Chapter 2. Linear Elastodynamics

2.1. Fundamental Boundary-Initial Value Problems in Elastodynamics

We begin with a brief description of the linear elastodynamic theory. Complete treatments of the topic including solution techniques and details of the classical solutions can be found in the monographs by Graff, (1975) Achenbach (1973) and Miklowitz (1978). Consider a body occupying the region $R$ with boundaries $\partial R$. Let the displacement vector $\mathbf{u}$ depend on the position vector $\mathbf{x}$ and time $t$ and be denoted by $\mathbf{u}(\mathbf{x},t)$. The strain tensor $\mathbf{\varepsilon}(\mathbf{x},t)$ is the symmetric gradient of $\mathbf{u}$:

$$
\mathbf{\varepsilon}(\mathbf{x},t) = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right].
$$

(1)

Here we consider the infinitesimal strain tensor and therefore neglect higher order terms involving higher powers of the gradient of $\mathbf{u}$. It is also assumed that $\mathbf{u}$ is a continuous function of $\mathbf{x}$, but it is not necessary to require that its derivatives with respect to $\mathbf{x}$ and $t$ be continuous as we shall see later. The material of the body is assumed to be homogeneous, isotropic and linearly elastic. Therefore, the stress tensor $\mathbf{\sigma}(\mathbf{x},t)$ is given by:

$$
\mathbf{\sigma}(\mathbf{x},t) = \lambda \mathbf{\varepsilon}_{kk} \mathbf{1} + 2 \mu \mathbf{\varepsilon}
$$

(2)

where $\mathbf{1}$ is the identity tensor and $\lambda$ and $\mu$ are the Lame constants. The balance of linear momentum results in the following equation of motion

$$
\nabla \cdot \mathbf{\sigma} + \rho \mathbf{f} = \rho \ddot{\mathbf{u}}
$$

(3)

where $\rho$ is the mass density, $\mathbf{f}$ is the body force per unit volume and the superdot indicates time derivatives. Symmetry of the stress tensor ensures the balance of angular momentum.

Substituting Eqs. (1) into (2) and the result into (3), the equations of motion can be obtained in terms of the displacements alone; these are the Navier's equations of motion:

$$
(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f} = \rho \ddot{\mathbf{u}}
$$

(4)

This is a system of three partial differential equations governing the motion of points in the body. In this chapter we shall assume that the body forces vanish and remove them in subsequent equations. To the set of equations (4), we must add initial conditions as well as boundary conditions. As in the quasi-static problem, there are three fundamental

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1 Standard index notation will be used throughout this chapter. Latin subscripts take the range 1,2,3 while Greek subscripts take the range 1,2. Repeated index implies summation over the range of the index and an index following a comma indicates partial differentiation with respect to the coordinate identified by that index.
problems that can be posed, depending on whether the displacements, tractions or some combination are prescribed on the boundaries. For the displacement boundary value problem

\[ u(x, t) = u^*(x, t) \]  

on \( \partial R \) for \( t > 0 \), where \( u^*(x, t) \) is a prescribed function. For the traction boundary value problem, the traction vector, \( s(x, t) \), is prescribed

\[ s(x, t) = \sigma(x, t)n = s^*(x, t) \]  

on \( \partial R \) for \( t > 0 \), where \( n \) is the unit outward normal and \( s^*(x, t) \) is a prescribed function.

The third problem is the mixed-boundary value problem for which the displacements are prescribed in a part of the boundary and tractions are prescribed over the remainder:

\[ u(x, t) = u^*(x, t) \quad \text{on } \partial_1 R \]
\[ s(x, t) = s^*(x, t) \quad \text{on } \partial_2 R \]  

(7)

For all three problems, initial conditions must be added to complete the formulation of the problems

\[ u(x, 0) = u_0(x) \]
\[ \dot{u}(x, 0) = \dot{u}_0(x) \]  

(8)

on \( R \), where \( u_0(x) \) and \( \dot{u}_0(x) \) are prescribed functions. It should be evident that even though we have written the governing equations in terms of displacements, the boundary conditions may be in terms of tractions, i.e., in terms of linear combinations of the derivatives of the displacement components and therefore complicating the solution of the problem.

Finally, the principle of conservation of energy (or the theorem of power expended) may be written as

\[ \int_{\partial R} \mathbf{s} \cdot \frac{\partial \mathbf{u}}{\partial t} \, dR + \int_R \rho \mathbf{F} \cdot \frac{\partial \mathbf{u}}{\partial t} \, dV = \frac{d}{dt} \int_R \left[ U(t) + T(t) \right] \, dV - \]  

(9)

where \( U(t) \) is the strain energy density and \( T(t) \) is the kinetic energy density per unit volume given by:

\[ U(t) = \int_R \frac{1}{2} \sigma \cdot \varepsilon \, dV \]  

(10)
\[ T(t) = \int_{R} \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \mathbf{u} dV \quad (11) \]

For problems in classical elastodynamics the conservation of energy provides a convenient way of approaching solutions; however, since there are no dissipative processes, it is seldom necessary to introduce the energy conservation equations explicitly into the problem formulation. On the other hand, in the fracture problems that are the focus of this chapter, dissipation is inherent in the problem. The fracture processes that occur in the crack tip region remove energy from the system and hence, we have to augment Eq. (9) to account for dissipation that occurs in the fracture process regions as we shall discuss later. Boundary-initial value problems posed within the context of linear elastodynamics above possess unique solutions (see Wheeler and Sternberg, 1968).

### 2.2. Bulk Waves

Equations (4) represent a hyperbolic system of partial differential equations and hence admit propagating wave solutions. The character of these waves can be obtained by considering special deformations. The Laplacian of \( \mathbf{u} \) in Eqs. (4) can be replaced using the following vector identity

\[ \nabla \times \nabla \times \nabla - \dot{\nabla} \nabla = 2 \nabla \]

so that:

\[ (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} = \rho \dot{\mathbf{u}}. \quad (12) \]

First, if we consider \( \mathbf{u} \) to be an irrotational deformation, \( \nabla \times \mathbf{u} = 0 \), we get:

\[ \nabla^2 \mathbf{u} = \frac{1}{C_d^2} \ddot{\mathbf{u}} \quad \text{with} \quad C_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \quad (13) \]

This deformation is seen to obey the standard wave equation with a characteristic speed \( C_d \). Such waves are called irrotational or dilatational waves. Next, if we consider the dilatation to be zero, \( \nabla \cdot \mathbf{u} = 0 \), we obtain the case of an equivoluminal deformation. In this case:

\[ \nabla^2 \mathbf{u} = \frac{1}{C_s^2} \ddot{\mathbf{u}} \quad \text{with} \quad C_s = \frac{\mu}{\sqrt{\rho}}. \quad (14) \]

Therefore, equivoluminal deformations also obey the standard wave equation but with a characteristic speed \( C_s \). Such waves are called equivoluminal or shear waves. Clearly \( C_d > C_s \); an observer at some distance from a source of these waves (such as an earthquake) will first receive the dilatational wave and then the equivoluminal waves; hence in the seismology literature these waves are called primary (P) waves and secondary (S) waves. While the wave speeds are expressed here in terms of the Lamé constants, materials are usually characterized in terms of the engineering constants \( E \) and \( \nu \), the modulus of elasticity and Poisson's ratio respectively. Conversion between these constants can be effected by using the following relationships:
\[ E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \]  

(15)

Now, the wave speeds may be expressed in terms of \( E \) and \( \nu \), but more importantly, the ratio of wave speeds depends only on the Poisson's ratio:

\[ \frac{C_d}{C_s} = \left( \frac{2 - 2\nu}{1 - 2\nu} \right)^{1/2} \equiv k \]  

(16)

Representative values of material properties are given in Table 1; these values are based on nominal values of modulus of elasticity, Poisson's ratio and density in order to provide an idea about the order of magnitude of the wave speeds. As can be seen from the values in the table, the shear wave speeds are typically about one-half of the dilatational wave speeds.

<table>
<thead>
<tr>
<th>Material</th>
<th>Modulus of elasticity ( E ) (GPa)</th>
<th>Poisson's Ratio ( \nu )</th>
<th>Density ( \rho ) (Mg/m(^3))</th>
<th>Dilatational Wave Speed ( C_d ) (m/s)</th>
<th>Distortional Wave Speed ( C_s ) (m/s)</th>
<th>Plate Wave Speed ( C_p ) (m/s)</th>
<th>Rayleigh Wave Speed ( C_R ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>High strength steel</td>
<td>200</td>
<td>0.3</td>
<td>7.8</td>
<td>5875</td>
<td>3140</td>
<td>5308</td>
<td>2913</td>
</tr>
<tr>
<td>Tungsten and alloys</td>
<td>406</td>
<td>0.3</td>
<td>13.4</td>
<td>6386</td>
<td>3414</td>
<td>5770</td>
<td>3167</td>
</tr>
<tr>
<td>Aluminum alloys</td>
<td>70</td>
<td>0.3</td>
<td>2.7</td>
<td>5908</td>
<td>3158</td>
<td>5338</td>
<td>2929</td>
</tr>
<tr>
<td>Alumina</td>
<td>390</td>
<td>0.22</td>
<td>3.9</td>
<td>10685</td>
<td>6402</td>
<td>10251</td>
<td>5858</td>
</tr>
<tr>
<td>Silicon Nitride</td>
<td>350</td>
<td>0.22</td>
<td>3.2</td>
<td>11175</td>
<td>6695</td>
<td>10721</td>
<td>6127</td>
</tr>
<tr>
<td>Silica glass</td>
<td>70</td>
<td>0.22</td>
<td>2.6</td>
<td>5544</td>
<td>3322</td>
<td>5319</td>
<td>3040</td>
</tr>
<tr>
<td>Homalite-100</td>
<td>4.5</td>
<td>0.34</td>
<td>1.2</td>
<td>2402</td>
<td>1183</td>
<td>2059</td>
<td>1104</td>
</tr>
<tr>
<td>Plexiglas</td>
<td>3.4</td>
<td>0.34</td>
<td>1.2</td>
<td>2088</td>
<td>1028</td>
<td>1790</td>
<td>960</td>
</tr>
<tr>
<td>Polycarbonate</td>
<td>2.6</td>
<td>0.40</td>
<td>1.2</td>
<td>1826</td>
<td>899</td>
<td>1565</td>
<td>839</td>
</tr>
<tr>
<td>Rubber</td>
<td>0.1</td>
<td>0.499</td>
<td>0.85</td>
<td>4434</td>
<td>198</td>
<td>396</td>
<td>189</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.499</td>
<td>0.85</td>
<td>1402</td>
<td>63</td>
<td>125</td>
<td>60</td>
</tr>
</tbody>
</table>

### 2.3. Lamé Solution

Since the problem considered here is linear, Eqs. (13) and (14) suggest that propagation of an arbitrary deformation that is a combination of dilatation and shear will be governed by both types of waves. This can be shown directly using the Lamé solution of the displacement equations of motion Eq. (4). Consider the following representation of the displacement vector \( \mathbf{u} \)

\[ \mathbf{u} = \nabla \varphi + \nabla \times \mathbf{\psi} \]  

(17)
where \( \varphi(x, t) \) is a scalar function and \( \psi(x, t) \) is a vector-valued function with \( \nabla \cdot \psi = 0 \). The displacement components obtained from Eqs. (17) will satisfy the differential equations Eq. (4) if \( \varphi(x, t) \) and \( \psi(x, t) \) are obtained as solutions of the following wave equations:

\[
\begin{align*}
\nabla^2 \varphi &= \frac{1}{C_d^2} \ddot{\varphi} \\
\nabla^2 \psi &= \frac{1}{C_s^2} \ddot{\psi}
\end{align*}
\]  

(18)

(19)

The scalar potential \( \varphi(x, t) \) corresponds to the dilatational wave and the vector potential \( \psi(x, t) \) corresponds to shear waves. The completeness of the Lamé decomposition of the displacement vector has been demonstrated by Clebsch, Somigliana and others; Sternberg (1960) provides a discussion of this decomposition.

It should be noted that the Lamé solution is really a reduction of the complicated hyperbolic system of equations for the displacement vector into two standard wave equations for the potentials \( \varphi(x, t) \) and \( \psi(x, t) \) coupled through the boundary conditions. As Miklowitz points out, the advantage of reformulating Navier's equations in terms of the Lamé potentials is that solutions and solution procedures developed for the standard wave equation can now be used to address problems associated with elastic wave propagation in solids.

### 2.4. Plane Waves

In order to gain insight into the wave character of the dynamic problem, consider the propagation of plane waves in the three dimensional solid. A plane is defined by \( x \cdot n = d \) where \( x \) represents the position vector, \( n \) is the normal to the plane and \( d \) is the distance from the origin to the plane along the normal. If the plane is assumed to move in the direction of the normal at a wave speed \( c \), then \( d = d_0 + ct \), where \( d_0 \) is the location of the plane at time \( t = 0 \) describes the propagation of the plane. Clearly, as the wave propagates, \( d_0 \) remains constant and is called the phase; the surface with constant phase (in this case the plane) is the wavefront. Now, applying this idea to elastodynamics, plane waves corresponding to dilatational and shear deformations can be represented as:

\[
\begin{align*}
\varphi &= \varphi(x \cdot n - C_d t) \\
\psi &= \psi(x \cdot n - C_s t)
\end{align*}
\]  

(20)

(21)

It is easily demonstrated that if \( \varphi(x, t) \) and \( \psi(x, t) \) are represented as above, they automatically satisfy the wave equations (18) and (19) respectively. To examine the plane waves further, without loss of generality, the direction of propagation can be taken to be the \( x_i \) axis; then using Eqs. (20) and (21) in Eq. (17), we obtain the displacement components:
The dilatational wave travels at the speed \( C_d \) and can only sustain a displacement \( u_1 \) in the direction of wave propagation, \( x_1 \); hence this is a longitudinal wave with the particle motion in the direction of the wave propagation. This is the \( P \) wave or dilatational; note that it could be a compressive or tensile wave depending on whether the particle motion is in the direction of motion or opposed to it. The shear wave travels at the speed \( C_s \) and can sustain displacement components \( u_2 \) and \( u_3 \), i.e., in the directions perpendicular to the wave propagation; hence these are transverse waves. As a consequence of our resolution of the vector along the Cartesian coordinates, the shear wave has been decomposed into two components, with particle motions in the \( x_2 \) and \( x_3 \). These are commonly called the SH and SV waves (for the horizontally polarized and vertically polarized shear waves where the \( x_3 \) axis points in the vertical direction).

2.5. Propagation of Discontinuities: Wavefronts and Rays

The displacement vector \( u(x,t) \) need not possess continuous derivatives; the governing equations \( (4) \) allow discontinuities in the derivatives of \( u(x,t) \) to exist along certain planes (called wavefronts) and to propagate along certain directions (called rays). Discontinuities in the spatial gradients of \( u(x,t) \) imply a discontinuity in the strains and stresses and discontinuities in the temporal gradients of \( u(x,t) \) indicate jumps in the particle velocity and/or acceleration. While such discontinuities cannot be sustained physically, rapid changes in \( u(x,t) \) that occur over a very short distances or time intervals are approximated as discontinuous jumps in the gradients. This representation is useful in characterizing the variations in the strains, stresses and velocities generated by suddenly applied loads. Love (1926) described the kinematic and dynamic conditions that must hold on a surface of discontinuity. With the normal to the discontinuity denoted by \( n \), the kinematic and dynamic jump conditions are

\[
\dot{u}_i = -c[u_{i,j}]n_j - \rho \sigma_{ij}n_j
\]  
\[
\rho \dot{u}_i = -\lambda \delta_{ij}[u_{k,k}]n_j - \mu [u_{i,j}]n_j - \mu [u_{j,j}]n_j
\]  

where \( \rho \) is the density, \( c \) is the appropriate wave speed and the square bracket around a quantity indicates the jump in that quantity across the discontinuity. Equations \( (23) \) and \( (24) \) can be interpreted using the equations of motion \( (4) \). Introducing \( (4) \) into \( (24) \) yields

\[
\rho \dot{u}_i = -\lambda \delta_{ij}[u_{k,k}]n_j - \mu [u_{i,j}]n_j - \mu [u_{j,j}]n_j
\]

which may be rearranged as follows:
\[
(\rho c^2 - \mu) \hat{u}_i = -c(\lambda + \mu)[u_{s,k}]n_j \tag{26}
\]

If we impose a velocity jump \( [\hat{u}_i] \) with zero a dilatation \( [u_{sk}] = 0 \), the jump propagates at a speed \( c = C_s \). In a similar manner, if we consider a velocity jump \( [\hat{u}_i] \) with \( \nabla \times \mathbf{u} = 0 \), equation (25) reduces to

\[
(\rho c^2 - (\lambda + 2\mu)) [\hat{u}_i] n_i = 0 \tag{27}
\]

which indicates that jumps in dilatation travel with the speed \( c = C_d \). Therefore, we might expect that if an arbitrary velocity jump is provided (through an external loading agent or from an internal source), both dilatational and shear waves propagate in the body carrying the appropriate jump discontinuities along both wavefronts.

The construction of wavefronts and rays is useful in understanding and interpreting the development stress fields in elastodynamic problems. So, we shall briefly outline the construction of the equations for the wavefronts and rays. The surface of discontinuity may be written as \( S(x,t) = \tau(x) - t = 0 \) or equivalently by \( t = \tau(x) \). At any point on the wavefront, the ray is normal to the wavefront; thus, the governing equation for the rays is obtained:

\[
\frac{dx}{dt} = cn = c \frac{\nabla \tau(x)}{\vert \nabla \tau(x) \vert} \tag{28}
\]

But,

\[
\frac{dS}{dt} = \frac{\partial \tau(x)}{\partial x_i} \frac{\partial x_i}{\partial t} - 1 = 0 \quad \text{or} \quad c\nabla \tau(x) \cdot n = 1
\]

Using the second expression in Eq. (28) and the above, we get the equation for the ray as:

\[
c\nabla \tau(x) = 1 \tag{29}
\]

This is called the *eikonal* equation, the terminology arising from geometrical optics. Using (29) in (28) results in the following equation for the rays:

\[
\frac{dx}{dt} = c^2 \nabla \tau(x) \tag{30}
\]

In the linearly elastic solid, the wave speeds are constant and therefore the rays are straight lines and the corresponding wavefronts are parallel surfaces. Therefore, the knowledge of the wavefront at some time \( t \) can be used to construct the wavefront at a later time \( t' \) simply by extending the rays along the normal to the wavefront by amount \( c(t' - t) \). In the optics literature, this is called *Huygens' principle* and Huygens' construction of wavefronts. This construction is quite useful in obtaining a quick, qualitative picture of the wave propagation event as we shall see in later examples.
Two-Dimensional Problems in Elastodynamics

So far, the three dimensional elastodynamic problem has been discussed. However, only two dimensional dynamic fracture problems are considered in this chapter. Therefore, the governing equations will now be reduced to the case of two dimensions for three special cases: anti-plane shear, plane strain and plane stress. The first two reductions are based on restrictions on the deformation while the third is based on an assumption regarding the stress tensor.

1.6.1. Anti-plane shear

It is assumed that the only nonzero displacement component is in the direction and further that it is a function only of and :

\[ u_\alpha = 0; \quad u_3 = u_3(x_1, x_2, t) \] (31)

Substituting into the governing equations (4) and setting the body forces to zero results in the following wave equation for :

\[ \nabla^2 u_3 = \frac{1}{C_s^2} \ddot{u}_3 \] (32)

The corresponding nonzero components of the stress tensor are given by:

\[ \sigma_{3\alpha} = \mu u_{3,\alpha} \] (33)

Clearly, only shear waves arise in this problem; these are the horizontally polarized shear waves. The wave motion is in the plane with the transverse particle motion directed towards . Note that the reduction in the equations of motion can also be effected at the level of the Lamé potentials, but this is not really necessary since the equation in terms of the nonzero displacement component is already quite simple.

1.6.2. Plane strain

It is assumed that is constant and further that the remaining components are independent of :

\[ u_\alpha = u_\alpha(x_1, x_2, t); \quad u_3 \propto x_3 \] (34)

As a consequence, the strain displacement relations in Eq. (1) yield:

\[ \varepsilon_{\alpha\beta}(x_1, x_2, t) = \frac{1}{2} \left( u_{\alpha,\beta} + u_{\beta,\alpha} \right), \quad \varepsilon_{33} = \text{const} \quad \varepsilon_{3\alpha} = 0; \] (35)

The stress-strain relations in Eq. (2) reduce to:
\[ \sigma_{\alpha\beta}(x_1, x_2, t) = \lambda \epsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \epsilon_{\alpha\beta} \]
\[ \sigma_{33}(x_1, x_2, t) = \nu \sigma_{\gamma\gamma} \]
\[ \sigma_{3\alpha}(x_1, x_2, t) = 0 \]

(36)

It is simpler in this case to use the Lamé solution; corresponding to the above assumption, we have

\[ \varphi = \varphi(x_1, x_2, t) \quad \text{and} \quad \psi = \psi(x_1, x_2, t)e_3 \]  

(37)

where \( e_3 \) is the unit vector in the \( x_3 \) direction. Therefore we have two scalar potentials in the plane-strain problem that satisfy

\[ \nabla^2 \varphi = \frac{1}{C_d^2} \ddot{\varphi} \]  

(38)

\[ \nabla^2 \psi = \frac{1}{C_s^2} \ddot{\psi} \]  

(39)

where \( \nabla^2 \) is the two dimensional Laplacian operator. From these equations, it is clear that under conditions of plane strain, there are still dilatational and shear waves that propagate with speeds \( C_d \) and \( C_s \) respectively as in the full three dimensional problem.

At this point, we record the relationship between the potential and the nonzero displacement and stress components:

\[ u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2} \]
\[ u_2 = \frac{\partial \varphi}{\partial x_2} - \frac{\partial \psi}{\partial x_1} \]  

(40)

and

\[ \sigma_{11} = \lambda \nabla^2 \varphi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right] \]
\[ \sigma_{22} = \lambda \nabla^2 \psi + 2\mu \left[ \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right] \]
\[ \sigma_{12} = \mu \left[ 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi}{\partial x_2^2} - \frac{\partial^2 \psi}{\partial x_1^2} \right] \]  

(41)

1.6.3. Plane stress

In the last of the plane problems, the following assumption is imposed on the stress components:
\begin{equation}
\sigma_{3t} = 0
\end{equation}
\begin{equation}
\sigma_{a\beta} = \sigma_{a\beta}(x_1, x_2, t)
\end{equation}

Introducing the above assumptions into equations (4), results in the following displacement equations for plane stress:

\begin{equation}
\frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)} \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} = \rho \ddot{\mathbf{u}}
\end{equation}

Comparing with equations (12), it can be shown that there are once again two waves as in the plane strain and three dimensional cases, a dilatational wave, with a speed \( C_d^p \) (the superscript \( p \) is for the plane stress dilatational wave speed) and a shear wave with a speed \( C_s \). The plane stress dilatational wave speed is given by:

\begin{equation}
C_d^p = \sqrt{\frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)}} = \sqrt{\frac{E}{\rho(1-v^2)}}
\end{equation}

For problems involving thin plates, \( C_d^p \) is the dilatational wave speed appropriate for plane stress conditions; Table 2.5.1 shows that this speed is slightly smaller than the bulk dilatational wave speed. Apart from this, the plane stress problem with the assumptions imposed in Eqs. (42) is indistinguishable from the plane strain problem. Of course, the strain component \( \varepsilon_{33} \) is given in terms of the in-plane components of stress:

\begin{equation}
\varepsilon_{33} = 0
\end{equation}
\begin{equation}
\varepsilon_{33} = -\frac{\nu \sigma_{aa}}{E}
\end{equation}

Lateral inertia in the \( x_3 \) direction has been ignored; this analysis is appropriate when the wavelengths of the disturbances are long in comparison to the plate thickness.

2.7. Surface Waves

So far we have considered bulk waves in the medium. Rayleigh discovered that when free surfaces are present a wave that travels along the surface and decays into the medium is generated. Consider the medium occupying the region \( x_2 > 0 \), with a traction free boundary at \( x_3 = 0 \). A surface wave propagating along the \( x_1 \) direction with a speed \( c \) and decaying into the \( x_2 \) direction can be represented as

\begin{equation}
\varphi = f(x_2) \exp\{i\kappa(x_1 - ct)\}
\end{equation}
\begin{equation}
\psi = g(x_2) \exp\{i\kappa(x_1 - ct)\}
\end{equation}
where \( i = \sqrt{-1} \), \( \kappa = \omega / c \), is the wave number and \( \omega \) is the frequency of a time harmonic disturbance. Substituting this into the wave equations in Eq. (38 and 39), results in two ordinary differential equations for \( f \) and \( g \):

\[
\frac{d^2 f}{dx_2^2} - \kappa^2 \alpha_d^2 f = 0
\]

\[
\frac{d^2 g}{dx_2^2} - \kappa^2 \alpha_s^2 g = 0
\]

where

\[
\alpha_d = \sqrt{1 - \frac{c^2}{C_d^2}} \quad \text{and} \quad \alpha_s = \sqrt{1 - \frac{c^2}{C_s^2}}.
\]

Solving these equations and retaining only the solution that decays as \( x_2 \to \infty \), the solution becomes:

\[
\varphi = A \exp(-\alpha_d \kappa x_2) \exp\{i\kappa(x_i - ct)\}
\]

\[
\psi = B \exp(-\alpha_s \kappa x_2) \exp\{i\kappa(x_i - ct)\}
\]

The stress components corresponding to this can be determined from Eqs. (41). Applying the traction free boundary conditions at \( x_2 = 0 \):

\[
(1 + \alpha_s^2)A + i2\alpha_s B = 0
\]

\[-i2\alpha_d A + (1 + \alpha_d^2)B = 0
\]

For a nontrivial solution the determinant of coefficients must be zero:

\[
R(c) = 4\alpha_d \alpha_s - (1 + \alpha_s^2)^2 = 0
\]

The function \( R(c) \) is called the Rayleigh function and its variation with \( c \) is shown in Figure 1. Letting \( C_r / C_s = k_r \), where \( C_r \) is the root of Eq. (51) and recalling from Eq. (16) that the ratio of the bulk wave speeds, \( k \), depends only on the Poisson's ratio, the above can be rewritten as:

\[
4\sqrt{1-k_r^2 / k^2} \sqrt{1-k_r^2} - (2-k_r^2)^2 = 0
\]

This is a cubic equation for \( k_r^2 \), and depends only on the Poisson's ratio. The roots of this equation indicate the propagation speed of the surface wave assumed in Eq. (46). Physically meaningful solutions to (51) must be in the range \( 0 < k_r < 1 \). For Poisson's ratio in the range of \( 0 < \nu < 0.5 \), at least one real solution exists. This solution
corresponds to the Rayleigh surface wave. Viktorov (1967) has developed an approximate representation for the Rayleigh wave speed, $C_R$:

$$k_R = \frac{C_R}{C_s} = \frac{0.862 + 1.14\nu}{1 + \nu}$$

(53)

Clearly, $C_R < C_s$; Table 1 lists values of the Rayleigh wave speed for selected materials. Eqs (48) and (49) indicate that the Rayleigh wave travels along the surface in the $x_1$ direction, and experiences an exponential decay along the $x_2$ direction.

In this section, we have posed the fundamental problems and described the nature of propagating waves within this theory; solving boundary value problems under dynamical loading still requires enormous effort; typically integral transform methods and Green’s function methods are used in effecting solutions. For propagating cracks, the further complication of moving boundary conditions must be addressed. In the following sections, we shall describe some of the methods used and solutions obtained for a few crack problems.

![Figure 1. Variation of the Rayleigh function $R(c)$ with speed.](image-url)
A PHOTOLEASTIC STUDY OF STRESS WAVE PROPAGATION IN A QUARTER-PLANE

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Fig. 4. Isochromatic fringe pattern showing the early response of the quarter-plane ($t=106$ μsec).