Student number ______

PhD Written Qualifying Exam 2011

Systems

SYSTEM THEORY 2011

1. For the dynamic system given by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 \end{aligned}$$

   (a) Formulate the third Liapunov Theorem on the linear approximations.

   (b) Find all equilibrium points of the system and determine the corresponding linear approximations ("abridged systems"). Classify the stability of these points.

   (c) Sketch a phase portrait of this system. Clearly label equilibrium points, trajectories and directions, and areas of stability (if any).

   (d) Find a suitable Liapunov function for this system and, if possible, find a region of stability using this function.

2. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} + \dot{x} = -\delta(t-1) \\ x(0) = \dot{x}(0) = 0 \, , \end{cases}$$

where δ denotes the Dirac's delta.

(a) Explain how the presence of the delta functional translates into an interface condition, and solve the problem using elementary calculus

(b) Define the Laplace transform for the delta distribution and compute it.

(c) Compute the Laplace transform of the initial-value problem.

(d) Use the attached Table of Laplace Transforms to compute the inverse Laplace transform and compare it with the solution obtained using the elementary calculus.

Table of
Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	t
4	$2!/s^3$	t
5	$3!/s^4$	t^3
6	$m!/s^{m+1}$	t^m
7	$1/(s+a)$	e^{-at}
8	$1/(s+a)^2$	te^{-at}
9	$1/(s+a)^3$	$\frac{1}{2!}t^2e^{-at}$
10	$1/(s+a)^m$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$a/(s^2+a^2)$	$\sin at$
18	$s/(s^2+a^2)$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$

3. Consider the problem of optimizing the performance index

$$J = xy$$

subject to the inequality constraint

$$x + y \geq 1.$$

Find and identify (max, min) the optimal points. Only consider solutions for which $x, y \geq 0$. Use a slack variable and a Lagrange multiplier to solve the problem.

4. Solve the following optimal control problem:

a. Find the control history $u(t)$ that minimizes the performance index

$$J = x_{3_f}$$

subject to the differential constraints

$$\dot{x}_1 = u$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = u^2/2$$

and the prescribed boundary conditions

$$t_0 = 0, \quad x_{1_0} = 0, \quad x_{2_0} = 0, \quad x_{3_0} = 0$$

$$t_f = 1, \quad x_{1_f} = 0, \quad x_{2_f} = 1$$

b. Apply the Legendre-Clebsch condition and the Weierstrass condition to verify that the solution can be a minimum.

c. What is the conjugate point condition? What does it mean if a conjugate point exists within the interval of integration?

5. (a) Starting from the batch equations:

$$\hat{x}_k = P_k(H_k^T W_k y_k + \bar{P}_k^{-1}\bar{x}_k) \qquad P_k = (H_k^T W_k H_k + \bar{P}_k^{-1})^{-1}$$

derive the sequential equations:

$$P_k = (I - K_k H_k)\bar{P}_k \qquad K_k = \bar{P}_k H_k^T (H_k \bar{P}_k H_k^T + W_k^{-1})^{-1}$$

$$\hat{x}_k = \bar{x}_k + K_k(y_k - H_k \bar{x}_k)$$

(b) Explain why H_k can be replaced by $\tilde{H}_k$ in the equations in (a) to obtain

$$P_k = (I - K_k \tilde{H}_k)\bar{P}_k \qquad K_k = \bar{P}_k \tilde{H}_k^T (\tilde{H}_k \bar{P}_k \tilde{H}_k^T + W_k^{-1})^{-1}$$

$$\hat{x}_k = \bar{x}_k + K_k(y_k - \tilde{H}_k \bar{x}_k)$$

(c) For processing a scalar observation using the sequential equations, what are the dimensions of K_k?

(d) In the sequential algorithm, the estimate and covariance are propagated to a different time using the state transition matrix, as follows:

$$\bar{x}(t) = \Phi(t, t_k)\hat{x}_k \qquad \bar{P}(t) = \Phi(t, t_k) P_k \Phi^T(t, t_k)$$

Starting from these equations, and given that

$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k)$$

derive the following alternate equations for propagating the estimate and covariance:

$$\dot{\bar{x}}(t) = A(t)\bar{x}(t) \qquad \dot{\bar{P}}(t) = A(t)\bar{P}(t) + \bar{P}(t)A^T(t)$$

6. Assume you have the linearized data equation

$$y = Hx + \epsilon$$

where y is $m \times 1$, H is $m \times n$, x is $n \times 1$ and is the true deviation away from the nominal trajectory, and ϵ is $m \times 1$ with

$$E[\epsilon] = 0 \qquad E[\epsilon\epsilon^T] = R$$

Assume $m < n$.

(a) Derive the equation for the Minimum Norm estimate of x.

(b) What is the expected value of the estimate derived in (a)?

(c) Is the Minimum Norm estimate an unbiased estimate? Explain.

(d) Derive the equation for the covariance of this estimate.

(e) Given the linearized data equation

$$y(t) = x_1 + tx_2 + t^2 x_3 + \epsilon(t)$$

and one observation y=42 at t=2, calculate the Minimum Norm estimate of

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$