Mathematics Qualifying Exam
2006

1. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} + \omega^2 x = F_0 \delta(t - t_0) \\ x(0) = \dot{x}(0) = 0\,, \end{cases}$$

where δ denotes the Dirac delta, $t_0 > 0$, and F_0 is a parameter. Solve this problem using either (a) elementary calculus or (b) the Laplace transform. If you take approach (a) explain how the delta distribution translates into interface conditions on the solution at $t = t_0$. If you take approach (b), first define the Laplace transform for the delta distribution and compute it, solve for the Laplace transform of the solution x, and use the Residue Theorem to compute the inverse Laplace transform.

2. Solve the following eigenvalue problem.

   - Derive the formula for the gradient in $\Re^2$ in polar coordinates,

   $$\boldsymbol{\nabla} u = \frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta$$

   - Use the formula for the gradient to derive the formula for the Laplacian in polar coordinates,

   $$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

   - Consider the eigenvalue problem,

   $$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

   where Ω is the segment of the unit disk,

   $$\Omega = \{(r, \theta) \,:\, 0 < r < 1,\ 0 < \theta < \frac{3}{2}\pi\}$$

   Prove that the eigenvalues λ must be real and positive.

   - Use separation of variables to solve the eigenvalue problem.

3. Let $\phi_1, \phi_2, \ldots$ be normalized eigenfunctions of a Sturm-Liouville problem on [a,b]; that is

$$(\phi_i, \phi_j)_w = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

where

$$(\phi, \psi)_w = \int_a^b w(x)\phi(x)\psi(x)dx$$

and $w > 0$. Let f be a square integrable function on [a,b].

- Determine the linear combination

$$f_N = \sum_{n=1}^{N} k_n \phi_n$$

which gives the 'best approximation' to f in the sense that the error $I_N(f) = (f - f_N, f - f_N)_w$ is minimized.

- Prove Bessel's inequality

$$\sum_{n=1}^{\infty} k_n^2 \leq (f, f)_w.$$

4. Let $\mathbf{F}(x, y, z) = -y\mathbf{i} + xy\mathbf{j} - xyz\mathbf{k}$ and let Σ be the surface consisting of the half-cone $z = \sqrt{x^2 + y^2}$ for $x^2 + y^2 \leq 9$ with $x \geq 0$. Sketch the surface Σ. Verify Stokes Theorem

$$\int_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma = \int_C \mathbf{F} \cdot d\mathbf{R}.$$

5. Given $\underline{w}$ is a left eigenvector of matrix A with real negative eigenvalue λ; that is $\underline{w}^T A = \lambda \underline{w}^T$:

   - Find an expansion for $\underline{w}^T e^{At}$ using the eigenvalue λ, where $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$.
   - Let $\alpha < \beta$. The set $\{\underline{z} \in \Re^N \mid \alpha \leq \underline{w}^T \underline{z} \leq \beta\}$ is referred to as a *slab*. Briefly explain this terminology based on the mathematical definition. Draw a picture in $\Re^2$.
   - Consider the system $\dot{\underline{x}} = A\underline{x}$. Show that for all $t \geq 0$, $\underline{x}(t)$ lies in the slab $\{\underline{z} \in \Re^N \mid 0 \leq \underline{w}^T \underline{z} \leq \underline{w}^T \underline{x}(0)\}$

6. Using the following facts about the function $f(t)$:

   - $f(t)$ is real.
   - $f(t)$ is periodic with period $T = 4$ and it has complex Fourier series coefficients a_k (that is, $f(t) = \sum_{-\infty}^{\infty} a_k e^{ik\pi t/2}$).
   - $a_k = 0$ for $|k| > 1$.
   - The function $g(t)$ with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd.
   - $\frac{1}{4} \int_4 |f(t)|^2 dt = \frac{1}{2}$, where $\int_4$ means integration over any period of f.

   What is $f(t)$?