

There are 5 problems of equal weight. Start each solution on a new page. Write your Student Number on each page.

Problem 1

Consider an elliptic orbit transfer from an initial elliptic orbit to a final elliptic orbit about a point mass with gravitational parameter μ . The transfer is executed with a delta-v combination consisting of a first delta-v vector $\Delta\bar{v}_1$ to leave the initial orbit and enter the transfer orbit and a second delta-v vector $\Delta\bar{v}_2$ to leave the transfer orbit and enter the final orbit. The semimajor axes and eccentricities of the initial, transfer, and final orbits are known: (a_i, e_i) , (a_t, e_t) , and (a_f, e_f) , where i refers to the initial orbit, t refers to the transfer orbit, and f refers to the final orbit. Assume the orbits are coplanar but do **not** have collinear lines of apsides.

Provide the steps and equations needed to determine the minimum delta-v combination ($\Delta v = |\Delta\bar{v}_1| + |\Delta\bar{v}_2|$) to complete the transfer. Define all variables and provide diagrams labeling the velocity vectors, delta-v vectors, and any other appropriate variables used in your calculations. State any additional information, if any, needed to solve the problem.

Problem 2

A rocket is launched in such a way that the trajectory up and down is a straight line perpendicular to the surface of the Earth. The rocket dry mass plus payload is M_R . The mass of the fuel and oxidizer at the launch time t_0 is $M_F(t_0)$. The constant exhaust speed of the vehicle rocket motor is V_E . Assume there is no atmosphere and there is a constant gravitational acceleration g , i.e., a flat Earth.

Derive from first principles an expression for the speed of the vehicle at any given time $V_V(t)$ as a function of V_E , $M_F(t_0)$, M_R , and g . Provide diagrams labeling the appropriate variables including the elemental mass dM_V .

Problem 3

The dynamics of a particle in the circular restricted 3-body problem with masses m_1 and m_2 ($m_1 \geq m_2$) are described by the scalar equations of motion:

$$\ddot{x} = +2\dot{y} + \frac{\partial U}{\partial x}, \quad \ddot{y} = -2\dot{x} + \frac{\partial U}{\partial y}, \quad \ddot{z} = + \frac{\partial U}{\partial z},$$

where $U \equiv$ potential function $= \frac{x^2 + y^2}{2} + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}$, and $\mu \equiv$ mass ratio parameter $= \frac{m_2}{m_1 + m_2} < 0.5$.

The coordinates x, y, z and their rates $\dot{x}, \dot{y}, \dot{z}$ (the velocity components) are referenced to a rotating coordinate system centered at the common center of mass with the x-axis pointing toward the smaller mass, the z-axis pointing along the angular velocity of the orbit of the smaller mass with respect to the larger mass, and the y-axis pointing toward the velocity of the small mass about the origin to complete the right-handed system. The distances between the particle and the larger mass and smaller mass are $r_1 = [(x + \mu)^2 + y^2 + z^2]^{1/2}$ and $r_2 = [(x - 1 + \mu)^2 + y^2 + z^2]^{1/2}$, respectively. The unit of time has been chosen so that the constant of gravitation is unity.

Derive an expression for the only known integral of the problem in terms of the coordinates x, y and the velocity components \dot{x}, \dot{y} . Using this result, and any necessary sketches, argue that a low altitude orbit about m_2 with any orbit plane orientation will remain bounded to it and never escape it.

Problem 4

Consider a spacecraft in orbit about the Moon, in which the restricted problem of three bodies is a satisfactory model (Earth-Moon-spacecraft) with the assumptions that the Earth-Moon orbit is circular and the Earth and Moon are spheres with uniform mass density. The spacecraft is placed in an orbit with semimajor of 1920 km (lunar radius is 1738 km) and small eccentricity, and such that at time $t = 0$ the inclination is 90 degrees with respect to the Moon-Earth orbital plane and the line of nodes is perpendicular to the Moon-Earth line.

Determine and describe the evolution with time of the spacecraft line of nodes over a long time scale (e.g., years) with respect to the Moon-Earth line. Provide a verbal description with supporting equations. If your description is approximate, assess the accuracy.

Problem 5

An impulsive velocity change maneuver $\Delta \mathbf{v}_0$ at time t_0 is applied to a particle that places it on an escape trajectory relative to a central body with gravitational parameter μ . The trajectory $\mathbf{x}(t)$ and state transition matrix $\Phi(t_f, t_0)$ from t_0 to t_f ($t_f > t_0$) are known:

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{pmatrix} \quad \text{and} \quad \Phi(t_f, t_0) = \begin{pmatrix} \frac{\partial \mathbf{r}_f}{\partial \mathbf{r}_0} & \frac{\partial \mathbf{r}_f}{\partial \mathbf{v}_0} \\ \frac{\partial \mathbf{v}_f}{\partial \mathbf{r}_0} & \frac{\partial \mathbf{v}_f}{\partial \mathbf{v}_0} \end{pmatrix}$$

where \mathbf{v}_0 is the velocity at t_0 after the velocity impulse is applied. Assume the presence of other perturbing accelerations such that specific energy and angular momentum are **not** conserved between t_0 and t_f .

What is the sensitivity of the magnitude of the hyperbolic excess velocity $v_{\infty f}$ at t_f with respect to $\Delta \mathbf{v}_0$? In other words, what is $\partial v_{\infty f} / \partial \Delta \mathbf{v}_0$?

Useful Information

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial D}{\partial M}$$

$$\frac{de}{dt} = \frac{(1-e^2)^{1/2}}{na^2 e} \left[(1-e^2)^{1/2} \frac{\partial D}{\partial M} - \frac{\partial D}{\partial \omega} \right]$$

$$\frac{di}{dt} = \frac{1}{h \sin i} \left(\cos i \frac{\partial D}{\partial \omega} - \frac{\partial D}{\partial \Omega} \right)$$

$$\frac{d\Omega}{dt} = \frac{1}{h \sin i} \frac{\partial D}{\partial i}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{h \sin i} \frac{\partial D}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2 e} \frac{\partial D}{\partial e}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e} \frac{\partial D}{\partial e} - \frac{2}{na} \frac{\partial D}{\partial a}$$

where $h = na^2(1-e^2)^{1/2}$.

$$\frac{da}{dt} = \left(\frac{2a^2 e}{h} \sin f \right) \hat{R} + \left(\frac{2a^2 h}{\mu r} \right) \hat{T}$$

$$\frac{de}{dt} = \frac{h}{\mu} \left[(\sin f) \hat{R} + \left(\frac{e + 2 \cos f + e \cos^2 f}{1 + e \cos f} \right) \hat{T} \right]$$

$$\frac{di}{dt} = \left[\frac{r}{h} \cos(\omega + f) \right] \hat{N}$$

$$\frac{d\Omega}{dt} = \left[\frac{r \sin(\omega + f)}{h \sin i} \right] \hat{N}$$

$$\frac{d\omega}{dt} = -\left(\frac{h}{\mu e} \cos f \right) \hat{R} - \left[\frac{r}{h} \cot i \sin(\omega + f) \right] \hat{N} + \left[\frac{(h^2 + r\mu) \sin f}{\mu e h} \right] \hat{T}$$

$$\frac{dM}{dt} = n - \left[\frac{1}{na} \left(\frac{2r}{a} - \frac{1-e^2}{e} \cos f \right) \right] \hat{R} - \left[\frac{1-e^2}{nae} \left(1 + \frac{r}{p} \right) \sin f \right] \hat{T}$$

where $\hat{R}, \hat{T}, \hat{N}$ are components of force in the radial, along-track and cross-track directions.

$$P_n(\cos S) = P_n(\sin \phi) P_n(\sin \phi') + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} \left[Q_{nm}(\phi, \lambda) Q_{nm}(\phi', \lambda') + W_{nm}(\phi, \lambda) W_{nm}(\phi', \lambda') \right]$$

where $\begin{cases} Q_{nm}(\phi, \lambda) = P_{nm}(\sin \phi) \cos m\lambda \\ W_{nm}(\phi, \lambda) = P_{nm}(\sin \phi) \sin m\lambda \end{cases}$ and $\begin{cases} Q_{nm}(\phi', \lambda') = P_{nm}(\sin \phi') \cos m\lambda' \\ W_{nm}(\phi', \lambda') = P_{nm}(\sin \phi') \sin m\lambda' \end{cases}$

$$(a/r) = 1 + e \cos M + O(e^2) \quad , \quad (a/r)^2 = 1 + 2e \cos M + O(e^2) \quad , \quad (a/r)^3 = 1 + 3e \cos M + O(e^2)$$

$$GM_{earth} = 398600.44 \text{ km}^3/\text{s}^2 \quad , \quad GM_{moon} = GM_{earth}/81.301$$

$$\int \ln(a+bx) dx = b^{-1} [(a+bx) \ln(a+bx) - (a+bx)] \quad , \quad \int (a+bx)^{-1} dx = b^{-1} \ln|a+bx| \quad , \quad \int b^x dx = b^x / \ln b$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \quad , \quad \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \quad , \quad \tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

$$\sin \alpha = \sin \beta \sin \gamma$$

