P} \cdot \mathbf{n} = \omega_b$	$P_i n_i = \omega_b$
$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	$D_i = \epsilon_0 E_i + P_i$
$\nabla \cdot \mathbf{D} = q_f$	$D_{i,i} = q_f$
$\mathbf{D} \cdot \mathbf{n} = -\omega_f$	$D_i n_i = -\omega_f$
$\delta W = \int_V \mathbf{E} \cdot \delta \mathbf{D}\, dV$	$\delta W = \int_V E_i\, \delta D_i\, dV$

Ultimately, these equations are not sufficient for the determination of the electrostatic fields in matter. Specifically, we still need to know the constitutive response of the material. For uncoupled situations this means that we need to know how the electric displacement is related to the electric field. When significant coupling exists in the material, the polarization and electric displacement may also depend on the temperature, strain, magnetic field and perhaps other variables. Furthermore, for irreversible material response, the constitutive behavior is given only in incremental form, such that only the increment (or rate of change) of electric displacement or material polarization is specified.

We will focus on constitutive modeling in more detail at a later time. For now we will look at the simplest relationship between the polarization and electric field, a linear one. For a general linear material response the components of the polarization are related to the components of the electric field as,

$$P_i = \epsilon_0 \chi_{ij} E_j$$

The material property $\boldsymbol{\chi}$ is the second rank tensor of *electrical susceptibility*. Given a linear electrical susceptibility, the relationship between electric displacement and electric field is also linear, and can be derived as follows,

$$D_i = \epsilon_0 E_i + P_i = \epsilon_0 E_i + \epsilon_0 \chi_{ij} E_j = \underbrace{\epsilon_0 \left(\delta_{ij} + \chi_{ij}\right)}_{\kappa_{ij}} E_j$$

The *dielectric permittivity* $\boldsymbol{\kappa}$ is also a second rank tensor. The quantity in parentheses is sometimes referred to as the *dielectric constant* which is very close to 1 for gases, about 5 for glass, 80 for water, and can be in the thousands for perovskite (a type of crystal structure) ceramics.

Example: Parallel plate capacitor
Consider a parallel plate capacitor consisting of two materials bonded together and sandwiched between two electrodes. The thickness and dielectric permittivity of material 1 are h_A and κ_A, and those of material 2 are h_B and κ_B. We would like to determine the capacitance for this capacitor.

First, the capacitance of a given capacitor is simply the total charge transferred between the capacitor electrodes divided by the electrical potential drop between the electrodes. Let's say that the charge on the top electrode is given as Q, then by charge conservation (given that initially the capacitor has no net charge) the charge on the bottom electrode is $-Q$. We can then take the bottom electrode to be grounded and the top electrode has a potential of Φ. We need to determine the relationship between Q and Φ. For simplicity, we will assume that our parallel plate capacitor is infinitely extended in two directions such that we only need to deal with the fields in the direction perpendicular to the electrodes. We will call this direction x, and any vector quantities listed will be assumed to have only x-components. In such situations the governing equations are,

$$E = -\frac{d\phi}{dx}, \quad \frac{dD}{dx} = q, \quad D = \kappa_A E \text{ in material 1, or } D = \kappa_B E \text{ in material 2}$$

Given that the free charge density $q = 0$, Gauss' law implies that the electric displacement must be a constant in each material (we will discuss the interface shortly). Let's call the electric displacement in each material D_A and D_B. Then, the material constitutive relationships imply that the electric field in each material must also be constants, $E_A = D_A / \kappa_A$ and $E_B = D_B / \kappa_B$. Finally, the electric potential through the capacitor must be a continuous piecewise linear function with a possible discontinuity in the slope at the interface between the two materials, $\phi = -E_A x$ for $0 \le x \le h_A$ and $\phi = -E_A h_A - E_B (x - h_A)$ for $h_A \le x \le h_A + h_B$. Note that the continuity of the electric

potential at $x = h_A$ has been enforced. At a material interface we must consider the jump in the normal component of the electric displacement. Specifically, the difference in the normal component of the electric displacement is precisely the free surface charge density on the interface. For our case we would have, $D_B - D_A = \omega_f$. Since we have not placed any free charge at the interface we have $\omega_f = 0$ and $D_B = D_A$. We now need to relate this uniform electric displacement to the free charge on the capacitor plates. Here we again use the fact that the normal component of electric displacement must be equal to the free surface charge density. The free surface charge density on the top plate is $\omega_f = Q/A$ and that for the bottom plate is $\omega_f = -Q/A$. Your intuition for determining how a free charge layer affects the *sign* of the normal component of electric displacement is that a positive charge layer forces the electric displacement (i.e. the positive end of the dipole) to point away from it and a negative charge layer pulls the normal component of electric displacement towards it. Hence, at the top electrode, where we are assuming that the outward pointing normal is in the positive x-direction, we have $D_B = -Q/A$, and on the bottom electrode $D_A = -Q/A$. Placing these results into our relationships for the electric fields, and then into the equation for the electric potential we obtain,

$$\phi = \begin{cases} \dfrac{Q}{\kappa_A A} x & \text{for } 0 \le x \le h_A \\ \dfrac{Q}{\kappa_A A} h_A + \dfrac{Q}{\kappa_B A}(x - h_A) & \text{for } h_A \le x \le h_A + h_B \end{cases}$$

Now we can determine the potential on the top electrode as,

$$\Phi = \phi(x = h_A + h_B) = \frac{Q h_A}{\kappa_A A} + \frac{Q h_B}{\kappa_B A},$$

and the capacitance is $C = Q/\Phi = \dfrac{\kappa_B \kappa_A A}{\kappa_B h_A + \kappa_A h_B}$.

For a capacitor with a homogeneous material we can check three cases,

$\kappa_A = \kappa_B = \kappa \rightarrow C = \kappa A / H$, where $H = h_A + h_B$

$h_A = H,\ h_B = 0,\ \kappa_A = \kappa \rightarrow C = \kappa A / H$

$h_B = H,\ h_A = 0,\ \kappa_B = \kappa \rightarrow C = \kappa A / H$

One of the important physical features of this problem is the need for the interface conditions. In a more general three-dimensional problem the interface conditions on the electric field and electric displacement come from two considerations. First, the continuity of the electric potential implies that the tangential components of the electric field must be continuous across any *equi-potential* interface, e.g. an electrode. Mathematically this can be stated as,

$$[\mathbf{E} - \mathbf{n}(\mathbf{E}\cdot\mathbf{n})]^{+} = [\mathbf{E} - \mathbf{n}(\mathbf{E}\cdot\mathbf{n})]^{-}$$

We then also need a normal condition at an interface, and this condition comes from Guass' law and choosing a small pillbox for a surface enclosing a small element of the interface,

$$\int_S \mathbf{D}\cdot\mathbf{n}\; dS = Q_f^{enclosed} \quad \rightarrow \quad (\mathbf{D}^{+} - \mathbf{D}^{-})\cdot\mathbf{n} = \omega_f$$

Note that for this second equation the unit normal to the interface is taken to point from the "–" side to the "+" side.

Magnetostatics

The second non-mechanical set of fields that we will study deal with magnetism. Specifically, we will only be concerned with magnetic fields associated with *steady* currents, such that the magnetic fields do not change in time, or at least change slowly such that we can use a quasi-static approximation. This is the study of magnetostatics. As you probably know, electricity and magnetism are intimately related and the full set of Maxwell's equations couple electrical and magnetic fields through their rates of change.

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Only when the rates of change vanish, do these equations decouple. Note that the vector $\mathbf{J}$ is the current density and $\mathbf{B}$ is what we will call the *magnetic induction.* (The history of this name is unfortunate as this and another quantity $\mathbf{H}$ are sometimes both called the magnetic field. For lack of an alternative and to differentiate between these fields we will stick to calling $\mathbf{B}$ the magnetic induction). The constant μ_0 is the magnetic permeability of free space, and is equal to $4\pi \times 10^{-7}$ N/A^2.

One approach to developing the equations of magnetostatics would be to introduce a magnetic version of Coulomb's law and proceed from there. In attempting to do so one would find a few differences. First, it is impossible to find the magnetic equivalent of a charge (at least to date no repeatable confirmation that a magnetic monopole exists has been made). So instead of charges/monopoles we would have to determine the interactions between magnetic dipoles. If we were careful, we would find that the force between two magnetic dipoles is identical to the forces between two electrical dipoles. The magnitude of the force between two dipoles is inversely proportional to the distance between them raised to the fourth power, and it also depends in a rather complicated way on the relative orientations of the dipoles. Instead of taking this approach, let's start from another result, the Lorentz force law for a point charge.

$$\mathbf{F} = Q\Big[\mathbf{E} + \big(\mathbf{v} \times \mathbf{B}\big)\Big]$$

The first part of this equation is our electrostatic result, though we arrived at the electric field $\mathbf{E}$ as a definition through the work done on two charges. The second part of this equation introduces the *magnetic induction* $\mathbf{B}$, and claims that the force on the charge is proportional to the cross product of its velocity $\mathbf{v}$ and the magnetic induction. But what is $\mathbf{B}$ and where does it come from? The answer is that $\mathbf{B}$ is itself generated by

moving charges. For an account of the experiments performed by André-Marie Ampère in 1820, go to

http://farside.ph.utexas.edu/teaching/em/lectures/node32.html

For the sake of simplicity we will start from Maxwell's laws. First, note that since we now have a moving charge we no longer have a "static" situation. However, we can have situations were large numbers of charge carriers are moving through a material such that while the charges themselves are not stationary, the average number of charges moving through a given surface per unit of time remains constant. If we consider the charge accumulation in a given volume of material we have,

$$\int_S \mathbf{J}\cdot\mathbf{n}\; dS = -\frac{d}{dt}\int_V q\; dV$$

where $\mathbf{J}$ is the current density and the left-hand side of this equation gives the flux of charge per unit of time flowing out of the surface S and the right-hand side is the amount of charge accumulated (*de*-accumulated?) in the volume. Applying the divergence theorem and recognizing that the equation holds for any arbitrary volume yields,

$$\nabla\cdot\mathbf{J} = -\frac{dq}{dt}$$

(On an interface with surface currents we would have $\left(\mathbf{J}^+ - \mathbf{J}^-\right)\cdot\mathbf{n} + \nabla\cdot\mathbf{K} = -\frac{d\omega}{dt}$.)

This result can also be obtained from Maxwell's equations,

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla\cdot(\nabla\times\mathbf{B}) = \mu_0\nabla\cdot\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla\cdot\mathbf{E}$$

$$0 = \mu_0\nabla\cdot\mathbf{J} + \mu_0\frac{\partial q}{\partial t}$$

$$\nabla\cdot\mathbf{J} = -\frac{\partial q}{\partial t}$$

In magnetostatic situations the rate of change of the charge densities must be zero. Furthermore, Maxwell's equations reduce to,

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

The first of these equations allows us to rewrite the magnetic induction as the curl of a vector potential field.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Note that any two vector potentials, say $\mathbf{A}_A$ and $\mathbf{A}_B$, can be used to represent the same magnetic induction field so long as they only differ by the gradient of a scalar field. In other words,

$\mathbf{A}_A = \mathbf{A}_B + \nabla\psi$, then

$$\mathbf{B} = \nabla \times \mathbf{A}_A = \nabla \times \left(\mathbf{A}_B + \nabla\psi\right) = \nabla \times \mathbf{A}_B + \underbrace{\nabla \times \nabla\psi}_{0} = \nabla \times \mathbf{A}_B$$

Applying the vector potential within the latter of Maxwell's equations listed above gives,

$$\begin{aligned}
\nabla \times \mathbf{B} &= \nabla \times \left(\nabla \times \mathbf{A}\right) \\
&= \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2\mathbf{A} \\
&= -\nabla^2\mathbf{A} \quad \text{if we take } \nabla \cdot \mathbf{A} = 0 \\
\\
&\Rightarrow \nabla^2\mathbf{A} = -\mu_0\mathbf{J}
\end{aligned}$$

Aside: Proof of second line using index notation

$$\begin{aligned}
\left[\nabla \times \left(\nabla \times \mathbf{A}\right)\right]_i &= \epsilon_{ijk}\epsilon_{klm}A_{m,lj} \\
&= \underbrace{\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)}_{\epsilon-\delta \text{ identity}}A_{m,lj} \\
&= A_{j,ij} - A_{i,jj} \\
&= \frac{\partial}{\partial x_i}\left(A_{j,j}\right) - \nabla^2 A_i \\
&= \frac{\partial}{\partial x_i}\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 A_i
\end{aligned}$$

Note that taking $\nabla \cdot \mathbf{A} = 0$ is *"legal"* since we can always choose the scalar field ψ such that $\nabla \cdot \nabla\psi = -\nabla \cdot \mathbf{A}_B \Rightarrow \nabla \cdot \mathbf{A}_A = 0$. So, given any $\mathbf{B}$ we will choose to represent it with the specific $\mathbf{A}$ that has $\nabla \cdot \mathbf{A} = 0$. Now, each component of our vector potential is governed by a partial differential equation that is directly analogous to its electrical

counterpart $\nabla^2\phi = -\frac{q}{\epsilon_0}$. For a given charge distribution we showed that the scalar potential could be written as

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{q(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}dV' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\omega(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}dS'$$

Then, by analogy the vector potential can be written as,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}dV' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{K}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}dS'$$

Here, $\mathbf{K}$ is the surface current density. We note that the divergence of $\mathbf{A}$ will only be zero if $\nabla\cdot\mathbf{J} = 0$ in the volume and $\mathbf{J}\cdot\mathbf{n} - \nabla\cdot\mathbf{K} = 0$ on the surface, but these must be the case for magnetostatics. The magnetic induction can then be written as,

$$\begin{aligned}\mathbf{B}(\mathbf{x}) = \nabla_{\mathbf{x}}\times\mathbf{A} &= \frac{\mu_0}{4\pi}\int_V \nabla_{\mathbf{x}}\times\frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}dV' + \frac{\mu_0}{4\pi}\int_S \nabla_{\mathbf{x}}\times\frac{\mathbf{K}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}dS' \\ &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|^3}dV' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{K}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|^3}dS'\end{aligned}$$

This equation is the *Biot-Savart* law for $\mathbf{B}$ in terms of the current densities.

We now approach the concept of work done by a magnetic field. Here we encounter a subtlety, which is the source of a great deal of confusion, because from the Lorentz force law it can be readily shown that the magnetic induction does no work on a charge.

$$\begin{aligned}dW &= \mathbf{F}\cdot d\mathbf{x} \\ &= Q(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt \\ &= 0 \quad \text{because } \mathbf{v}\times\mathbf{B} \text{ is perpendicular to } \mathbf{v}\end{aligned}$$

OK, but this does not change the fact that it does take energy to magnetize an object, and if that process is reversible then it is possible to get that energy back. Let's see if we can understand this by looking at the forces on a magnetic dipole. We are investigating dipoles because they are the simplest magnetostatic element and all current distributions can be built up from combinations of differential current loops (which are magnetic dipoles). First, the vector potential for a current loop is,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{r^3}$$

where $\mathbf{m}$ is the dipole for the loop carrying a current I with differential area $d\mathbf{a}$ (direction chosen using the right-hand rule with the current direction) given as $\mathbf{m} = I\,d\mathbf{a}$. First, let's consider this dipole in a uniform field. The current I can be represented as a number N of charge carriers Q passing through a point on the loop traveling with velocity $\mathbf{v} = v\mathbf{e}_t$ in the loop, where $\mathbf{e}_t$ is the unit tangential direction in the loop such that $\mathbf{e}_t \cdot d\mathbf{a} = 0$. Then $I = NQv / L$ where L is the arc length of the loop. The force on each one of the charge carriers is,

$$d\mathbf{F} = Qv\mathbf{e}_t \times \mathbf{B}$$

The total force on all of the charge carriers in the loop is then,

$$\mathbf{F} = \frac{NQv}{L}\oint_{loop} \mathbf{e}_t\, dl \times \mathbf{B} = I\oint_{loop} \mathbf{e}_t\, dl \times \mathbf{B}$$

If the loop is in fact closed, then the integral of the tangential unit vector around the loop will be zero.

$$\mathbf{e}_t = \frac{d\mathbf{x}}{dl} \quad \Rightarrow \quad \int \frac{d\mathbf{x}}{dl}\,dl = \int_{\mathbf{x}_A}^{\mathbf{x}_B} d\mathbf{x} = \mathbf{x}_B - \mathbf{x}_A = 0 \text{ when } \mathbf{x}_B = \mathbf{x}_A$$

So, in a *uniform* field the force on a dipole is zero. How about a non-uniform field? To analyze non-uniform fields we will resort to a rectangular loop for the sake of simplicity. Take the opposite sides of the loop to be lengths a along the x-direction and b along the y-direction, and the current circling in the counterclockwise direction such that the dipole is given as $\mathbf{m} = Iab\,\mathbf{e}_z$. The total force on the wire is the sum of the contributions from each charge carrier.

$$\mathbf{F} = I\oint_{loop} \mathbf{e}_t \times \mathbf{B}\, dl$$

Expanding the B-field into a Taylor series up through all first order terms yields,

$$
\begin{aligned}
\mathbf{F} &= I\left\{\int_0^a\left[\mathbf{e}_x\times\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial x}x\right)\right]dx+\int_0^b\left[\mathbf{e}_y\times\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial x}a+\frac{\partial\mathbf{B}}{\partial y}y\right)\right]dy\right.\\
&\qquad\left.+\int_a^0\left[-\mathbf{e}_x\times\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial x}x+\frac{\partial\mathbf{B}}{\partial y}b\right)\right]\left(-dx\right)+\int_b^0\left[-\mathbf{e}_y\times\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial y}y\right)\right]\left(-dy\right)\right\}\\
&= I\left(\mathbf{e}_y\times\frac{\partial\mathbf{B}}{\partial x}ab-\mathbf{e}_x\times\frac{\partial\mathbf{B}}{\partial y}ab\right)\\
&= m\left(\mathbf{e}_y\times\frac{\partial\mathbf{B}}{\partial x}-\mathbf{e}_x\times\frac{\partial\mathbf{B}}{\partial y}\right)\\
&= m\left(\frac{\partial B_z}{\partial x}\mathbf{e}_x-\frac{\partial B_x}{\partial x}\mathbf{e}_z+\frac{\partial B_z}{\partial y}\mathbf{e}_y-\frac{\partial B_y}{\partial y}\mathbf{e}_z\right)\\
&= m\left(\frac{\partial B_z}{\partial x}\mathbf{e}_x+\frac{\partial B_z}{\partial y}\mathbf{e}_y+\frac{\partial B_z}{\partial z}\mathbf{e}_z\right)\quad\text{using } B_{i,i}=0\\
&= \nabla(\mathbf{m}\cdot\mathbf{B})
\end{aligned}
$$

Here we can interpret $-\mathbf{m}\cdot\mathbf{B}$ as the potential energy of the dipole and of course the force on the dipole is just the negative gradient of the potential energy. The increment of work done by this force to bring the dipole into place is just,

$$dW = \mathbf{F}\cdot d\mathbf{x}_L = \nabla(\mathbf{m}\cdot\mathbf{B})\cdot d\mathbf{x}_L$$

However, we must be careful because this is not the total energy of the dipole. You see, during the process of moving the dipole through the **B** field we have to impart a velocity on the loop, which in turn gives rise to forces on the charges, which if left unchecked will *change* the current in the loop. But our analysis considered only steady currents in the loop, so we must also determine the energy required to maintain the steady currents. We have already calculated the contribution to the force due to the *current velocity*. This is exactly what the result above is. Now we need to calculate the work due to the *loop placement* velocity. To do this we need to compute,

$$\mathbf{F} = \frac{NQ}{L}\mathbf{v}_L\times\oint_{loop}\mathbf{B}\,dl$$

When we dot this with $\mathbf{v}_L dt$ we of course get zero. However, we also have to dot this with $v\mathbf{e}_t dt$. Note that in the work computed previously it looks like we only dotted our force result with $\mathbf{v}_L dt$, which perhaps is true, but when we also include the $v\mathbf{e}_t dt$ this second contribution will be zero. Maybe you have already recognized that if we carry out this second dot product we will obtain $\mathbf{m}\cdot\mathbf{B}$ for the work done. Let's do it anyway.

$$
\begin{aligned}
dW &= \mathbf{F}\cdot v\mathbf{e}_t dt = \frac{NQ}{L}\left[\oint_{loop}\left(\mathbf{v}_L\times\mathbf{B}\right)\cdot v\mathbf{e}_t\ dl\right]dt \\
&= I\oint_{loop}\left(d\mathbf{x}_L\times\mathbf{B}\right)\cdot\mathbf{e}_t\ dl \\
&= I\left\{d\mathbf{x}_L\times\int_0^a\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial x}x\right)dx\cdot\mathbf{e}_x + d\mathbf{x}_L\times\int_0^b\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial x}a+\frac{\partial\mathbf{B}}{\partial y}y\right)dy\cdot\mathbf{e}_y\right. \\
&\qquad \left. -d\mathbf{x}_L\times\int_0^a\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial x}x+\frac{\partial\mathbf{B}}{\partial y}b\right)dx\cdot\mathbf{e}_x - d\mathbf{x}_L\times\int_0^b\left(\mathbf{B}_{0,0}+\frac{\partial\mathbf{B}}{\partial y}y\right)dy\cdot\mathbf{e}_y\right\} \\
&= Iab\left[\left(d\mathbf{x}_L\times\frac{\partial\mathbf{B}}{\partial x}\right)\cdot\mathbf{e}_y - \left(d\mathbf{x}_L\times\frac{\partial\mathbf{B}}{\partial y}\right)\cdot\mathbf{e}_x\right] \\
&= m\left(\frac{\partial B_x}{\partial x}dz - \frac{\partial B_z}{\partial x}dx + \frac{\partial B_y}{\partial y}dz - \frac{\partial B_z}{\partial y}dy\right) \\
&= -m\left(\frac{\partial B_z}{\partial x}dx + \frac{\partial B_z}{\partial y}dy + \frac{\partial B_z}{\partial z}dz\right) \\
&= -\nabla(\mathbf{m}\cdot\mathbf{B})\cdot d\mathbf{x}_L
\end{aligned}
$$

This is exactly the opposite of the previous work term that we calculated. The previous work that we computed is the work done by the external force required to move the loop into place while the currents in the loop and in the source of **B** are maintained. This is usually called the mechanical work. This second work contribution is the work required to maintain the current in the loop at its steady value while the loop is being moved into place. This work can be thought of as being done by a battery and is usually called the electrical work. So overall we are back to the situation where no net work has been done. However, we are not quite done accounting for all of the work that has been done. You see, we not only have to maintain the currents in the loop, but we also have to maintain the currents at the source. A quick way to determine what is going on at the source comes from Feynman's Lecture Notes. Instead of moving the test loop into place, change the frame of reference so that it appears that we are moving the source into place. For quasi-static situations there are equal but opposite reactions between the source and the test loop, so the source requires the same work contributions as we calculated for the test loop. Finally, to determine the total work done in either of these processes we must recognize that the electrical work at both the source and the test loop must be supplied in order to maintain the currents, but the mechanical work is done on only the test loop or the source depending on your frame of reference. Hence, the total work required to bring a test dipole into a **B** field is,

$$W = \mathbf{m}\cdot\mathbf{B}$$

The derivation of this result sheds some light on the subtleties associated with determining the "magnetic work" done on a system. When we look at the work done to magnetize an object we will use Maxwell's laws directly to calculate the magnetic work through the electric field.

