Student Number ______

PhD Written Qualifying Exam 2011

Mathematics

ASE/EM Math Qual Exam 2011

1. (a) Let X and Y be two Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$. Let $A : X \to Y$ be a linear operator. Consider the linear equation,

$$Ax = f$$

Formulate the sufficient and necessary condition (in terms of adjoints) for the right-hand side f, for a solution x to exist.

(b) Use the criterion to determine the value of α for which the linear system

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 1 \end{pmatrix}$$

has a solution. Is the solution unique? *Hint:* You may solve first the problem using elementary means but you are expected to demonstrate how to solve the problem using adjoints.

2. Consider a dynamic system given by:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2\end{aligned}$$

(a) Formulate the third Liapunov Theorem on the linear approximations.

(b) Find all equilibrium points of the system and determine the corresponding linear approximations ("abridged systems"). Classify the stability of these points.

(c) Sketch a phase portrait of this system. Clearly label equilibrium points, trajectories and directions, and areas of stability (if any).

(d) Find a suitable Liapunov function for this system and, if possible, find a region of stability using this function.

3. (a) State the Gauss Divergence Theorem relating volume and surface integrals.

(b) Verify the theorem for a cylindrical domain:

$$x^2 + y^2 \leq 2, \quad 0 \leq z \leq 1$$

and the vector field:

$$\boldsymbol{v} = (2, 0, 3z^4)$$

4. (a) State the Laurent Expansion Theorem.

(b) Expand

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

around $z = 0$ in the domain $1 < |z| < 2$.

5. Use whatever means you can, to solve the Poisson equation in the square domain:

$$\begin{cases} -\Delta u &= 1 \quad \text{in } \Omega = (0,\pi)^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

6. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} + \dot{x} = -\delta(t-1) \\ x(0) = \dot{x}(0) = 0\,, \end{cases}$$

where δ denotes the Dirac's delta.

(a) Explain how the presence of the delta functional translates into an interface condition, and solve the problem using elementary calculus

(b) Define the Laplace transform for the delta distribution and compute it.

(c) Compute the Laplace transform of the solution to the initial-value problem.

(d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the "Laplace domain" and compare it with the solution obtained using the elementary calculus.