Solid Mechanics Written PhD Qualifying Exam

June 15, 2007

9 am to 12 pm

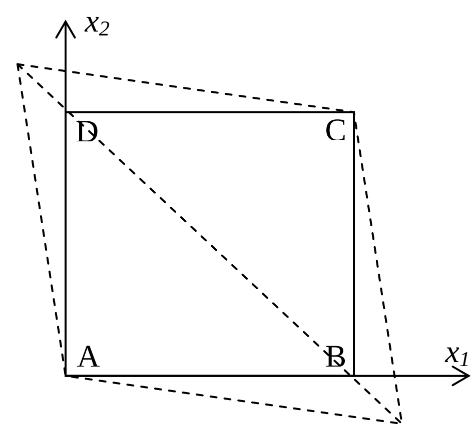
Please attempt all five problems

1. A thin, square sheet is subjected to uniform tractions along its four edges. The length of the sheet is L. It is linearly elastic and isotropic with a Young's modulus E and Poisson's ratio ν. The Airy stress function for the problem is $\phi = a(x_1^2 + x_2^2) + bx_1x_2$ and the deformed shape (exaggerated for clarity) of the sheet is shown below.

   a) Determine the tractions.

   b) Determine the in-plane displacements u_1 and u_2 given that the length of the diagonal AC is unchanged under the applied tractions.

   c) If the diagonal BD increases in length by 0.2% and $\nu = 1/3$, determine the tractions in terms of the shear modulus $G = \dfrac{E}{2(1+\nu)}$.

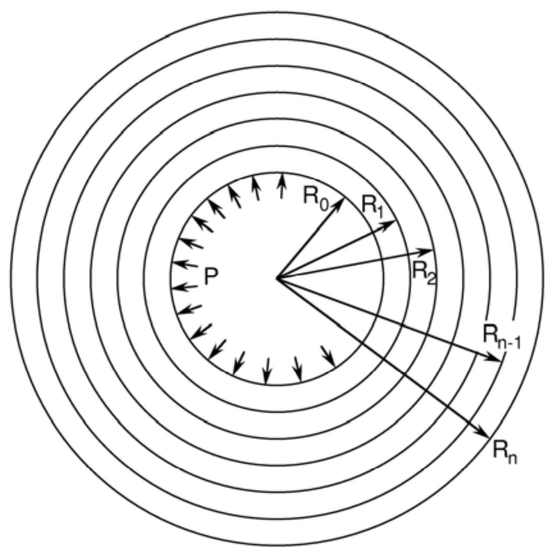


2. A pressure vessel is made up of n contacting cylinders of the same material and internal and external radii of

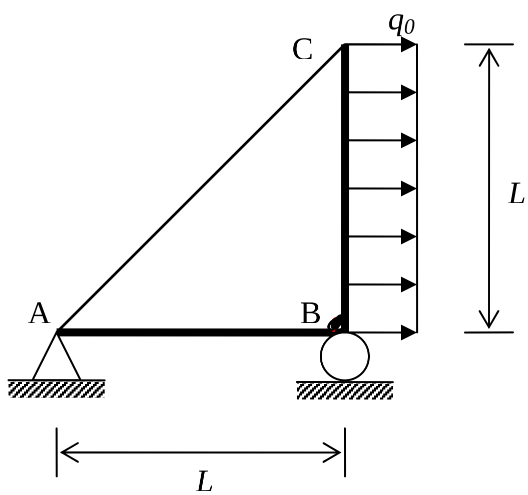
$$(r_o, r_1),\ (r_1, r_2), \ldots\ldots\ldots\ldots (r_{n-1}, r_n).$$

The vessel carries an internal pressure of P. Assume that in this design all cylinders are expected to yield on their inner surface simultaneously. Find a recursion expression between the applied pressure, the yield stress (σ_o) and the ratios of the radii (r_{k-1}/r_k, $k = 1, n$). You can neglect the axial stress and assume that the material yields according to the Tresca criterion.

Hint: The simultaneous yielding is an artificial situation hard to realize in practice, but do not be concerned about it.

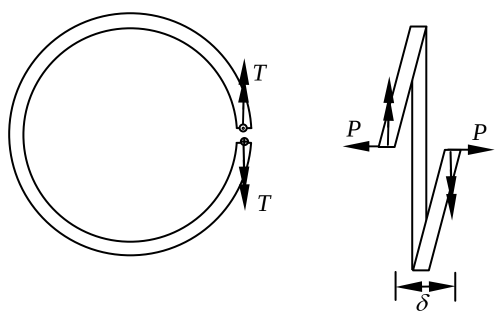


3. The frame ABC consists of beams of bending rigidity EI and length L. The frame is reinforced with a cable with an axial rigidity EA that connects A to C. BC is loaded with a uniformly distributed load q_o per unit length. Use an energy method to determine the horizontal deflection of point C in terms of the problem variables.



4. A nearly complete loop of a ring with radius R and cross sectional diameter d is made of a material with Young's modulus E. A force P and a torque T are applied as indicated in the figure:

   a) Determine the variation of the bending and twisting moment along the perimeter of the ring.
   b) Determine the gap δ between the loads (assume that the deflections are small).
   c) Explain how the above analysis may be used in evaluating the spring constant of a spring. Indicate, through appropriate free-body-diagrams, how the analysis should be modified if initial pitch angle of the spring is not zero.



5. A thin layer of material 1 is sandwiched between two thick blocks of material 2 (see Fig. 1). Material 1 has a smaller coefficient of thermal expansion than material 2 (i.e., $\alpha_1 < \alpha_2$). Upon cooling ($\Delta T < 0$), material 1 develops a biaxial compressive stress of magnitude σ_T in the plane of the laminate, while the two thick blocks are nearly stress free. Of course, this statement is only valid far away from the surface $x_1 = 0$. In fact, it was observed in experiments that the thin layer cracked near the edge along the x_1 axis, indicating that a tensile stress σ_{22} developed there. Linear superposition can be used to find the magnitude of this tensile stress, as illustrated in Fig. 2. In Problem A, there is a temperature change $\Delta T < 0$ and we apply a compressive traction of magnitude σ_T on the edge of the thin layer, so that the stress field in the thin layer in Problem A is uniform and equi-biaxial. In problem B, we apply a tensile traction on the edge of the thin layer. The original problem is the superposition of Problem A and Problem B. For simplicity, assume that the two materials have identical elastic properties (E and ν).

   a) Find the thermal stress σ_T far away from the edge, in terms of α_1, α_2, ΔT, E and ν.
   b) Find the stresses σ_{11} and σ_{22} in the thin layer, along the x_1 axis.

Hint: The solution to Problem B can be found by using the solution to the Flamant Problem with a concentrated line force acting on a half space, for which the Airy stress function takes the form, $\phi(r,\theta) = Ar\theta\sin\theta$*, where A is a constant to be determined, and* $\theta = \tan^{-1}(x_2/x_1)$ *for the coordinates in Fig. 1. Note that you only need to determine the stresses along* $x_2 = 0$ *and hence you may find it convenient to convert an integration with respect to* x_2 *to an integration with respect to* θ*, i.e., along* $x_2 = 0$, $dx_2 = -x_1 d\theta / \cos^2\theta$.

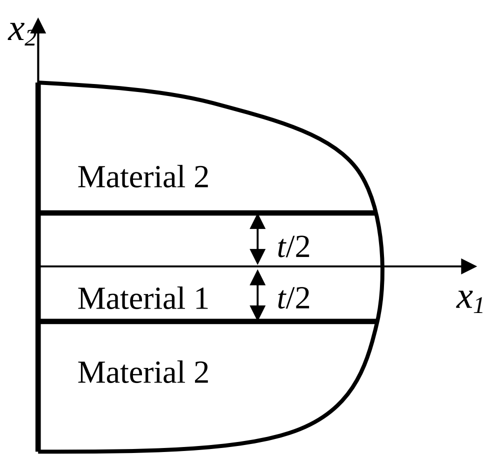


Figure 1

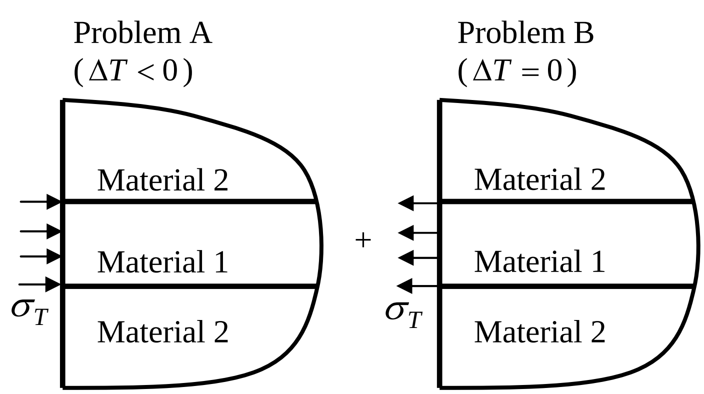


Figure 2

Equations

$$\sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} \qquad \overset{n}{T}_i = \sigma_{ji}n_j$$

$$\sigma_{ij,j} + f_i = 0$$

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{2\,\sigma_{zr}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + f_r = 0$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{z\theta}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + f_\theta = 0$$

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + f_z = 0$$

$$\varepsilon_{rr} = u,_r\,; \quad \varepsilon_{\theta\theta} = \frac{1}{r}(u + v,_\theta); \quad \varepsilon_{zz} = w,_z$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}u,_\theta + v,_r - \frac{v}{r}\right); \quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}w,_\theta + v,_z\right); \quad \varepsilon_{zr} = \frac{1}{2}\left(u,_z + w,_r\right)$$

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\,\theta,_{ij} = -\frac{-\nu}{1-\nu}\delta_{ij}\,f_{k,k} - (f_{i,j} + f_{j,i})$$

$$\sigma_{11} = \phi,_{22} \qquad \sigma_{22} = \phi,_{11} \qquad \sigma_{12} = -\phi,_{12}$$

$$\sigma_{rr} = \frac{1}{r}\phi,_r + \frac{1}{r^2}\phi,_{\theta\theta}\,; \; \sigma_{\theta\theta} = \phi,_{rr}\,; \sigma_{r\theta} = -\left(\frac{1}{r}\phi,_\theta\right),_r$$

$$\nabla^4\phi = 0 \text{ for } f_i = 0$$

$$\nabla^4\phi = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\phi,_{rr} + \frac{1}{r}\phi,_r + \frac{1}{r^2}\phi,_{\theta\theta}\right)$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

$$\sigma_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}$$