ASE/EM Math Qualifying Examination
Date: 06/09/2008.

Work all six problems

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1. Consider the mechanical system consisting of three vibrating masses m connected with elastic springs with the same stiffness k, shown in Fig.1. Formulate the system of three ordinary differential equations describing the motion of the mechanical system, and find its general solution.

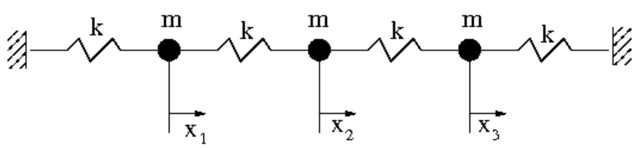

Figure 1: A mechanical system with 3 degrees of freedom

2. Consider the matrix eigenvalue problem,

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

where $\boldsymbol{A}$ is the 3×3 matrix,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Determine the eigenvalues and eigenvectors of the matrix. What is the *algebraic* multiplicity and *geometric* multiplicity (number of linearly independent eigenvectors) of each eigenvalue. Are they equal to each other in this case? For what types of matrices must they be equal?

b) Find the general solution of the system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{1}$$

c) Recall the notion of a generalized eigenvector:

$\boldsymbol{g}$ is a generalized eigenvector of matrix $\boldsymbol{A}$ corresponding to eigenpair $(\lambda, \boldsymbol{e})$ [1] if,

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{g} = \boldsymbol{e}$$

[1] The notion can be generalized recursively to $\boldsymbol{e}$ being itself a generalized eigenvector.

Determine the generalized eigenvector(s) of matrix $\boldsymbol{A}$ under consideration. Select a generalized eigenvector that is orthogonal to eigenvectors corresponding to $\lambda = 1$ and discuss how the generalized eigenvectors enable solution of an arbitrary ODE system (1). *Hint:* Recall Step 2 of this problem.

3. Use calculus of variations to demonstrate that the geodesics on the sphere are the great circles, by carrying out the following argument. Given arbitrary points A, B on a sphere with radius R, align spherical coordinates θ, ψ for the sphere in such a way that $\theta_A = \theta_B = 0, -\pi/2 \le \psi_A < \psi_B \le \pi/2$ (see Fig. 2 for the definition of the coordinates). Write down then the formula for the length of an *arbitrary* curve connecting points A and B using the parametrization:

$$\theta = f(\psi), \quad \psi \in [\psi_A, \psi_B]$$

Formulate the corresponding minimization problem for function $f(\psi)$, and solve it.

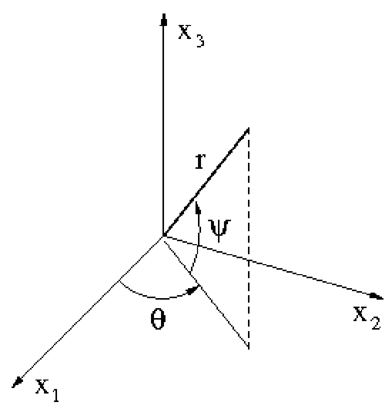

Figure 2: Definition of spherical coordinates (r, θ, ψ)

4. Consider the function $f(x) = x$ for $-\pi < x \le \pi$ and extended periodically.

   a) Find the Fourier Series for f.

   b) Does the series converge to f, and if so, in what sense?

   c) If we differentiate the Fourier Series term-by-term, does this series converge to $f'(x)$?

5. Consider the vibrating string problem with damping proportional to the velocity:

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} - 2\frac{\partial y}{\partial t}, &\qquad 0 < x < 1,\ t > 0 \\ y(0,t) = y(1,t) = 0, &\qquad t > 0, \\ y(x,0) = f(x),\ y_t(x,0) = 0, &\qquad 0 < x < 1 \end{aligned}$$

Solve using separation of variables.

6. Consider the real integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2 - 2x + 6} dx$$

a) Without evaluating it, prove that this integral is finite.

b) Evaluate it using any means you wish.