Student number ______

PhD Written Qualifying Exam 2011

Controls

Summer 2011
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Justify all the important steps in your solution process

1. (10 points) Given constant $n \times n$ matrices M and N, show that the state-transition matrix for the unforced linear system

$$\dot{\boldsymbol{x}}(t) = e^{-Mt} N e^{Mt} \boldsymbol{x}(t), \quad \boldsymbol{x}(t_o) = x_o$$

is given by

$$\Phi(t, t_o) = e^{-Mt} e^{(M+N)(t-t_o)} e^{Mt_o}$$

2. (20 points) For an $n \times n$ matrix $A(t)$, suppose there exist finite positive constants α and μ such that for all t, we have $\|A(t)\| \leq \alpha$ and the pointwise-in-time eigenvalues of $A(t)$ satisfy $\mathrm{Re}[\mathrm{eig}_A(t)] \leq -\mu$. If $P(t)$ is the unique symmetric positive-definite solution for

$$A^T(t)P(t) + P(t)A(t) = -I_{n \times n}$$

then show that the linear system

$$\dot{\boldsymbol{x}}(t) = \frac{1}{2}[2A(t) - P^{-1}(t)\dot{P}(t)]\boldsymbol{x}(t)$$

is uniform exponentially stable (UES).

3. (20 points) Consider the single-input, single-output linear system

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t), \quad y(t) = [1 \;\; 1]\, \boldsymbol{x}(t)$$

Design a two-dimensional state observer such that the state estimation error decays exponentially at a rate $\lambda = 10$. Following this, you are also required to design a reduced-order observer for this system while satisfying the same error convergence rate requirement.

4. (20 points) Consider a simple pendulum, illustrated in Figure 1, formed by a massless rod of length l and a particle of mass m attached to its free end. The pendulum pivots such that the path of the mass traces a curve on an invariant plane. The pendulum is subject to a control torque, $M(t)$. If rotational damping and stiffness are also present, the equation of motion

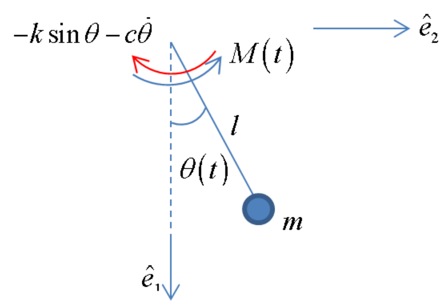

Figure 1: Simple Planar Pendulum w/ Control Torque

may be derived as

$$\ddot{\theta}(t) + \frac{c}{ml^2}\dot{\theta}(t) + \frac{k}{ml^2}\sin\theta(t) = \frac{M(t)}{ml^2} \tag{1}$$

where $c \geq 0$ and $k \geq 0$ are the damping and stiffness coefficients, respectively. A sensor, mounted at the pivot, is able to measure the angular deflection of the pendulum directly.

a. Linearize the equations of motion about some arbitrary controlled equilibrium point (i.e. $M_e \to \theta_e$) and find expressions for the Jacobian matrices of the corresponding linear state space model.

b. For $ml^2 = 1$, identify the range of c and k for which the open loop system is asymptotically stable, marginally stable, and unstable. Also, identify the range of c and k for which the system will exhibit oscillatory behavior. Justify all your answers.

c. Suppose the plant is driven by a controller, represented by transfer function $K(s) = k_p$, where $k_p > 0$. Find the closed loop transfer function associated with the corresponding unity feedback system.

d. If the reference input is a unit step, is it possible for the closed loop system to achieve zero steady state error? If so, identify the range of c, k, and k_p required to guarantee this outcome.

5. (10 points) A closed loop system has a characteristic equation of the form

$$10s^3 + 5s^2 + k_1 s + k_2 = 0 \tag{2}$$

a. Determine the range of k_1 and k_2 that guarantees asymptotic stability. Also, identify the ratio of k_1 to k_2 where the closed loop system becomes marginally stable and the associated crossover frequency.

b. Provide a reasonably accurate or representative sketch of the root locus for the case when $k_1/k_2 = 10$ and $k_1/k_2 = 2$.

6. (20 points) The stability of a unity feedback control system is dictated by a proportionality constant, K, such that the closed loop characteristic equation is given by:

$$1 + KG(s) = 0 \tag{3}$$

with open loop poles at 0, -2 and -10 and open loop zeros at $4 \pm 4j$. The associated root locus and bode plots are illustrated in Figures 2 and 3, respectively.

a. Identify the exact value of the control gain, K, where the dominant closed loop poles are indicative of a critically damped system response.

b. From the bode plot, measure the gain margin (in dB's) and the phase margin (in degrees). Label your measurements on the plot.

c. Identify the crossover frequency on the bode plot and the associated controller gain.

d. What stability information about the system can you derive from the gain margin?

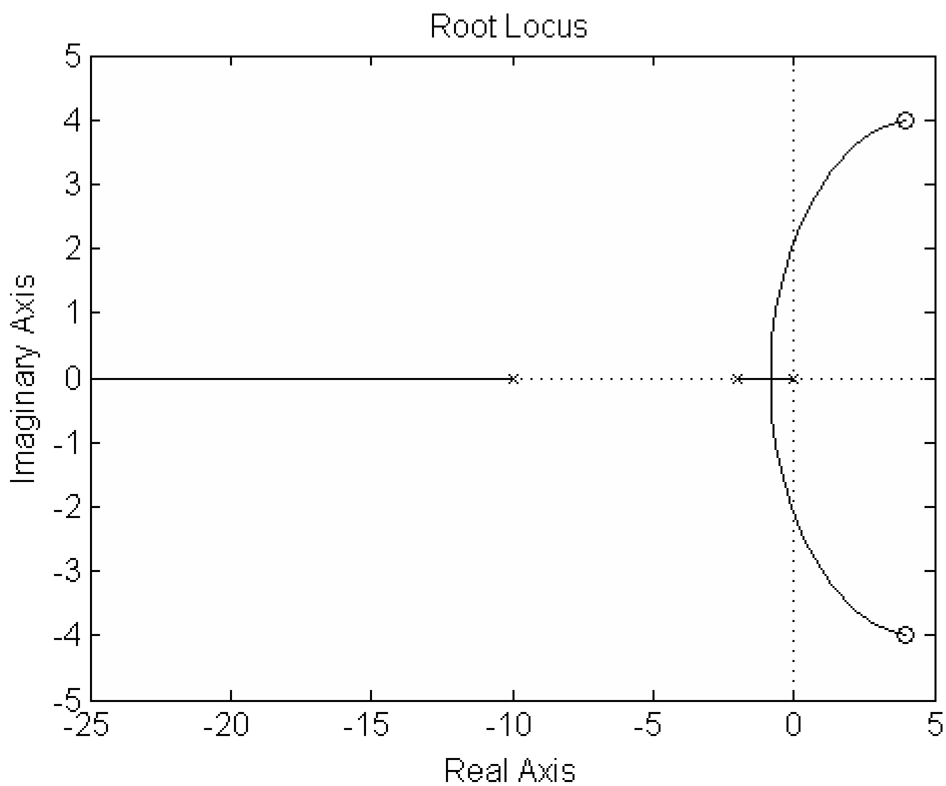

Figure 2: Root Locus

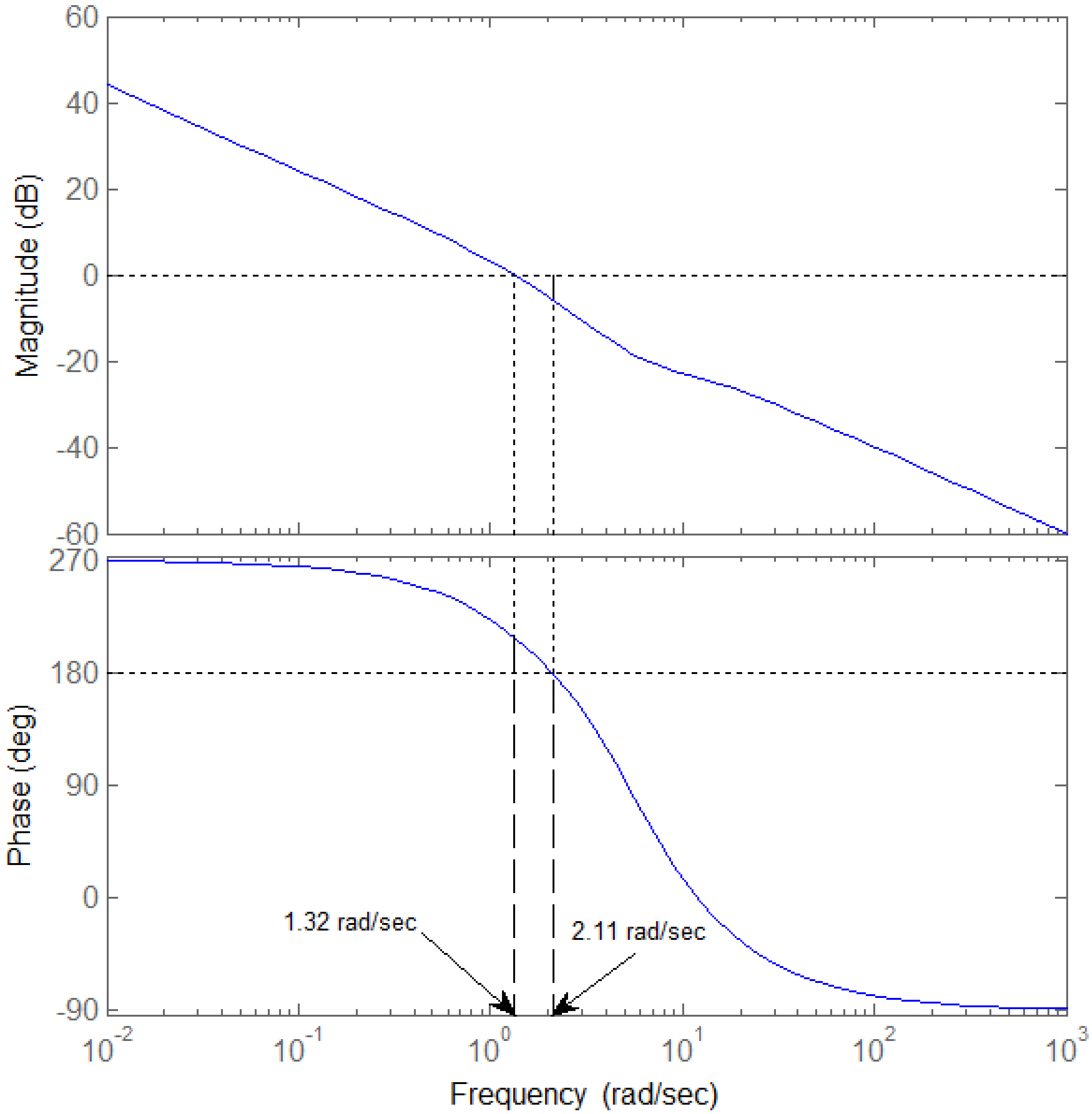

Figure 3: Bode Plot