SYSTEM THEORY

Problem 1.

For the nonlinear equation:

$$ml^2\ddot{\theta} + mgl\sin\theta = k(\beta - \theta)$$

where g, m, l, k, and β are known constants,

(a) Determine the linear approximation for small deviations $\delta\theta$ about $\theta = \pi/2$.

(b) Determine the linear approximation for the general case of small deviations $\delta\theta$ about $\theta = \theta_0$, subject to the equilibrium condition:

$$k(\beta - \theta_0) = mgl\sin\theta_0$$

Express your final result in simplest form.

Problem 2.

A sequence is given by the recursion formula:

$$x_{n+1} = x_n(2 - x_n) \qquad\qquad 0 < x_0 < 1$$

(a) Does the series converge? Why or why not?

(b) If the series converges, compute the limit.

3. Find the point x_1, x_2 which optimizes the performance index

$$J = x_1^2 + x_1 x_2 + x_2^2$$

subject to the constraint

$$x_1 + x_2 = 0$$

using a Lagrange multiplier. Show that this point satisfies the sufficient condition for a minimum.

4. Consider the motion of a vehicle whose acceleration can be controlled, that is, $\ddot{x} = a$ where the acceleration is bounded as $0 \leq a \leq 1$. If the system equation is rewritten as $\dot{x}_1 = x_2$ and $\dot{x}_2 = a$, the boundary conditions are chosen to be $t_0 = 0, x_{1_0} = 0, x_{2_0} = 0$ and $t_f = 1$. Find the control history that maximizes the final speed x_{2_f} of the vehicle.

   (a) List the first-order conditions, the Legendre-Clebsch condition, and the Weierstrass condition, and apply them to this problem.

   (b) The optimal path is composed of three possible subarcs. First, determine which subarc must be in effect at the final point. Then show that there cannot be a jump from either of the other subarcs to the final subarc so that the final subarc holds over the whole optimal.

   (c) Verify that the Legendre-Clebsch condition and the Weierstrass condition are both satisfied along the optimal path.

5. Assume you are given the linearized data equations $y = Hx + \epsilon$ where y is $m \times 1$, H is $m \times n$, x is $n \times 1$, and ϵ is $m \times 1$, where $m >$n.

(a) Using a geometric argument, explain why, in general, Hx cannot exactly equal y. Continue using this geometric approach to derive the normal equations for the least squares estimate of x. Carefully explain each step. Define any new variables you introduce.

(b) For this set of equations, assume

$$y = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

For this y, one of the following vectors is a valid value for $H\hat{x}$, where $\hat{x}$ is the least squares estimate.

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

Which of these vectors is a valid value for $H\hat{x}$? Explain your answer.

(c) If $(H^T H)^{-1}$ exists, the normal equations can be solved to give $\hat{x} = (H^T H)^{-1} H^T y$. Under what condition on the observations does $(H^T H)^{-1}$ exist? Assuming this condition is met, prove that $(H^T H)^{-1}$ does exist. Carefully explain each step. Define any new variables you introduce.

6. (a) Assume you are estimating n parameters using m observations, where $m < n$. Assume the linearized data equations are $y = Hx + \epsilon$, where y is $m \times 1$, H is $m \times n$, x is $n \times 1$ and is the true deviation away from the nominal trajectory, and ϵ is $m \times 1$ with $E[\epsilon] = 0$ and $E[\epsilon\epsilon^T] = R$. Assume you use the Minimum Norm method to obtain an estimate. What is the expected value of this estimate?

(b) Is the Minimum Norm estimate unbiased? Explain your answer.

(c) Assume you now combine the data in (a) with an *apriori* estimate $\bar{x}$, where $\bar{x}$ is an unbiased estimate of x with covariance $\bar{P}$. Assume there is no correlation between η and ϵ. Thus,

$$\bar{x} = x + \eta \qquad E[\eta] = 0 \qquad E[\eta\eta^T] = \bar{P} \qquad E[\epsilon\eta^T] = 0$$

Combine the data from (a) with this *apriori* estimate into one data equation. Explain why you can now use the Linear Unbiased Minimum Variance equations to calculate an estimate using this combined data. Use the Linear Unbiased Minimum Variance equations to write out the equation for the estimate.

(d) Show that the estimate in (c) is unbiased.

(e) Let the estimate from (a) be $\hat{x}_a$. Let the estimate from (c) be $\hat{x}_c$. Calculate the correlation between $\hat{x}_a$ and $\hat{x}_c$. That is, calculate $E[(\hat{x}_a - E[\hat{x}_a])(\hat{x}_c - E[\hat{x}_c])^T]$.