

Summer 2007  
PhD Qualifying Examination  
Control Theory

1. (15 points) Consider dynamical system

$$\dot{\mathbf{x}} = A\mathbf{x} + q(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_o$$

where  $\mathbf{x}(t) \in \mathcal{R}^n$  and the matrix  $A$  is Hurwitz. We assume  $q : \mathcal{R}^n \rightarrow \mathcal{R}^n$  is sufficiently regular to ensure existence and uniqueness of solutions. Further, we have  $\|q(\mathbf{x})\| \leq \alpha\|\mathbf{x}\|$  for all  $\mathbf{x}$  but  $q(\mathbf{x})$  is otherwise an unknown function. Determine (largest possible) finite constant  $\bar{\alpha}$  such that for all  $\alpha \leq \bar{\alpha}$  and for all initial conditions  $\mathbf{x}_o \in \mathcal{R}^n$ , the state solution  $\mathbf{x}(t)$  remains uniformly bounded and  $\lim_{t \rightarrow \infty} \mathbf{x}(t) = 0$ . Your derivation must establish explicit conditions on  $\bar{\alpha}$ , i.e., it should be easily calculated using standard matrix operations (eigenvalues, singular values, solutions to Lyapunov or Riccati equations, etc.), and the problem data.

2. (20 points) Consider a single-input LTI system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Suppose we are interested in transferring the system state from initial value  $\mathbf{x}(0) = [0, 2]^T$  to  $\mathbf{x}(2) = [2, 0]^T$ . Your tasks are as follows:

- (a) (10 points) Determine the *optimal* (minimum energy) control signal  $u^*(t)$  for the desired finite-time state transfer which minimizes the integral performance metric

$$J = \int_0^2 u^2(t) dt$$

- (b) (10 points) Find a piece-wise constant input of the form

$$u(t) = \begin{cases} \alpha, & 0 \leq t < 1, \\ \beta, & 1 \leq t \leq 2 \end{cases}$$

for some suitable values for scalar constants  $\alpha$  and  $\beta$  such that the stated state transfer objective is accomplished. Compare the control costs associated with this piece-wise constant control signal with the optimal controller  $u^*(t)$  according to the same the integral performance metric listed in part (a) above.

3. (15 points) Consider an inverted pendulum formed by a massless rod of length  $l$ , and a point mass  $m$ . The rod pivots about point  $O$  while mass  $m$  moves with the free end of the rod. At the pivot, the system is subject to a control torque ( $T$ ), frictional losses that are proportional to the rod's rate of rotation, and stiffness that is proportional to the angular displacement. When the rod is aligned with the vertical (up) axis, the mass is located directly below a range sensor that measures the inertial distance from the sensor to the mass.
- (a) (5 points) Derive a linear model that describes motion of the pendulum about an arbitrary controlled equilibrium orientation.
- (b) (5 points) Identify the plant transfer function.
- (c) (5 points) For the plant in (d), assuming unity feedback, provide a suitable argument to justify whether or not a closed loop system can achieve zero steady state tracking error when subject to a step reference input and a proportional control law.
4. (15 points) Figure 1 illustrates a Bode plot of magnitude for a plant of unknown form. Using only the information from this plot, deduce a representative form for the plant transfer function. Provide adequate justification for the origin of each term in your answer.

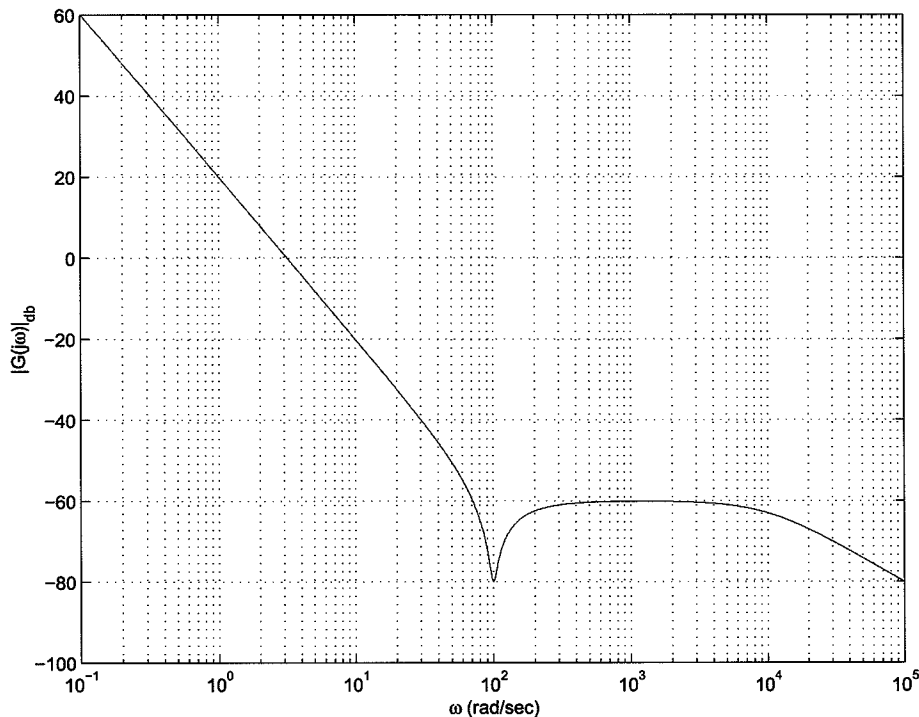


Figure 1: Bode Plot - Magnitude (dB) vs. Frequency (rad/sec)

5. (15 points) Consider the unity feedback system shown in Figure 2. The system has two parameters, the controller gain  $K$  and the constant  $K_1$  in the process.

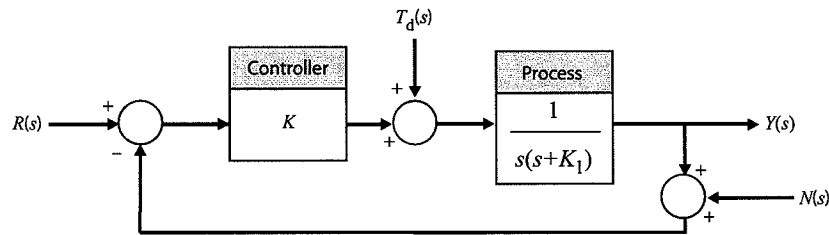


Figure 2: Closed-loop feedback system with two parameters,  $K$  and  $K_1$ .

- (3 points) Calculate the sensitivity of the closed-loop transfer function to changes in  $K_1$ .
  - (3 points) At what frequency does the sensitivity calculated in (a) reach a maximum?
  - (3 points) How would you select a value for  $K$  to minimize the effects of external disturbances,  $T_d(s)$ ?
  - (3 points) How would you select a value for  $K$  to minimize the effects of the measurement noise,  $N(s)$ ?
  - (3 points) Is it possible to reduce the sensitivity of the closed-loop system to changes in  $K_1$  at low frequencies, and simultaneously reduce the effects of external disturbances and noise on the system response? Explain your answer.
6. (20 points) A unity feedback system with a time delay is shown in Figure 3. The time delay is given by  $T$  (seconds). The controller is a proportional-derivative controller with gains  $K_p = 10$  and  $K_d = 2$ .

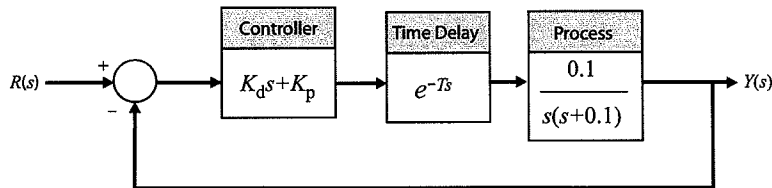


Figure 3: Unity feedback system with a time delay and PD controller.

- (10 points) Estimate the phase margin of the system for  $T=0$  seconds.
- (5 points) What happens to the phase margin as the time delay increases from  $T=0$  to  $T=0.1$  second?
- (5 points) What is the (approximate) maximum time delay allowed before the system becomes unstable?