Department of Aerospace Engineering and Engineering Mechanics
The University of Texas at Austin

Orbital Mechanics Written Qualifying Exam

June 10 2004

Student Number______

Instructions. Answer all problems. Use separate sheets for each problem. There are 5 problems. The last page contains additional information and data. Each problem is worth 20 points This is a closed book and notes examination. If you want the solution to a problem to be evaluated, you must write clearly and legibly.

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1. An initial position vector $\mathbf{r}_0$ at time t_0 and a final position vector $\mathbf{r}_f$ at time t_f are specified. Assume two body motion is appropriate and that the position vectors are given, for example, in an planet-centered inertial frame with xyz axes (such as, Earth-centered inertial, ECI):

   (a) with this information, what types of problems can be solved and/or what are the applications?

   (b) derive expressions for the inclination and ascending node location in the xyz frame

   (c) on an ellipse with semi-major axis a, what is the distance between the vacant focus and the point on the orbit at $\mathbf{r}$? (the distance between the focus and this point is r)

   (d) For an elliptic arc between the specified position vectors, state Lambert's theorem symbolically or in words. What is(are) the unknown quantity(ies) in this problem?

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2. Two gravitational point masses with masses m_1 and m_2 orbit each other in circular orbits, with their relative distance characterized by the distance a.

   (a) Derive the equations of motion of a third particle with extremely small mass in comparison to m_1 and m_2. The equations should be expressed in a frame such that the x-axis remains coincident with the direction from m_1 and m_2, the z-axis coincident with the angular momentum vector of the m_1, m_2 system, the y-axis completes the right handed system, and the **origin is coincident with m_1**. The equations of motion can be expressed in vector form or in scalar form. If in scalar form, assume three degrees of freedom.

   (b) Derive any known integral(s) of motion for the particle and discuss its (their) significance.

Define any quantities if they are different than what is already given.

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3. Consider the acceleration per unit mass for a particle

$$\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r}) = \left(\frac{\partial U(r)}{\partial \mathbf{r}}\right)^{\top}$$

where $U(r)$ is a scalar potential function of $r \equiv |\mathbf{r}|$. At a given time t_0, the position vector is $\mathbf{r}(t_0) = \mathbf{r}_0$ and the velocity vector is $\mathbf{v}(t_0) = \dot{\mathbf{r}}(t_0) = \mathbf{v}_0$. A linearization along a solution of this system provides a relationship between position and velocity deviations at t_0 ($\delta\mathbf{r}_0$ and $\delta\mathbf{v}_0$) and position and velocity deviations at other times t ($\delta\mathbf{r}(t)$,$\delta\mathbf{v}(t)$). Let t be confined to the range $t_0 \leq t \leq t_f$.

(a) Derive the linearized equations of motion for $\delta\mathbf{r}(t)$ and $\delta\mathbf{v}(t)$.

(b) Assume that the linearized equations of motion for the position and velocity perturbations have been used to compute the state transition matrix $\Phi(t_f, t_0)$ for a particular solution,

$$\Phi(t_f, t_0) = \begin{pmatrix} \partial\mathbf{r}(t_f)/\partial\mathbf{r}(t_0) & \partial\mathbf{r}(t_f)/\partial\mathbf{v}(t_0) \\ \partial\mathbf{v}(t_f)/\partial\mathbf{r}(t_0) & \partial\mathbf{v}(t_f)/\partial\mathbf{v}(t_0) \end{pmatrix}$$

If $\delta\mathbf{r}_f$ and $\delta\mathbf{v}_f$ are known at t_f provide an expressions, free of any matrix inversions, for $\delta\mathbf{r}_0$ and $\delta\mathbf{v}_0$ in terms of $\Phi(t_f, t_0)$ and/or any of the submatrices of $\Phi(t_f, t_0)$.

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4. It is required that the ground track of an Earth satellite exactly repeats within a specified interval, referred to as the "repeat cycle." Assume the Earth is a sphere of constant density, the orbit inclination is 65 degrees and the eccentricity is 0.1. Determine the semimajor axis, a, for a repeat cycle of 8 days ($\pm$ 0.1 day), such that the semimajor axis lies between 6955 km and 6975 km.

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5. Assume the Earth, Moon and Sun are spheres of constant density with gravitational parameters GM_e, GM_m, GM_s respectively.

   (a) derive the equations of motion of the Moon with respect to the Earth. Identify the term(s) that are induced by the Sun.

   (b) the solar terms in part a.) can also be obtained by taking the gradient of a scalar disturbing function, R. Determine R.

   (c) It can be shown that R (from part b.) can be expanded in an infinite series in powers of the ratio of the Earth-Moon distance to the Earth-Sun distance. The lead term in this expansion is

   $$R = \frac{GM_s}{a_s^3} a^2 \left(\frac{-3}{8} \right) (\sin^2 i)$$

   where a_s is the Earth-Sun distance, a is the semimajor axis of the Moon's orbit, and i is the inclination of the Moon's orbit with respect to the ecliptic plane. This expression has assumed that the eccentricity of the Earth's orbit about the Sun and the Moon's orbit about the Earth are small. Given this approximation for the disturbing function, characterize the Sun's effect on the Moon's orbit in terms of temporal changes in the orbit elements, i.e., discuss qualitatively the evolution of each of the orbital elements.

   (d) The period of the Moon's node vector is 18.6 years; use this approximate model to verify this observation.

Lagrange's Planetary Equations from W.M. Kaula "Satellite Geodesy" Dover Publications 2000 page 29 Equations 3.38-3.40

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{de}{dt} &= \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} \\
\frac{d\omega}{dt} &= -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} \\
\frac{di}{dt} &= \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{dM}{dt} &= n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}
\end{aligned}
$$

where a is the semi-major axis, n is the mean motion, e is the eccentricity, i is the inclination, ω is the argument of periapsis, Ω is the ascending node, M is the mean anomaly, and R is the disturbing function. The mean motion is $n = \sqrt{\mu/a^3}$, where μ is the gravitational parameter of the central body.

Data:

$$
\begin{aligned}
GM_s &= 1.327 \times 10^{11}\text{km}^3/\text{s}^2 \\
GM_e &= 3.986 \times 10^{5}\text{km}^3/\text{s}^2 \\
GM_m &= 4.903 \times 10^{3}\text{km}^3/\text{s}^2 \\
a_s &= 1.496 \times 10^{8}\text{km} \\
a &= 3.844 \times 10^{5}\text{km} \\
i &= 5^\circ \\
e &= 0
\end{aligned}
$$