SYSTEM THEORY 2008

1. Consider the mechanical system consisting of three vibrating masses m connected with elastic springs with the same stiffness k, shown in Fig. 1. Formulate the system of three ordinary differential equations describing the motion of the mechanical system, and find its general solution.

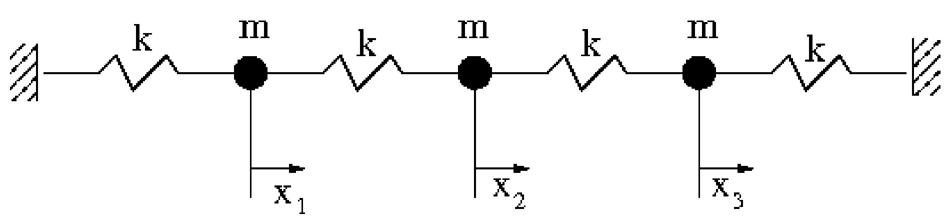

Figure 1: A mechanical system with 3 degrees of freedom

2. Consider the function $f(x) = x$ for $-\pi < x \leq \pi$ and extended periodically.

   a) Find the Fourier Series for f.

   b) Does the series converge to f, and if so, in what sense?

   c) If we differentiate the Fourier Series term-by-term, does this series converge to $f'(x)$?

3. Minimize the performance index

$$J = \frac{1}{2} x^T x$$

subject to the constraint

$$y - Hx = 0$$

where the dimensions of x, y, H are $x \times 1, m \times 1, m \times n$. The vector y and the matrix H are known (constant) and $n > m$. Find the optimal solution and show that it is a minimum by analyzing the second differential. Note that the x's are not independent.

4. A car accelerates from zero position and zero velocity to a final position. It is desired to find the acceleration history that minimizes the final time. Limits on the acceleration are handled by a penalty function. The problem is then to find the control history $u(t)$ that minimizes the performance index (k is a positve constant)

$$J = t_f + k \int_{t_0}^{t_f} u^2 \, dt$$

subject to the dynamics

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

and to the boundary conditions

$$t_0 = 0, \;\; x_{1_0} = 0, \;\; x_{2_0} = 0$$

$$t_f \; free, \;\; x_{1_f} = 1, \;\; x_{2_f} \; free.$$

Once you have found the optimal control, show that the Legendre-Clebsch and Weierstrass tests for a minimum are satisfied.

5. Given the dynamic system

$$\ddot{Z} = -k(1 + \alpha Z)Z - \beta \dot{Z}^2 + \lambda t$$

where k, α, β, and λ are constants, assume range and rangerate observations, ρ and $\dot{\rho}$ are taken at times t_i , $i = 1, ..., m$. The equation for the range model ρ is

$$\rho = \sqrt{Z^2 + h^2}$$

where h is a constant. Assume Z, $\dot{Z}$, k, α, β, and h are to be estimated. Define the state vector and set up the necessary equations to estimate the state of the system. That is, for this system, give the following:

$$X, \;\; F(X,t), \;\; A, \;\; G(X,t), \;\; \tilde{H}$$

You do not need to solve for the elements of Φ.

6. Assume you are calculating a Linear Unbiased Minimum Variance Estimate at t_0. Thus, you have

$$y = Hx_0 + \epsilon \qquad E[\epsilon] = 0 \qquad E[\epsilon\epsilon^T] = R$$

$$\hat{x}_0 = (H^T R^{-1} H)^{-1}(H^T R^{-1} y)$$

where y is $m \times 1$, H is $m \times n$, x_0 is $n \times 1$ and is the true deviation away from the nominal trajectory at t_0, and $m > n$.

(a) Show that this is an unbiased estimate.

(b) Show that the covariance for $\hat{x}_0$ is $P = (H^T R^{-1} H)^{-1}$

(c) Define $\hat{X}_0$ to be the updated value of the nominal trajectory:

$$\hat{X}_0 \equiv X_0^* + \hat{x}_0$$

where X_0^* is the starting value for the nominal trajectory. Let $P_{\hat{X}_0}$ be the covariance for $\hat{X}_0$. Show that $P_{\hat{X}_0}$ is equal to P.

(d) The estimate $\hat{x}_0$ can be mapped to time t_k using the state transition matrix as follows:

$$\bar{x}_k = \Phi(t_k, t_0)\hat{x}_0$$

Show that $\bar{x}_k$ is an unbiased estimate of x_k.

(e) Derive an expression for the covariance of $\bar{x}_k$.