

ASE/EM Math Qual Exam 2012

1. (a) If

$$L = p(t) \frac{d^2}{dt^2} + q(t) \frac{d}{dt} + r(t)$$

and we use the standard inner L^2 -inner product (real spaces only),

$$(x, y) := \int_a^b x(t)y(t) dt$$

demonstrate that the *formal adjoint* of operator L is given by the formula,

$$L^* = p \frac{d^2}{dt^2} + (2p' - q) \frac{d}{dt} + (p'' - q' + r)$$

- (b) Determine sufficient conditions on coefficients p, q, r (the more general, the better) for the operator L to be self-adjoint.
- (c) Provide sample boundary conditions for operator L under which the operator is self-adjoint.

2. (a) Discuss shortly the relation between Jordan Decomposition Theorem for a matrix A and solution of system of ODEs (with constant coefficients) of the form

$$\dot{x} = Ax$$

- (b) Find the general solution for the system

$$\begin{cases} 2\dot{x} = 5x + y \\ 2\dot{y} = -x + 3y \end{cases}$$

3. (a) Recall the general formulas for gradient and curl in an arbitrary curvilinear system of coordinates ξ_j in \mathbb{R}^3 ,

$$\begin{aligned}\nabla u &= \frac{\partial u}{\partial \xi_j} \mathbf{a}^j \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{E}}{\partial \xi_j} \times \mathbf{a}^j\end{aligned}$$

where \mathbf{a}^j are the co-basis vectors. Specialize the formulas to the standard cylindrical system of coordinates (r, θ, z) .

- (b) Verify that the force field (given in polar coordinates),

$$\mathbf{F} = \theta \mathbf{e}_r + \mathbf{e}_\theta + \mathbf{e}_z$$

is conservative. Here $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ denote the cylindrical coordinates unit vectors.

- (c) Determine the scalar potential of \mathbf{F} . Is it unique ?

4. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} + \dot{x} = -\delta(t - 2) \\ x(0) = 1, \dot{x}(0) = 0, \end{cases}$$

where δ denotes the Dirac's delta "function".

- (a) Define precisely delta functional and reinterpret its action in terms of appropriate jump conditions.
- (b) Solve the problem using elementary means.
- (c) Define the Laplace transform. Apply it to both sides of the equation and find the solution in the Laplace domain.
- (d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the Laplace domain and compare it with the solution obtained using the elementary calculus.

5. Use separation of variables, superposition and method of images to solve the problem shown in Fig. 1

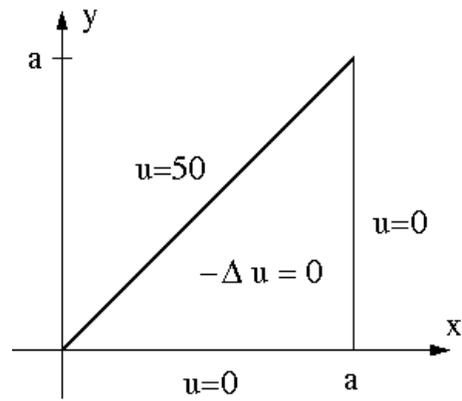


Figure 1: Steady-state heat conduction in a triangle.

6. Use whatever means necessary to solve the heat conduction problem:

$$\begin{cases} \alpha^2 u_{xx} = u_t & 0 < x < \infty, t > 0 \\ u(x, 0) = 0 & x > 0 \\ u(0, t) = 50 & t > 0 \end{cases}$$